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Acoustic scattering by a semi-infinite angular sector with impedance boundary conditions, II: the far-field asymptotics

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This work is a natural continuation of our recent study devoted to the scattering of a plane incident wave by a semi-infinite impedance sector. We develop an approach that enables us to compute different components in the far-field asymptotics. The method is based on the Sommerfeld integral representation of the scattered wave field, on the careful study of singularities of the integrand and on the asymptotic evaluation of the integral by means of the saddle point technique. In this way, we describe the waves reflected from the sector, diffracted by its edges or scattered by the vertex as well as the surface waves. Discussion of the far-field in the so-called singular directions (or in the transition zones) is also addressed.

Keywords: far-field asymptotics, diffraction by an impedance sector.

1. Introduction

In our work (Lyalinov, 2018), we studied the problem[1] of diffraction of a plane wave by a semi-infinite angular sector S with impedance boundary conditions (Fig. 1) and developed a self-consistent approach to its solution. To this end, we made use of the Watson–Bessel (WB) integral representation and formulated the problem for the so-called spectral function. This problem is a problem for the Laplace–Beltrami operator on the unit sphere with the cut $\overline{AB} = S \cap S^2$ (Fig. 1) with complex boundary conditions on its sides $\sigma_{\pm}$, $\mathrm{mes}(\sigma_{\pm}) = 2a < \pi$. By means of the theory of extensions of sectorial sesquilinear forms we investigated solvability of this problem in appropriate functional classes. Then the WB representation has been transformed to the Sommerfeld integral. An analytic function in the integrand of the Sommerfeld integral, which is called Sommerfeld transformant, was proved to be **a** holomorphic function of the integration variable in a subdomain of the complex plane. However, in the complementary domains (the semi-strips in Fig. 9 of Lyalinov, 2018) the Sommerfeld transformant has singularities, namely poles or those of the branch-point type. It was remarked that these singularities played a crucial role in calculation of the far-field asymptotics (see discussion in Section 7 of Lyalinov, 2018 and in Lyalinov, 2013). Indeed, the asymptotics of the Sommerfeld integral representation of the solution as $kr \to \infty$ is computed by use of the saddle point technique; (r, ϑ, φ) is the spherical coordinate system attributed to $x = (x_1, x_2, x_3)$ in Fig. 1, $x_1 = r\cos\varphi\sin\vartheta$, $x_2 = r\sin\varphi\sin\vartheta$, $x_3 = r\cos\vartheta$.

To this end, the contour of integration is deformed into the steepest descent (SD) paths. In the process of such deformation different singularities of the integrand can be crossed. The location of these

[1] A closed formulation of the diffraction problem at hand is carefully discussed in Lyalinov (2018).

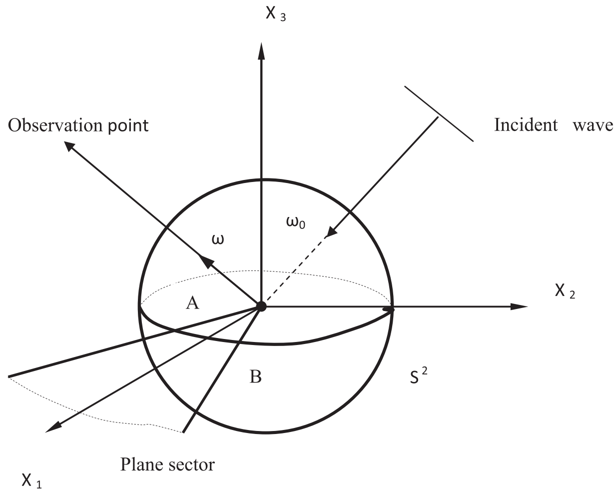

FIG. 1. Diffraction by an impedance sector.

sigularities on the complex plane depends on the direction of observation ω of the far field, $\omega := (\vartheta, \varphi)$. The singularities migrate provided the observation direction ω varies. In a particular case, when ω belongs to the so-called oasis Ω_0, $\omega \in \Omega_0$,[2] the singularities are not captured and the scattered field is specified by the contribution of the saddle points only, i.e. by the spherical wave from the vertex. As a result, the saddle points give rise to the spherical wave in the far field, whereas the contributions of the singularities are responsible for the other components like reflected or diffracted waves, including those concentrated near the surface of the sector which are naturally called surface waves. It is well known that an impedance surface can support the surface waves provided $\chi_\pm := \Im \eta_\pm < 0$, where $\eta_\pm = \mathrm{i}\chi_\pm$ are the surface impedances of the faces $S_\pm$ of the sector S corresponding to $\sigma_\pm$. The impedances are implied to be independent of the wave number k. These restrictions for the impedances are implied in what follows. These waves vanish exponentially, as $kr \to \infty$, outside the surface; however, they are only bounded on the surface if $\Re \eta_\pm = 0$.

In this work we study the so-called primary diffracted waves. This means that the 'secondary' diffracted waves from the edges are not considered. Such waves, for instance, arise due to the diffraction of the primary diffracted cylindrical wave from the edge[3] A that goes to and diffracts on the edge B producing the secondary diffracted cylindrical wave. In the general case, such waves have higher order with respect to the main small parameter $\frac{1}{kr}$ in comparison with the primary waves because the corresponding secondary singularities are of the lower order than those primary.

Let us also assume that the direction which the incident plane wave[4] comes from is specified by $\omega_0 = (\vartheta_0, \varphi_0)$,

$$U_i(kr, \vartheta, \varphi) = \exp\{-\mathrm{i}kr \cos\theta_i(\omega, \omega_0)\}\,. \tag{1}$$

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[2] Description of different characteristic domains, which are specified by the far-field components that are present in the asymptotics, is given in Section 2.2 of Lyalinov (2018).

[3] We denote the edges of the sector S and the ends of the cut $\overline{AB}$ by the same letters A and B, which doesn't lead to ambiguity.

[4] $\mathrm{e}^{-\mathrm{i}\widehat{\omega}t}$ time dependence is implied and omitted throughout the paper.

ω_0 is not close to the cut AB. We have

$$\cos\theta_i(\omega,\omega_0) = \cos\vartheta\cos\vartheta_0 + \sin\vartheta\sin\vartheta_0\cos[\varphi-\varphi_0];$$

$\theta_i(\omega,\omega_0)$ coincides with the geodesic distance on S^2 between two points ω and ω_0 denoted also $\theta(\omega,\omega_0)$, $\theta(\omega,\omega_0) = \theta_i(\omega,\omega_0)$.

The wave field $U(kr,\omega,\omega_0) + U_i(kr,\omega,\omega_0)$ is the sum of the scattered and incident fields, U fulfils the Helmholtz equation

$$(\triangle + k^2)U(kr,\omega,\omega_0) = 0, \tag{2}$$

$k > 0$ is the wave number. Let $\Sigma = S^2 \setminus \overline{AB}$ be the exterior of the cut on S^2 and $\sigma = \partial\Sigma$, $\sigma = S \cap S^2$ its boundary, $\sigma = \sigma_+ \cup \sigma_-$ and $\sigma_\pm$ are two sides of the cut AB. The impedance boundary condition

$$\frac{1}{r}\frac{\partial(U_i+U)}{\partial\mathcal{N}_\pm}\bigg|_{S_\pm} - \mathrm{i}k\eta_\pm\,(U_i+U)|_{S_\pm} = 0 \tag{3}$$

is satisfied on the faces $S_\pm$ of the sector S. The vectors $\mathcal{N}_\pm$ are in the tangent plane to S^2 at the points of $\sigma_\pm$, are orthogonal to $\sigma_\pm$ and point out to the 'exterior' of Σ.

The total field satisfies the Meixner's conditions and the radiation conditions at infinity ($kr \to \infty$). The latter are expressed in the form of the asymptotics described by the formulae (2.8)–(2.13) in Section 2.2 of Lyalinov (2018). It can be easily shown that for the problem at hand the solution depends on the product kr, whereas *a priori* one might expect the separate dependence on r and k. Our principal goal, in this work, is to describe a procedure that enables us to develop expressions for the different components in the far-field asymptotics from the Sommerfeld integral representation.

In the next section we discuss the Sommerfeld integral representations of the wave field and formulate the problems for the Sommerfeld transformants. Then we carefully study various singularities of the transformants and attribute them to the different components in the far field. In this way, we separately study the singularities corresponding to the reflected wave from the sector and to those attributed to the diffracted waves by the edges of the sector. The corresponding contributions to the asymptotics are calculated by means of the asymptotic evalutaion of the Sommerfeld integral. The mentioned singularities are on the real axis and give rise to the space waves. However, there are complex singularities that specify the surface waves which are negligible provided the observation point is not close to the surface of the sector. The asymptotic contribution of the complex singularities, as $kr \to \infty$, is also addressed.

However, the analysis becomes more complex provided that the singularities coalesce with each other or (and) with the saddle points of the Sommerfled integral. Such directions ω of observation (a singularity is close to a saddle point) are called "singular" because for these ω the diffraction coefficient of the spherical wave is singular and cannot be properly defined in a classical sense. In the asymptotically small angular vicinities of these directions (transition zones) the asymptotics are determined by special functions (integrals). From the physical point of view in the transition zones the fronts of the different waves are almost tangent, having approximately the same phases, and the waves interfere with each other producing complex asymptotic behaviour of the wave field in these directions. We specify such transition zones and study the wave field behaviour there. In the Conclusion we discuss our results and fix some remaining problems for the further studies.

It is worth remarking that our study has several common features with those in the previous works devoted to the ideal cones or sectors (see Lyalinov, 2013; Cheeger & Taylor, 1982; Smyshlyaev, 1991;

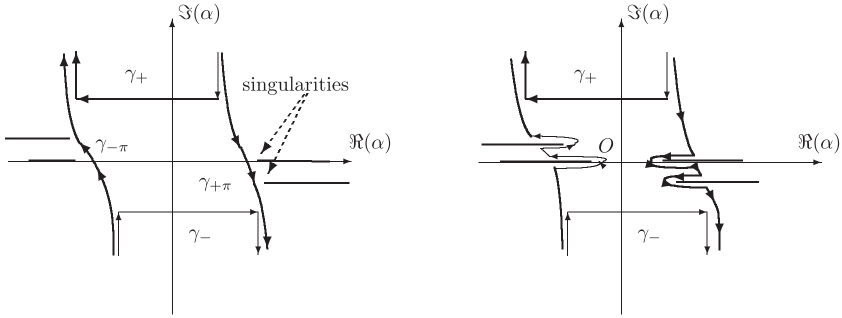

FIG. 2. Deformation of the Sommerfeld contour into the SD paths $\gamma_\pi \cup \gamma_{-\pi}$: (left) singularities are not captured, $\omega \in \Omega_0$; (right) singularities are captured, $\omega \notin \Omega_0$.

Smyshlyaev, 1990; Babich *et al.*, 2000; Bonner *et al.*, 2005; Assier & Peake, 2012a; Assier & Peake, 2012b; Shanin, 2011; Assier *et al.*, 2016, Assier & Shanin, 2019) and also to conical scatterers with impedance boundary conditions (Bernard, 2014; Bernard & Lyalinov, 2001; Bernard *et al.*, 2008; Lyalinov *et al.*, 2010; Lyalinov & Zhu, 2012; Babich *et al.*, 2008; Lyalinov, 2018). However, it seems that, except for the paper (Lyalinov, 2018), there are no works dealing with the acoustic diffraction by an impedance sector. More discussion of the relevant literature and motivation might be found in our recent works (Lyalinov, 2013, 2018).

2. Sommerfeld integral representations

It has been shown in our work (Lyalinov, 2018) that the Sommerfeld representations for the scattered field have the form

$$U(r,\vartheta,\varphi) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{\mathrm{e}^{-\mathrm{i}kr\cos\alpha}}{\sqrt{-\mathrm{i}kr}}\, \Phi(\alpha,\omega,\omega_0)\,\mathrm{d}\alpha \tag{4}$$

or

$$U(r,\vartheta,\varphi) = \frac{\sqrt{-\mathrm{i}kr}}{2\pi \mathrm{i}} \int_\gamma \mathrm{e}^{-\mathrm{i}kr\cos\alpha}\, \sin\alpha\, \widetilde{\Phi}(\alpha,\omega,\omega_0)\,\mathrm{d}\alpha \tag{5}$$

with

$$\Phi(\alpha,\omega,\omega_0) = \frac{\partial \widetilde{\Phi}(\alpha,\omega,\omega_0)}{\partial\alpha},$$

where $\gamma = \gamma_+ \cup \gamma_-$ is the Sommerfeld double-loop contour, Fig. 2. As we remarked the singularities of the Sommerfeld transformants $\Phi(\alpha,\omega,\omega_0)$, $\widetilde{\Phi}(\alpha,\omega,\omega_0)$ play a crucial role in studying of the far-field asymptotics by use of the Sommerfeld integral representations.

2.1 *Problems for the Sommerfeld transformants* $\Phi(\alpha,\omega,\omega_0)$, $\widetilde{\Phi}(\alpha,\omega,\omega_0)$

It is known that $\Phi(\alpha,\omega,\omega_0)$ is the Fourier transform of the spectral function $u_\nu(\omega,\omega_0)$ (see Section 6.2 in Lyalinov, 2018)

$$\Phi(\alpha,\omega,\omega_0) = -\sqrt{2\pi}\int_{\mathrm{i}R} \nu\, \mathrm{e}^{-\mathrm{i}\nu\alpha} u_\nu(\omega,\omega_0)\,\mathrm{d}\nu\,,$$

where the integration is conducted along the imaginary axis.

The asymptotics of $u_\nu(\omega,\omega_0)$ as $\nu \to \mathrm{i}\infty$ has a semi-classical ('high frequency') behaviour (see discussion in Section 3.3 of Lyalinov, 2018) so that its Fourier transform $\Phi(\alpha,\omega,\omega_0)$ has the singularities corresponding to the different 'high frequency' components in the asymptotics of $u_\nu(\omega,\omega_0)$. Location of the singularities depends on ω and α and varies as α changes, which can be interpreted as propagation of singularities. In practice, it is simpler to study 'propagation' of singularities of $\Phi(\alpha,\omega,\omega_0)$ than to use the asymptotics of $u_\nu(\omega,\omega_0)$.[5]

To this end, we shall actually use local expansions which are analogous to those in the non-stationary version of the 'ray method' in order to describe 'propagation' of singularities on the unit sphere with the cut AB. For real α the Sommerfeld transformant $\Phi(\alpha,\omega,\omega_0)$ satisfies a hyperbolic equation. However, together with the real singularities the transformant has also complex singularities which are responsible for the surface waves. In order to compute all these singularities we formulated the corresponding problem for $\Phi(\alpha,\omega,\omega_0)$, Section 6.3 in Lyalinov (2018).

The equation for the Sommerfeld transformant is

$$(\triangle_\omega - \partial_\alpha^2 - 1/4)\Phi(\alpha,\omega,\omega_0) = 0\,, \tag{6}$$

as $\omega \in \Sigma$, with the boundary condition

$$\left.\frac{\partial(\Phi+\Phi_i)}{\partial\mathcal{N}_\pm}\right|_{\sigma_\pm} = -\eta_\pm \frac{\partial}{\partial\alpha}\left.\frac{(\Phi+\Phi_i)}{\sin\alpha}\right|_{\sigma_\pm}. \tag{7}$$

In order to have a complete formulation of the problem for this equation on $S^2\setminus\sigma$ with the boundary condition (7) as $\alpha > 0$ we should add initial conditions, for instance, at $\alpha = 0$. For any $\alpha \in [0,\tau_0)$ we have

$$\Phi(\alpha,\omega,\omega_0) + \Phi_i(\alpha,\omega,\omega_0) = \mathrm{i}\sqrt{2\pi}\int_{\mathrm{i}R} \nu\,\sin(\nu\alpha)u_\nu(\omega,\omega_0)\,\mathrm{d}\nu \;-\; \frac{\sqrt{\pi}}{4}\frac{\sin\alpha}{(\cos\alpha-\cos\theta(\omega,\omega_0))^{3/2}} \tag{8}$$

which is a regular function in the strip $|\Re(\alpha)| < \tau_0$, Fig. 3, where $\Phi_i(\alpha,\omega,\omega_0) = -\frac{\sqrt{\pi}}{4}\frac{\sin\alpha}{(\cos\alpha-\cos\theta(\omega,\omega_0))^{3/2}}$ is the transformant specifying the incident plane wave. The branch of $(\cos\alpha - \cos\theta(\omega,\omega_0))^{-1/2}$ is specified as follows. The cuts are conducted from the points $\pm\theta$ to $\pm\infty$ correspondingly along the rays going to infinity along the real axis, $[\cos\alpha-\cos\theta]^{-1/2} > 0$ as $-\theta < \alpha < \theta$. Such a choice specifies a holomorphic function $[\cos\alpha-\cos\theta]^{-1/2}$ in the strip $|\Re(\alpha)| < \pi$ outside the cuts.

[5] Nevertheless, the approach based on the asymptotic behaviour of $u_\nu(\omega,\omega_0)$ was also exploited for the ideal cone (Shanin, 2011).

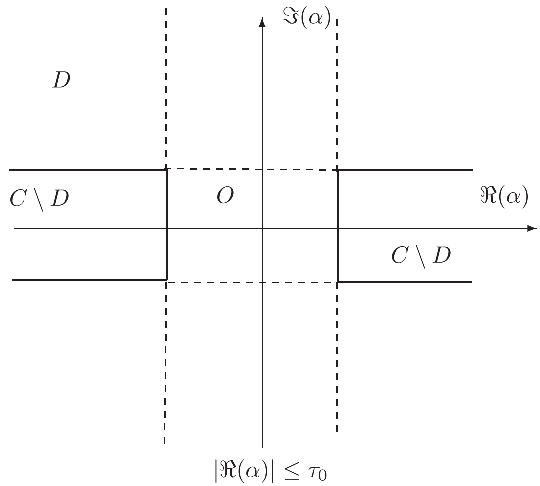

FIG. 3. Domain of regularity D.

Therefore, assuming that $u_\nu(\omega, \omega_0)$ is known in the integrand, we arrive at the initial conditions

$$\Phi(0, \omega, \omega_0) = 0, \quad \frac{\partial \Phi(\alpha, \omega, \omega_0)}{\partial \alpha}\Big|_{\alpha=0} = \mathrm{i}\sqrt{2\pi} \int_{\mathrm{i}R} \nu^2 u_\nu(\omega, \omega_0)\, \mathrm{d}\nu. \tag{9}$$

The mixed boundary value problem (6)–(9) can be used in order to determine singularities of $\Phi(\alpha, \omega, \omega_0)$ as $\alpha \in \mathbb{C} \setminus D$, Fig. 3, $\Phi(\alpha, \omega, \omega_0) = -\Phi(-\alpha, \omega, \omega_0)$.

A similar problem is valid for $\widetilde{\Phi}(\alpha, \omega, \omega_0)$, $\Phi_i(\alpha, \omega, \omega_0) = \frac{\partial}{\partial \alpha}\widetilde{\Phi}_i(\alpha, \omega, \omega_0)$,

$$(\triangle_\omega - \partial_\alpha^2 - 1/4)\widetilde{\Phi}(\alpha, \omega, \omega_0) = 0 \tag{10}$$

as $\omega \in S^2 \setminus \sigma$ and

$$\sin\alpha \, \frac{\partial\left(\widetilde{\Phi} + \widetilde{\Phi}_{\mathrm{i}}\right)}{\partial \mathcal{N}_\pm}\Big|_{\sigma_\pm} = -\eta_\pm \frac{\partial\left(\widetilde{\Phi} + \widetilde{\Phi}_i\right)}{\partial \alpha}\Big|_{\sigma_\pm}, \tag{11}$$

$$\widetilde{\Phi}(0, \omega, \omega_0) = -\mathrm{i}\sqrt{2\pi} \int_{\mathrm{i}R} u_\nu(\omega, \omega_0)\, \mathrm{d}\nu, \quad \frac{\partial \widetilde{\Phi}(\alpha, \omega, \omega_0)}{\partial \alpha}\Big|_{\alpha=0} = 0, \tag{12}$$

$\widetilde{\Phi}_i(\alpha, \omega, \omega_0) = -\frac{\sqrt{\pi}}{2}(\cos\alpha - \cos\theta(\omega, \omega_0))^{-1/2}$.

It has been shown (Lyalinov, 2018) that there exists an analytic function $\Phi(\alpha,\omega,\omega_0)$ which is holomorphic in the domain D, Fig. 3. In particular, for real $\alpha \in [0,\tau_0)$ this function solves the problem (6)–(8). For the strip $|\Re(\alpha)| < \tau_0(\omega,\omega_0)$ it is also specified by the equality

$$\Phi(\alpha,\omega,\omega_0) = \mathrm{i}\sqrt{2\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \nu\,\sin(\nu\alpha)u_\nu(\omega,\omega_0)\,\mathrm{d}\nu\,.$$

Similar properties are valid for $\widetilde{\Phi}(\alpha,\omega,\omega_0)$.

3. Singularities of the Sommerfeld transformants and application of the saddle point technique

The singularities of $\widetilde{\Phi}(\alpha,\omega,\omega_0)$ are located as $\alpha \in C\setminus D$, Fig. 3. The Sommerfeld transformant corresponding to the scattered field $U(r,\omega,\omega_0)$ can be represented as a sum of the singular components, which are united in groups, and possibly of the regular part (denoted $\widetilde{\Phi}_{reg}(\alpha,\omega,\omega_0)$)

$$\begin{aligned}&\widetilde{\Phi}(\alpha,\omega,\omega_0) = \widetilde{\Phi}_r(\alpha,\omega,\omega_0) + \{\widetilde{\Phi}_A(\alpha,\omega,\omega_0) + \widetilde{\Phi}_B(\alpha,\omega,\omega_0)\}+\\ &\{\widetilde{\Phi}^s_A(\alpha,\omega,\omega_0) + \widetilde{\Phi}^s_B(\alpha,\omega,\omega_0)\} + \widetilde{\Phi}_{sw}(\alpha,\omega,\omega_0)+\\ &\{\widetilde{\Phi}_{AB}(\alpha,\omega,\omega_0) + \widetilde{\Phi}_{BA}(\alpha,\omega,\omega_0)\} + \cdots + \widetilde{\Phi}_{reg}(\alpha,\omega,\omega_0)\,,\end{aligned} \tag{13}$$

where $\widetilde{\Phi}_r$ corresponds to the reflected wave, $\widetilde{\Phi}_A, \widetilde{\Phi}_B$ give rise to the edge waves from edges A and B, $\widetilde{\Phi}_{AB}$, etc. correspond to the other (secondary) singularities, i.e. attributed to the secondary diffracted waves,

$$\begin{aligned}&\widetilde{\Phi}_r(\alpha,\omega,\omega_0) = \frac{R}{\sqrt{\cos\alpha-\cos\theta_r(\omega,\omega_0)}}\,,\quad \alpha\sim\theta_r\\ &\widetilde{\Phi}_A(\alpha,\omega,\omega_0) = (A_0(\omega,\omega_0) + A_1(\omega,\omega_0)[\cos\alpha-\cos\theta_A(\omega,\omega_0)] + \dots)\log[\cos\alpha-\cos\theta_A(\omega,\omega_0)],\\ &\qquad \alpha\sim\theta_A,\\ &\widetilde{\Phi}_B(\alpha,\omega,\omega_0) = (B_0(\omega,\omega_0) + B_1(\omega,\omega_0)[\cos\alpha-\cos\theta_B(\omega,\omega_0)] + \dots)\log[\cos\alpha-\cos\theta_B(\omega,\omega_0)],\\ &\qquad \alpha\sim\theta_B,\\ &\widetilde{\Phi}_{AB}(\alpha,\omega,\omega_0) = (C_1(\omega,\omega_0) + C_2(\omega,\omega_0)[\cos\alpha-\cos\theta_{AB}(\omega,\omega_0)] + \dots)[\cos\alpha-\cos\theta_{AB}(\omega,\omega_0)]^{3/2},\\ &\qquad \alpha\sim\theta_{AB},\\ &\widetilde{\Phi}_{BA}(\alpha,\omega,\omega_0) = (D_1(\omega,\omega_0) + D_2(\omega,\omega_0)[\cos\alpha-\cos\theta_{BA}(\omega,\omega_0)] + \dots)[\cos\alpha-\cos\theta_{BA}(\omega,\omega_0)]^{3/2},\\ &\qquad \alpha\sim\theta_{BA},\end{aligned} \tag{14}$$

and[6] the singularities which are responsible for the surface waves,

$$\widetilde{\Phi}^s_A(\alpha,\omega,\omega_0) = A^s_0(\omega,\omega_0)[\cos\alpha - \cos\theta^A_s(\omega,\omega_0)]^{-1/2} + \ldots, \quad \alpha \sim \theta^s_A,$$

$$\widetilde{\Phi}^s_B(\alpha,\omega,\omega_0) = B^s_0(\omega,\omega_0)[\cos\alpha - \cos\theta^B_s(\omega,\omega_0)]^{-1/2} + \ldots, \quad \alpha \sim \theta^s_B,$$

$$\widetilde{\Phi}_{sw}(\alpha,\omega,\omega_0) = (c_1(\omega,\omega_0) + c_2(\omega,\omega_0)[\cos\alpha - \cos\theta_{sw}(\omega,\omega_0)] + \ldots)\log[\cos\alpha - \cos\theta_{sw}(\omega,\omega_0)],$$
$$\alpha \sim \theta_{sw}\,. \tag{15}$$

It is worth commenting on the defintion of a branch of an analytic function. Consider, for instance, $[\cos\alpha - \cos\theta]^{-1/2}$. We conduct the cuts from the points $\pm\theta$ to $\pm\infty$ correspondingly along the rays going to infinity parallel to the real axis, assuming that $[\cos\alpha - \cos\theta]^{-1/2} > 0$ as $-\theta < \alpha < \theta$. This choice specifies a holomorphic function $[\cos\alpha - \cos\theta]^{-1/2}$ in the strip $|\Re(\alpha)| < \pi$ outside the cuts. The branch of log is defined in a similar manner. It is worth mentioning that the formulae (13)–(15) that are actually motivated by the analogous results discussed in our work on diffraction by a perfect sector (Lyalinov, 2013). The only new singular terms in the expansion deal with the surface waves which are excited due to the impedance boundary conditions.

The representation (13) shows that $\widetilde{\Phi}(\alpha,\omega,\omega_0)$ is an analytic function having real singularities at $\theta_r, \theta_A, \ldots$ described by the local expansions (14) and also complex singularities at $\theta^s_A, \theta^s_B, \theta_{sw} \ldots$, which are responsible for the surface waves, with local expansions (15). $\widetilde{\Phi}^A_s(\alpha,\omega,\omega_0)$, $\widetilde{\Phi}^B_s(\alpha,\omega,\omega_0)$ are the singular parts being responsible for the surface waves from the edges, whereas $\widetilde{\Phi}_{sw}(\alpha,\omega,\omega_0)$ gives rise to the surface wave from the vertex.

3.1 *The reflected wave*

We shall make use of the following simple lemma.

LEMMA 1 Let $\widehat{\theta}(\omega)$ be a solution (real or complex) of the eikonal equation $(\nabla_\omega \widehat{\theta})^2 = 1$ $(\nabla_\omega = \mathbf{e}_\vartheta \frac{\partial}{\partial\vartheta} + \frac{\mathbf{e}_\varphi}{\sin\vartheta}\frac{\partial}{\partial\varphi})$ which additionally satisfies the equation $\triangle_\omega \widehat{\theta}(\omega) = \cot\widehat{\theta}(\omega)$ and A is a constant then

$$\Phi(\alpha,\omega) = \frac{A}{[\cos\alpha - \cos\widehat{\theta}(\omega)]^{1/2}}$$

is a solution of the equation, as $\alpha \neq \pm\widehat{\theta}(\omega) + 2\pi m, \;\; m = 0, \pm 1, \ldots,$

$$(\triangle_\omega - \partial^2_\alpha - 1/4)\Phi(\alpha,\omega) = 0\,.$$

[6] Logarithmic terms in the formulae written above give rise to the polar singularities after differentiation, $\Phi(\alpha,\omega,\omega_0) = \frac{\partial\widetilde{\Phi}(\alpha,\omega,\omega_0)}{\partial\alpha}$. The latters, however, are traditionally resposible for the edge waves in the formalism of the Sommerfeld integral.

The verification is performed by the direct substitution and obvious reductions

$$(\triangle_\omega - \partial_\alpha^2 - 1/4)\Phi(\alpha,\omega) = -\frac{3}{4}\frac{A(\sin^2\alpha - \sin^2\widehat{\theta}(\omega)[\nabla_\omega\widehat{\theta}(\omega)]^2)}{[\cos\alpha - \cos\widehat{\theta}(\omega)]^{5/2}} -$$

$$-\frac{1}{2}\frac{A(\cos\alpha + \cos\widehat{\theta}(\omega)[\nabla_\omega\widehat{\theta}(\omega)]^2 + \sin\widehat{\theta}(\omega)\triangle_\omega\widehat{\theta}(\omega))}{[\cos\alpha - \cos\widehat{\theta}(\omega)]^{3/2}} -$$

$$\frac{1}{4}\frac{A}{[\cos\alpha - \cos\widehat{\theta}(\omega)]^{1/2}} = 0\,.$$

REMARK 1 This lemma is closely connected with the following simple facts. The expression $U(x) = \mathcal{A}\exp(ik[ax_1 + bx_2 + cx_3])$ is a solution ('plane wave') of the Helmholtz equation for some complex a, b, c, $a^2 + b^2 + c^2 = 1$, which implies that $\tau(x) = ax_1 + bx_2 + cx_3$ solves the eikonal equation ($\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\nabla_\omega$)

$$(\nabla\tau)^2 = 1\,.$$

The eikonal can be written in the spherical coordinates $\tau(x) = -r\cos\widehat{\theta}(\omega)$, $\omega = (\vartheta, \varphi)$ with

$$\cos\widehat{\theta}(\omega) = \cos\vartheta\cos\vartheta_0 + \sin\vartheta\sin\vartheta_0\cos[\varphi - \varphi_0]$$

for some complex ϑ_0, φ_0 specified by a, b, c. As a result, we find that

$$\sin^2\widehat{\theta}(\omega)[\nabla_\omega\widehat{\theta}(\omega)]^2 + \cos^2\widehat{\theta}(\omega) = 1$$

and $(\nabla_\omega\widehat{\theta})^2 = 1$. In the same fashion we conclude that $\triangle(ax_1 + bx_2 + cx_3) = 0$ is followed by the equation $\triangle_\omega\widehat{\theta}(\omega) = \cot\widehat{\theta}(\omega)$.

Let us now look for the component Φ_r, related to the singularity θ_r in (13), in the form

$$\widetilde{\Phi}_r(\alpha,\omega,\omega_0) = -\frac{\sqrt{\pi}}{2}\frac{R}{(\cos\alpha - \cos\theta_r(\omega,\omega_0))^{1/2}}\,, \tag{16}$$

where R and θ_r are still unknown. Assuming that R is a constant and θ_r satisfies the equations

$$(\nabla_\omega\theta_r)^2 = 1 \tag{17}$$

and $\triangle_\omega\widehat{\theta}_r(\omega) = \cot\widehat{\theta}_r(\omega)$, from Lemma 1 we conclude that $\widetilde{\Phi}_r(\alpha,\omega,\omega_0)$ solves the equation (10).

From the boundary condition (11) on σ_+ we find $\theta_r|_{\sigma_+} = \theta_i|_{\sigma_+}$, ($< \nabla_\omega\theta_r|_{\sigma_+}, s > = < \nabla_\omega\theta_i|_{\sigma_+}, s >$, where s is the unit vector tangent to σ_+,) and the reflection coefficient is

$$R = \frac{\langle\nabla_\omega\theta_i|_{\sigma_+}, N_+\rangle\sin\theta_i|_{\sigma_+} - \eta_+}{\langle\nabla_\omega\theta_i|_{\sigma_+}, \mathcal{N}_+\rangle\sin\theta_i|_{\sigma_+} + \eta_+}\,,$$

where $\langle\, , \,\rangle$ is scalar product of vectors in a tangent plane to the unit sphere. It is easily verified that θ_r is found from the eikonal equation (17) on $S^2 \setminus \sigma$ and from the boundary conditions on σ_+. Namely,

$$\theta_r(\omega, \omega_0) = \min_{l \in \sigma_+}\{\theta(\omega, l) + \theta(\omega_0, l)\}$$

is the the length of the 'broken' geodesic; the geodesic (ray) on the unit sphere goes from the source ω_0, encounters σ_+ then reflects in accordance with geometrical optics (the angle of incidence is equal to that of reflection, $\langle \nabla_\omega \theta_r|_\sigma, \mathcal{N}_+\rangle = -\langle \nabla_\omega \theta_i|_\sigma, \mathcal{N}_+\rangle$) and arrives at ω. From a simple physical argumentation it is reasonable to expect that the plane surface S_+ of the sector S reflects the incident plane wave which gives rise to the reflected plane wave $U_r(kr, \omega, \omega_0)$. We verify this by the direct asymptotic evaluation of the Sommerfeld integral, namely by the contribution of its singilar component $\widetilde{\Phi}_r(\alpha, \omega, \omega_0)$.

Indeed, in order to evaluate the Sommerfeld integral representation asymptotically as $kr \to \infty$ we ought to deform the Sommerfeld double-loop contour γ into the SD paths $\gamma_{\pm\pi}$, see Fig. 2. In the process of such deformation some singularities of the transformant can be captured. Contributions of these singularities (Fig. 2, right) give rise to the corresponding components of the far field such as reflected wave, edge waves (see also Lyalinov, 2013 for the Dirichlet boundary conditions) as well as the surface waves which may be excited near the impedance surface of the sector. The saddle points of the integral are $\alpha = \pm\pi$; however, in view of the parity of the transformant we can only consider $\alpha = \pi$. For the reflected wave we have $(kr \to \infty)$[7]

$$U_r(kr, \omega, \omega_0) = \frac{\sqrt{-\mathrm{i}kr}}{\pi \mathrm{i}} \int\limits_{\gamma_r} \mathrm{e}^{-\mathrm{i}kr\cos\alpha} \sin\alpha\, \widetilde{\Phi}_r(\alpha, \omega, \omega_0)\, \mathrm{d}\alpha =$$

$$\frac{\sqrt{-\mathrm{i}kr}}{\pi \mathrm{i}} \int_{\gamma_r} \mathrm{e}^{-\mathrm{i}kr\cos\alpha} \left(-\frac{\sqrt{\pi}}{2}\right) \frac{R\,\sin\alpha}{(\cos\alpha - \cos\theta_r(\omega, \omega_0))^{1/2}}\, \mathrm{d}\alpha\,,$$

where γ_r comprises a part of the branch cut from θ_r to ∞ in the process of the mentioned deformation of the Sommerfeld contour γ.[8] We use the change of the integration variable in the latter integral, $t = \mathrm{i}kr\cos\alpha$, then after simple asymptotic reductions and estimates we arrive at

$$U_r(kr, \omega, \omega_0) = R\,\frac{\exp\{-\mathrm{i}kr\cos\theta_r(\omega, \omega_0)\}}{\sqrt{\pi}} \int\limits_0^\infty \frac{\mathrm{e}^{-t}}{\sqrt{t}}\, \mathrm{d}t + o([kr]^{-\infty}) = R\,\exp\{-\mathrm{i}kr\cos\theta_r(\omega, \omega_0)\}\,.$$

As was expected the reflected wave is nothing more but a plane wave with the reflection coefficient R.

We computed the reflected wave which is one of the components (Fig. 4) in the domain of the directions ω from $\Omega_r \cap (\Omega_B \cup \Omega_A)$, see definition of the domains in (Lyalinov, 2018, Section 2.2); the leading terms with respect to kr consist of the reflected, spherical and diffracted (from the edges A and

[7] The Sommerfled transformant $\widetilde{\Phi}(\alpha, \omega, \omega_0)$ is an odd function of α.

[8] The contour γ_r coincides with the part of γ_π which is contained in the circle shown in Fig. 5, left, $\theta = \theta_r$.

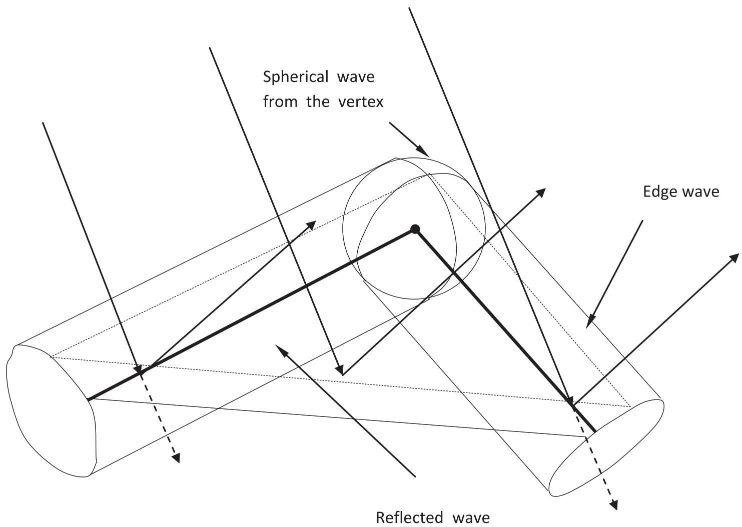

FIG. 4. Wave fronts of the reflected, spherical and edge waves.

B) waves (as well as the surface ones),

$$\begin{aligned} U(kr,\omega,\omega_0) &= R\,\exp(-\mathrm{i}kr\cos\theta_r(\omega,\omega_0)) + D(\omega,\omega_0)\tfrac{\exp(\mathrm{i}kr)}{-\mathrm{i}kr}\left(1+O\left(\tfrac{1}{kr}\right)\right)+ \\ & d_A(\omega,\omega_0)\,\frac{\exp(-\mathrm{i}kr\cos\theta_A(\omega,\omega_0))}{\sqrt{-\mathrm{i}kr\sin\psi_A}}\left(1+O\left(\frac{1}{kr\sin\psi_A}\right)\right)+ \\ & d_B(\omega,\omega_0)\,\frac{\exp(-\mathrm{i}kr\cos\theta_B(\omega,\omega_0))}{\sqrt{-\mathrm{i}kr\sin\psi_B}}\left(1+O\left(\frac{1}{kr\sin\psi_B}\right)\right)+V_s(r,\omega,\omega_0)\,+\ldots\,; \end{aligned} \tag{18}$$

$V_s(r,\vartheta,\varphi)$ is related to the surface waves. Now we turn to the calculation of the wave components corresponding to the edges A and B.

3.2 *The singularities specifying the edge waves*

The singularity corresponding to the edge wave has the form ($\alpha \sim \theta_A$ in (13), consider the edge A)

$$\widetilde{\Phi}_A(\alpha,\omega,\omega_0) = (A_0(\omega,\omega_0)+A_1(\omega,\omega_0)[\cos\alpha-\cos\theta_A(\omega,\omega_0)]+\ldots)\log[\cos\alpha-\cos\theta_A(\omega,\omega_0)],\ \ \alpha\sim\theta_A,$$

then the following lemma is valid.

LEMMA 2 Provided the eikonal θ_A satisfies

$$(\nabla_\omega\theta_A)^2 = 1$$

and A_0, $A_1 \ldots$ are solutions of a recurrent system of equations, in particular,

$$2\langle \nabla_\omega \theta_A(\omega,\omega_0), \nabla_\omega A_0(\omega,\omega_0)\rangle + \triangle_\omega \theta_A(\omega,\omega_0) A_0(\omega,\omega_0) = 0\,, \tag{19}$$

$\widetilde{\Phi}_A(\alpha,\omega,\omega_0)$ solves the equation (10), $\alpha \neq \pm\theta_A + 2\pi m,\ \ m = 0, \pm 1, \ldots.$

To verify Lemma 2 we substitute the local expansion for $\widetilde{\Phi}_A(\alpha,\omega,\omega_0)$ into the equation (10),

$$\frac{-\sin^2\alpha + \sin^2\theta_A(\nabla_\omega\theta_A)^2}{[\cos\alpha - \cos\theta_A(\omega,\omega_0)]^2} A_0(\omega,\omega_0) -$$
$$\frac{\cos\alpha A_0(\omega,\omega_0) + (\cos\theta_A(\nabla_\omega\theta_A)^2 + \sin\theta_A \triangle_\omega\theta_A)A_0(\omega,\omega_0) + 2\sin\theta_A\langle\nabla_\omega\theta_A, \nabla_\omega A_0(\omega,\omega_0)\rangle}{[\cos\alpha - \cos\theta_A(\omega,\omega_0)]} +$$
$$\frac{\sin^2\alpha + \sin^2\theta_A(\nabla_\omega\theta_A)^2 - 2\sin^2\theta_A(\nabla_\omega\theta_A)^2}{[\cos\alpha - \cos\theta_A(\omega,\omega_0)]} A_1(\omega,\omega_0)\ + O(\log[\cos\alpha - \cos\theta_A(\omega,\omega_0)]) + \cdots = 0\,,$$

where the dots denote regular terms ($\alpha \sim \theta_A$). We obtain

$$A_0(\omega,\omega_0)\frac{(\nabla_\omega\theta_A)^2 - 1}{[\cos\alpha - \cos\theta_A(\omega,\omega_0)]^2} + A_0(\omega,\omega_0)\frac{\cos^2\alpha - (\nabla_\omega\theta_A)^2\cos^2\theta_A(\omega,\omega_0)}{[\cos\alpha - \cos\theta_A(\omega,\omega_0)]^2} -$$
$$\frac{[2\cos\theta_A(\omega,\omega_0) + \sin\theta_A(\omega,\omega_0)\triangle_\omega\theta_A(\omega,\omega_0)]A_0(\omega,\omega_0) + 2\sin\theta_A(\omega,\omega_0)\langle\nabla_\omega\theta_A(\omega,\omega_0), \nabla_\omega A_0(\omega,\omega_0)\rangle}{[\cos\alpha - \cos\theta_A(\omega,\omega_0)]} +$$
$$+O(\log[\cos\alpha - \cos\theta_A(\omega,\omega_0)]) + \cdots = 0$$

and arrive at the desired equations equating zero coefficients at the leading singular terms.

It is convenient, instead of the spherical coordinates (ϑ,φ), to use the coordinates (ψ_A, χ_A) on the sphere S^2 attributed to the end A of the cut AB; ψ is the geodesic distance on the sphere between ω and A, χ is the angle between this geodesic and the line being the continuation of the cut σ across the point A, $|\chi| \leqslant \pi$. The sides $\sigma_\pm$ of the cut AB correspond to $\chi = \pm\pi,\ 0 \leqslant \psi < \pi$. (We omit the subscripts for (ψ_A, χ_A) in this section.)

The eikonal equation $(\nabla_\omega\theta_A)^2 = 1$ is supplemented by the condition $\theta_A|_{\omega=A} = \theta_i|_A$, $(\theta_i|_A = \theta(\omega_0, A))$ then the solution takes the form

$$\theta_A(\omega,\omega_0) = \psi + \theta(\omega_0, A)|_{(\psi,\chi)\to(\vartheta,\varphi)}. \tag{20}$$

For any point $\omega = (\vartheta,\varphi)$ we univalently connect it with A by the geodesic of the length ψ. The geodesic is emanated from the point A under the angle χ and arrives at ω then the sum $\psi + \theta(\omega_0, A)$ gives the value of the eikonal $\theta_A(\omega,\omega_0)$ in (20). Now we turn to the integration of the transport equation (19). We can write it in the coordinates (ψ,χ), taking into account that $\triangle_\omega\theta_A = \frac{1}{\sin\psi}\frac{\partial\sin\psi}{\partial\psi} = \cot\psi$. Thus, we obtain

$$2\,\frac{\partial A_0}{\partial\psi} + \cot\psi\, A_0 = 0\,,$$

and

$$A_0(\omega,\omega_0) = \left.\frac{a_0(\chi)}{\sqrt{\sin\psi}}\right|_{(\psi,\chi)\to(\vartheta,\varphi)}. \tag{21}$$

The coefficient $a_0(\chi)$ in (21) is still unknown; it will be specified below.

However, we first determine the contribution of the singularity

$$\Phi_A(\alpha,\omega,\omega_0) = \frac{\partial\widetilde{\Phi}_A(\alpha,\omega,\omega_0)}{\partial\alpha} = \frac{A_0(\omega,\omega_0)(-\sin\alpha)}{[\cos\alpha-\cos\theta_A(\omega,\omega_0)]} + O(\log[\cos\alpha-\cos\theta_A(\omega,\omega_0)]),\ \ \alpha\sim\pm\theta_A,$$

into the far-field asymptotics. After deformation of γ into the SD paths (Fig. 2) substitute the latter expression into the Sommerfeld integral. The singularities at $\alpha=\pm\theta_A$ are captured as $\alpha\in\Omega_A$ (see Section 2.2 in Lyalinov (2018) for the definition of Ω_A) and we arrive at the desired component in the asymptotics

$$U_A(kr,\omega,\omega_0) = \frac{1}{\pi\mathrm{i}}\int_{\gamma_A}\frac{\mathrm{e}^{-\mathrm{i}kr\cos\alpha}}{\sqrt{-\mathrm{i}kr}}\,\frac{A_0(\omega,\omega_0)\,(-\sin\alpha)}{(\cos\alpha-\cos\theta_A(\omega,\omega_0))}\,\mathrm{d}\alpha + \dots,$$

where γ_A is the circumference around the pole at $\alpha=\theta_A$. We apply the theorem on residues; in the leading approximation one has ($\omega\in\Omega_A$)

$$U_A(kr,\omega,\omega_0) = \left.\frac{2a_0(\chi)}{\sqrt{\sin\psi}}\right|_{(\psi,\chi)\to\omega}\frac{\mathrm{e}^{-\mathrm{i}kr\cos\theta_A(\omega,\omega_0)}}{\sqrt{-\mathrm{i}kr}}\left(1+O\left(\frac{1}{kr\sin\psi}\right)\right). \tag{22}$$

3.2.1 *Localization principle and derivation of a_0.* We now turn to the derivation of the unknown $a_0(\chi)$. One of the ways is to solve the problem of diffraction of the singularity corresponding to $\alpha=\theta_A$ by the end A of the cut σ on the unit sphere then to extract the desired information from the solution due to the localization principle. However, such a problem is not a simple one and we shall follow a different way. Namely, we compare the expression (22) with that obtained from the problem of diffraction of a skew incident plane wave by an impedance half-plane relying upon the high frequency localization principle. In the leading approximation, as $kr\to\infty$, the diffraction coefficient of the cylindrical wave obtained from this canonical problem is implied to be the same as that in the expression (22). Remark that such an approach has been sucessfully used in order to derive diffraction coefficients in the problem of diffraction by a perfect quarter-plane (Assier & Peake, 2012b).

To this end, we consider the cylindrical coordinate system ρ,ϕ,z, attributed to the edge A of the sector, assuming that the Z-axis coincides with the edge and directed from the vertex, X-axis is in the plane of the sector Fig. 7. We obviously have $\rho=r\sin\psi,\ \phi=\chi,\ z=r\cos\psi$ and

$$-kr\cos\theta_A(\omega,\omega_0) = -kr\cos[\psi+\theta(\omega_0,A)] = kz\cos[\pi-\theta(\omega,A)] + k\rho\sin[\pi-\theta(\omega_0,A)].$$

Now we can write the cylindrical wave (22) from the edge A as follows:

$$U_A(kr,\omega,\omega_0) = 2a_0(\phi)\,\mathrm{e}^{\mathrm{i}k''z}\sqrt{\sin[\pi-\theta(\omega_0,A)]}\,\frac{\mathrm{e}^{\mathrm{i}k'\rho+\mathrm{i}\pi/4}}{\sqrt{k'\rho}}\left(1+O\left(\frac{1}{k'\rho}\right)\right), \tag{23}$$

where $k' = k\sin[\pi - \theta(\omega_0, A)]$, $k'' = k\cos[\pi - \theta(\omega_0, A)]$. Remark that $\pi - \theta(\omega_0, A)$ is the angle of skew incidence at the edge A.

Due to the localization principle we can compare the expression (23) with that extracted from the asymptotics (A.1) for the explicit solution of the problem of diffraction of a plane wave which is skew incident under the angle $\beta = \pi - \theta(\omega_0, A)$ on the edge of the half-plane $x \leqslant 0,\ y = 0$, see the Appendix, Fig. 8. Thus, we have

$$a_0(\chi) = \frac{s(\phi - \pi) - s(\phi + \pi)}{2\sqrt{2\pi \sin[\pi - \theta(\omega_0, A)]}},$$

where the function $s(z)$ is specified in the Appendix, and

$$U_A(kr, \omega, \omega_0) = \frac{s(\chi - \pi) - s(\chi + \pi)}{\sqrt{2\pi \sin[\pi - \theta(\omega_0, A)]}} \frac{\mathrm{e}^{-\mathrm{i}kr\cos\theta_A(\omega,\omega_0)}}{\sqrt{-\mathrm{i}kr\sin\psi}} \left(1 + O\left(\frac{1}{kr\sin\psi}\right)\right)\bigg|_{(\psi,\chi)\to\omega}. \tag{24}$$

The expression (24) is valid provided $\omega \in \Omega_A$ and outside the singular directions of the diffraction coefficient of the spherical wave from the vertex ($\theta_A \sim \pi$). It specifies the third summand in the right-hand side of (18). In the same manner the components U_B (the fourth summand in the right-hand side of (18)) in the asymptotics is constructed.

3.2.2 *Coalescence of the singularities θ_i and θ_A and of θ_r and θ_A and the uniform asymptotics.* However, the formula (18) should be modified appropriately if the direction of observation corresponds to a transition region, where the singularities corresponding to the reflected and edge waves or those for the incident and edge waves are close to each other, $\theta_r \sim \theta_A$ or $\theta_i \sim \theta_A$. In these direction the asymptotics is described by the Fresnel integrals. The wave fronts of the reflected and cylindrical waves as $\theta_r \sim \theta_A$ and of the incident and cylindrical waves as $\theta_i \sim \theta_A$ are almost tangent, Fig. 4. The uniform w.r.t. ϕ formula for the sum of the incident, reflected from S_+ and scattered from the edge A (cylindrical), takes the form

$$U_i(kr, \omega, \omega_0) + U_r(kr, \omega, \omega_0) + U_A(kr, \omega, \omega_0) =$$
$$\mathrm{e}^{\mathrm{i}k''z}\Bigg\{\mathrm{e}^{-\mathrm{i}k'\rho\cos[\phi-\phi_0]} + R\,\mathrm{e}^{-\mathrm{i}k'\rho\cos[\phi+\phi_0]} + \frac{\mathrm{e}^{\mathrm{i}k'\rho+\mathrm{i}\pi/4}}{\sqrt{2\pi k'\rho}}\Bigg[s(\phi-\pi) - s(\chi+\pi) -$$
$$\sqrt{\frac{\mathrm{i}}{2}}\frac{1 - \mathrm{F}_{\mathrm{KP}}\left(\mathrm{i}k'\rho s_i^2\right)}{s_i} - R\sqrt{\frac{\mathrm{i}}{2}}\frac{1 - \mathrm{F}_{\mathrm{KP}}\left(\mathrm{i}k'\rho s_r^2\right)}{s_r}\Bigg]\Bigg\}\Bigg|_{\rho,\phi,z\to r,\omega}, \tag{25}$$

where $s_i = -\sqrt{2\mathrm{i}}\cos\frac{\phi-\phi_0}{2}$, $s_r = -\sqrt{2\mathrm{i}}\cos\frac{2\pi-\phi-\phi_0}{2}$ and

$$\mathrm{F}_{\mathrm{KP}}(z^2) = \pm 2\mathrm{i}z\mathrm{e}^{\mathrm{i}z^2}\int_{\pm z}^{\infty} \mathrm{e}^{-\mathrm{i}t^2}\mathrm{d}t, \quad \text{the lower sign for } \frac{\pi}{4} < \arg z < \frac{5\pi}{4}$$

is the Fresnel type integral. The Fresnel integral and its plots are carefully considered in an Appendix of Babich *et al.* (2008). Analogous to (25) formulae are valid for the second edge B and are to be properly combined with (25). In a close vicinity of the surface of the sector surface waves should be taken into account for the total field.

3.3 *The surface waves and the corresponding singularities*

The incident plane wave interacts with the edges A and B, which gives rise to the cylindrical edge waves described in the presious section and to the surface waves. The surface waves from the edges propagate along the faces $S_\pm$ of the sector and exponentially attenuate in the orthogonal directions to the faces as $kr \to \infty$; however, they are only bounded on the surface. There exists another type of the surface wave that propagates from the vertex of the sector. This surface wave is not only localized near the surface (not considering vicinities of the edges) but also vanishes as $O(1/\sqrt{kr})$ with increasing the distance from the vertex. The contribution of the surface waves to the far field is denoted by $V_s(kr,\omega,\omega_0)$ in (18).

The surface waves of the first type (i.e. propagating from the edges) are analogous to those generated by the edge of an infinite impedance half-plane when the incident plane wave illuminates the edge, see Appendix. We first specify the expressions for the surface waves from the edges of the sector then the corresponding excitation coefficients will be found by means of the localization principle. To this end, we make use of the Lemma 1 and look for the corresponding singular expression in the form

$$\widetilde{\Phi}_s^A(\alpha,\omega,\omega_0) = \frac{C_s^A}{[\cos\alpha - \cos\theta_s^A(\omega,\omega_0)]^{1/2}}\,,$$

where C_s^A is independent of ω and $\theta_s^A(\omega,\omega_0)$ is a complex-valued solution of the eikonal equation $(\nabla_\omega \theta_s^A)^2 = 1$.

We exploit the coordinates (ψ,χ) attributed to the point A and defined in Section 3.2 and determine the complex-valued eikonal θ_s^A in an implicit form for the face S_+

$$\cos\theta_s^A(\omega,\omega_0) = \cos\psi\cos\theta(\omega_0,A) + \sin\psi\sin\theta(\omega_0,A)\cos[\pi+\zeta'_+ - \chi]|_{(\psi,\chi)\to\omega}$$

noticing that $\theta_s^A(\omega,\omega_0)|_{\omega=A} = \theta(\omega_0,A)$ and using the notations $\zeta'_\pm$ in accordance with

$$\sin\zeta'_\pm = \frac{\eta_\pm}{\sin\beta},\quad \beta = \pi - \theta(\omega_0,A).$$

It remains to verify that $\widetilde{\Phi}_s^A(\alpha,\omega,\omega_0)$ satisfies the boundary condition (11) on σ_+,

$$\left(\frac{\sin\alpha}{\sin\psi}\frac{\partial}{\partial\chi} + \eta_+\frac{\partial}{\partial\alpha}\right)\widetilde{\Phi}_s^A(\alpha,\omega,\omega_0)\bigg|_{\chi=\pi} = 0\,,$$

which follows from a simple calculation

$$-\frac{1}{2}\left(\frac{\sin\alpha}{\sin\psi}\frac{\partial(-\cos\theta_s^A(\omega,\omega_0))}{\partial\chi} - \eta_+\sin\alpha\right)\frac{C_s^A(\omega_0)}{[\cos\alpha - \cos\theta_s^A(\omega,\omega_0)]^{3/2}}\bigg|_{\chi=\pi} = 0,$$

where C_s^A is still unknown.

We substitute the obtained expression for $\widetilde{\Phi}_s^A(\alpha,\omega,\omega_0)$ into the Sommerfeld integral then following the calculations of the Section 3.1 we evaluate the integral asymptotically and arrive at the expression for the surface wave originated at the edge A

$$U_s^A(kr,\omega,\omega_0) = C(\omega_0)\ \exp\{-\mathrm{i}kr\cos\theta_s^A(\omega,\omega_0)\}, \tag{26}$$

$C_s^A(\omega_0) = -C(\omega_0)\,\frac{\sqrt{\pi}}{2}$.

The unknown excitation coefficient $C(\omega_0)$ is found by means of the localization principle. We compare the expression (26), written in the form

$$U_s^A(kr,\omega,\omega_0) = C(\omega_0)\,\mathrm{e}^{\mathrm{i}k''z+\mathrm{i}k'\rho\cos[\pi+\zeta'_+-\phi]}$$

($\phi = \chi,\ \ x = \rho\cos\phi,\ y = \rho\sin\phi$), with the expression for the surface wave from the edge of an impedance half-plane (see Appendix, (A.2)) thus obtaining

$$C(\omega_0) = R_s^+ := \frac{\widetilde{s}(2\pi+\zeta'_+)}{\Psi_0(\phi_0)}\,\Psi(\zeta'_+,\zeta'_-)\psi_\pi(5\pi/2+2\zeta'_+)\psi_\pi(-\pi/2)\,.$$

3.3.1 *The surface wave from the vertex that propagates over S_+.* The wave field near the vertex is of particular importance because the corresponding domain may be considered as that being responsible for generation of waves of the various physical nature like diffracted from the tip or those surface waves propagating along $S_\pm$ from the vertex. In this domain the parameter kr is only bounded and tends to zero provided the observation point approachs the vertex. Near the vertex the integral representation can be approximated by its ‘low frequency’ asymptotics which is derived by means of the Sommerfeld integral representation for the Bessel functions, see Section 5.2.2 in Lyalinov & Zhu (2012). In this way, the Meixner’s conditions can be verified, see (2.5) in Lyalinov (2018). However, at far distances, as $kr \to \infty$, the SD method is applied so that the abovementioned singularities play a crucial role for the asymptotics. It should be remarked that the corresponding diffraction coefficient is a non-local characteristic. Contrary to the diffraction coefficients of the surface waves propagating from the edges, it is a functional of the total solution of the problem.

The vertex of the sector can be considered as an imaginary source of the surface wave that is expected to have ‘circular’ wave front on S_+. It is natural to forsee that it can be really isolated in the asymptotics provided that the observation point is in the vicinity of S_+, however outside some close neighbourhood of the edges.[9]

We are looking for the singularity responsible for this wave in the form

$$\widetilde{\Phi}_{sw}(\alpha,\omega,\omega_0) = (A_0^s(\omega,\omega_0) + A_1^s(\omega,\omega_0)[\cos\alpha-\cos\theta_{sw}(\omega,\omega_0)] + \ldots)\log[\cos\alpha-\cos\theta_{sw}(\omega,\omega_0)],$$
$$\alpha\sim\theta_{sw},$$

where in accordance with Lemma 2 we assert that $\theta_{sw}(\omega,\omega_0)$ and $A_0^s(\omega,\omega_0)$ should solve the equations

$$(\nabla_\omega\theta_{sw})^2 = 1$$

[9] The behaviour of this wave near the edges and its possible interference with the other waves require special study.

and

$$2\langle \nabla_\omega \theta_{sw} s(\omega,\omega_0), \nabla_\omega A_0^s(\omega,\omega_0)\rangle + \triangle_\omega \theta_{sw}(\omega,\omega_0) A_0^s(\omega,\omega_0) = 0. \tag{27}$$

Remark that θ_{sw} is a complex solution of the eikonal equation on S^2. The desired solution has a simple form

$$\theta_{sw}(\omega,\omega_0) = \pi + \zeta_+ - \vartheta + \pi/2,$$

which can be understood from the analysis of the surface wave excited by a real point source placed on an impedance plane, $\eta_+ = \sin\zeta_+$. The transport equation (27) for $A_0^s(\omega,\omega_0)$ takes the form

$$2\frac{\partial A_0^s}{\partial\vartheta} + \cot\vartheta A_0^s = 0$$

and is supplemented by the boundary condition on σ_+

$$\left.\frac{\partial A_0^s}{\partial\vartheta}\right|_{\vartheta=\pi/2-0} = 0$$

that follows from (11). Simple integration leads to

$$A_0^s(\omega,\omega_0) = \frac{C_0^s(\varphi)}{\sqrt{\sin\vartheta}}$$

and to the local expression

$$\widetilde{\Phi}_{sw}(\alpha,\omega,\omega_0) = \left(\frac{C_0^s(\varphi)}{\sqrt{\sin\vartheta}} + \dots\right)\log[\cos\alpha - \cos\theta_{sw}(\omega,\omega_0)], \quad \alpha \sim \theta_{sw}.$$

We substitute the latter expression into the Sommerfeld integral recalling that $\Phi(\alpha,\omega,\omega_0) = \frac{\partial\widetilde{\Phi}(\alpha,\omega,\omega_0)}{\partial\alpha}$ and compute contribution of the singularity

$$U_{sw}(kr,\omega,\omega_0) = \frac{C_0^s(\varphi)}{\pi\mathrm{i}\sqrt{\sin\vartheta}} \int\limits_{\gamma_{sw}} \frac{\mathrm{e}^{-\mathrm{i}kr\cos\alpha}}{\sqrt{-\mathrm{i}kr}} \frac{(-\sin\alpha)}{(\cos\alpha - \cos\theta_{sw}(\omega,\omega_0))}\,\mathrm{d}\alpha + \dots,$$

where γ_{sw} is the circumference around the pole at $\alpha = \theta_{sw}$. We apply the theorem on residues; in the leading approximation one has

$$U_{sw}(kr,\omega,\omega_0) = \frac{2C_0^s(\varphi)}{\sqrt{\sin\vartheta}} \frac{\mathrm{e}^{-\mathrm{i}kr\cos\theta_{sw}(\omega,\omega_0)}}{\sqrt{-\mathrm{i}kr}}\left(1 + O\left(\frac{1}{kr}\right)\right). \tag{28}$$

The same construction is valid for the surface wave from the vertex on the face S_-. It is worth commenting on the expression (28). The excitation coefficent of the surface wave from the vertex is specified by $C_0^s(\varphi)$; however, contrary to that from the previous section, the latter characteristic cannot be found from the localization principle. The determination of $C_0^s(\varphi)$ requires solution of the total problem for $\widetilde{\Phi}$ because this coefficient is a non-local characteristic of the scattering. In the leading approximation contribution of the surface waves in (18) is given by

$$V_s(kr,\omega,\omega_0) = U_s^A(kr,\omega,\omega_0) + U_s^B(kr,\omega,\omega_0) + U_{sw}(kr,\omega,\omega_0)\,,$$

where U_{sw} is of $O\left(\frac{1}{\sqrt{kr}}\right)$ in comparison with U_s^A, U_s^B as $kr \to \infty$.

Together with $C_0^s(\varphi)$ the scattering diagram $D(\omega,\omega_0)$ of the spherical wave from the vertex (see (18))

$$U_{sph}(kr,\omega,\omega_0) = D(\omega,\omega_0)\frac{\exp(\mathrm{i}kr)}{-\mathrm{i}kr}\left(1 + O\left(\frac{1}{kr}\right)\right)$$

is also a non-local characteristic of the scattering that can be found by means of solution of the problem for $\Phi(\alpha,\omega,\omega_0)$ and is specified by the values of the Sommerfled transformant at the saddle points $\pm\pi$ (see Lyalinov, 2018; Lyalinov & Zhu, 2012, Chapter 5)

$$D(\omega,\omega_0) = -\sqrt{\frac{2}{\pi}}\,\Phi(\pi,\omega,\omega_0)\,. \tag{29}$$

The latter characteristic (29) is correctly defined and the spherical wave can be separated in the asymptotics provided that there are no any singularities of the Sommerfled transformant located in some close vicinity of the saddle points. This implies that the observation direction ω is outside the transition zones.

The experience gained in problems of diffraction by perfect or impedance cones (see e.g. Babich *et al.*, 2000; Bonner *et al.*, 2005; Bernard *et al.*, 2008; Lyalinov & Zhu, 2012; Lyalinov *et al.*, 2010) enables one to expect that in order to compute the diffraction coefficient in the 'oasis' Ω_0 the formula (Smyshlyaev, 1990's type representation)

$$D(\omega,\omega_0) = -2\mathrm{i}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \nu\,\sin(\nu\pi)u_\nu(\omega,\omega_0)\,\mathrm{d}\nu$$

can be exploited. To this end, it is important to develop an efficient algorithm to calculate the spectral function $u_\nu(\omega,\omega_0)$ numerically. We expect that it can be done by means of the incomplete separation of variables for the problem on the unit sphere making use of the sphero-conical coordinates there. Numerical evalution of the diffraction coefficients outside the oasis Ω_0 and outside singular directions requires additional efforts. In particular, an appropriate integral representation for the diffraction coefficient $D(\omega,\omega_0)$ ought to be developed. This is equivalent to analytic continuation of the representation

$$\Phi(\alpha,\omega,\omega_0) = -\sqrt{2\pi}\int_{\mathrm{i}R} \nu\,\mathrm{e}^{-\mathrm{i}\nu\alpha}u_\nu(\omega,\omega_0)\,\mathrm{d}\nu$$

for the Sommerfeld transformant from the strip $|\Re(\alpha)| < \tau_0$ to a wider strip $|\Re(\alpha)| < \pi + \delta$ for some positive δ and $\tau_0 < \pi$.

4. Coalescence of the singularities and saddle points. The far field in the transition zones

We now turn to the case when some singularities can be located in a vicinity of the saddle points.

4.1 *The singularities at $\alpha = \pm\theta_A$ are near the saddle points $\pm\pi$*

We computed contributions of these singularities at $\alpha = \pm\theta_A$ provided that they are separated from the saddle points $\pm\pi$[10] i.e. are not located in $O(1/[kr]^{1/2-\delta})$-vicinities of $\pm\pi$ for some small positive δ. However, as we assume $\theta_A \sim \pi$ (or $\theta_B \sim \pi$ but not simultaneously) the contributions of the saddle points and θ_A (or θ_B) cannot be separated.[11]

We briefly describe the far field asymptotics in the case, $\theta_A \sim \pi$ implying that the other singularities are not close to $\pm\pi$. The details are very similar to those in Section 7 of Lyalinov (2013). The directions $\theta_A = \pi$ (θ_B is not close to π) correspond to those in which the front of the spherical wave is tangent to that of the cylindrical wave from the edge A. (They correspond to the directions near the big circle attributed to A on the sphere in Fig. 4 and outside the vicinity of intersections with the second circle corresponding to B.) It is reasonable to name the domain of ω such that $\omega \in \{\omega : \ |\theta_A(\omega,\omega_0) - \pi| < \text{const}/[kr]^{1/2-\delta}\}$ transition zone.

The total field can be then represented as

$$U_i(kr,\omega,\omega_0) + U(kr,\omega,\omega_0) = U_{go}(kr,\omega,\omega_0) + W_B(kr,\omega,\omega_0) + U_{cs}(kr,\omega,\omega_0)\,, \tag{30}$$

where the geometrical optics part $U_{go}(kr,\omega,\omega_0) = U_i(kr,\omega,\omega_0)$ as $\theta_r > \pi$ and $\theta_i < \pi$, $U_{go}(kr,\omega,\omega_0) = U_i(kr,\omega,\omega_0) + U_r(kr,\omega,\omega_0)$ as $\theta_r < \pi$ and $\theta_i < \pi$. (Analogously, $W_B(kr,\omega,\omega_0) = U_B(kr,\omega,\omega_0)$ as $\theta_B < \pi$, otherwise it is zero.)

For the component $U_{cs}(kr,\omega,\omega_0)$ attributed to the cylindrical and spherical waves we use the following motivation. We add and subtract the main singular term in the integrand $\Phi(\alpha,\omega,\omega_0)$ of the Sommerfeld integral assuming that $\alpha \sim \theta_A \sim \pi$. It is useful to notice that the following representation is valid

$$\Phi_1(\alpha,\omega,\omega_0)\log[\cos\alpha - \cos\theta_A(\omega,\omega_0)] =: \left(\frac{\Phi(\alpha,\omega,\omega_0)}{-\sin\alpha} - \frac{A_0(\omega,\omega_0)}{\cos\alpha - \cos\theta_A(\omega,\omega_0)}\right),$$

where $\Phi_1(\alpha,\omega,\omega_0)$ is holomorphic in a vicinity of $\alpha \sim \pi$. Provided that ω belongs to the transition zone, integration over the vicinities of $O(1/[kr]^{1/2-\delta})$ of $\pm\pi$, shown by the circle in Fig. 5 (left), gives the leading contribution to the asymptotics ($kr \to \infty$). We consider the $O(1/[kr]^{1/2-\delta})$-vicinity of the point π in Fig. 5 (left) then exploit change of the integration variable $\zeta(\alpha) = \sqrt{\mathrm{i}(1+\cos\alpha)}$. After some reductions analogous to those in Section 7 of Lyalinov (2013) we arrive at

$$U_{cs}(kr,\omega,\omega_0) = \frac{A_0(\omega,\omega_0)}{\pi\mathrm{i}}\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{-\mathrm{i}kr}}\int_\Gamma \mathrm{e}^{-t^2}\frac{2t\,\mathrm{d}t}{t^2 - \mathrm{i}kr(1+\cos\theta_A(\omega,\omega_0))} +$$
$$\frac{1}{\pi\mathrm{i}}\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{-\mathrm{i}kr}}\int_\Gamma \mathrm{d}t\,\mathrm{e}^{-t^2}\left(\frac{\mathrm{i}\sqrt{2}\,\Phi(\alpha(t/\sqrt{kr}),\omega,\omega_0)}{\sqrt{-\mathrm{i}kr}} - \frac{2t A_0(\omega,\omega_0)}{t^2 - \mathrm{i}kr(1+\cos\theta_A(\omega,\omega_0))}\right)\left(1 + O((kr)^{-1})\right), \tag{31}$$

[10] The upper or lower signs are only taken simultaneously.
[11] Recall that $\Phi(\alpha,\omega,\omega_0)$ is an odd function of α so that we speak about $\theta_A \sim \pi$ and $-\theta_A \sim -\pi$.

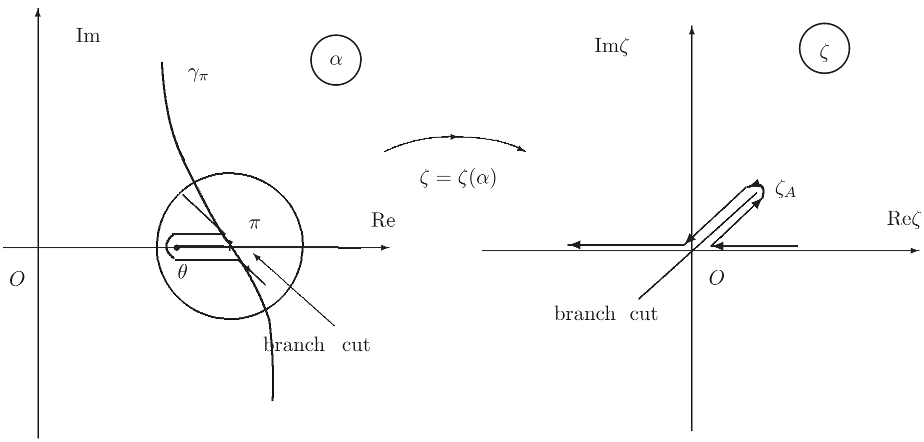

FIG. 5. Conformal mapping, $\zeta(\alpha) = \sqrt{\mathrm{i}(1+\cos\alpha)}$.

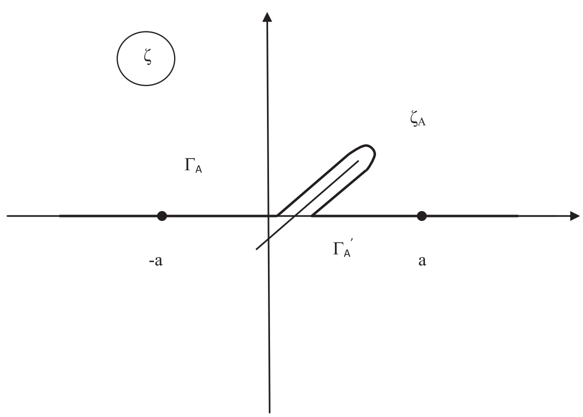

FIG. 6. The contour $\varGamma_A$, $\zeta_A = \sqrt{\mathrm{i}(1+\cos\theta_A)}$.

where $\varGamma$ is obtained from $\varGamma_A$ shown in Fig. 6 and in Fig. 5 (right) in accordance with $t = \sqrt{kr}\zeta$, $t(\alpha) = \sqrt{\mathrm{i}kr(1+\cos\alpha)} = \mathrm{e}^{\mathrm{i}\pi/4}\sqrt{2kr}\cos(\alpha/2) = \sqrt{kr}\zeta(\alpha)$.[12] Remark that

$$\alpha(t/\sqrt{kr}) = \pi - \frac{2t}{\sqrt{2\mathrm{i}kr}} + \ldots$$

for bounded t and large kr, where $\alpha(t)$ is the inverse to $t(\alpha)$.

[12] The conformal mapping $\zeta(\alpha) = \mathrm{e}^{\mathrm{i}\pi/4}\sqrt{2}\cos(\alpha/2)$ is illustrated in Fig. 5.

It is useful to write the second integral in (31) in the asymptotically equivalent form

$$\frac{1}{\pi \mathrm{i}} \frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{-\mathrm{i}kr}} \int_{\Gamma} 2t\,\mathrm{e}^{-t^2}\, \frac{\Phi_1(\alpha(t/\sqrt{kr}), \omega, \omega_0)}{\mathrm{i}kr}\, \log\left(\frac{t^2}{\mathrm{i}kr} - (1+\cos\theta_A(\omega,\omega_0))\right) \mathrm{d}\,t,$$

where $\Phi_1(\alpha, \omega, \omega_0)$ is holomorphic w.r.t. α in $O(1/[kr]^{1/2-\delta})$-vicinity (Fig. 5.) of the saddle point π ($\delta > 0$ is small). The branch cut for $\log(\cdot)$ goes from $\theta = \theta_A$ to infinity and is shown in Fig. 5 (left) and in Fig. 6. The latter expression is estimated by $O(\log(kr)/[kr]^{3/2})$ provided $kr(1+\cos\theta_A) = O(1)$, which means that the first integral in (31) plays the leading role in this case.

However, provided ω moves to the exterior of the transition region, both integrals contribute to the asymptotics. Indeed, it is verified that, provided $\theta_A < \pi - C/[kr]^{1/2-\delta}$, $C > 0$ and $kr(1+\cos\theta_A) = O([kr]^{\delta}) \to \infty$, the singularities can be separated from the saddle points; from (31) one has

$$U_{cs}(kr, \omega, \omega_0) = U_A(kr, \omega, \omega_0) - \sqrt{\frac{2}{\pi}}\, \Phi(\pi, \omega, \omega_0)\, \frac{\mathrm{e}^{\mathrm{i}kr}}{(-\mathrm{i}kr)} (1 + O((kr)^{-1}))\,,$$

i.e. the sum of the edge and spherical waves.

4.2 *Three singularities are near the saddle point π*

Now two transition zones under consideration are correspondingly described by

$$\omega \in \left\{ \omega :\ |\theta_A(\omega,\omega_0) - \pi| < \frac{\text{const}}{[kr]^{1/2-\delta}},\quad |\theta_B(\omega,\omega_0) - \pi| < \frac{\text{const}}{[kr]^{1/2-\delta}},\quad |\theta_r(\omega,\omega_0) - \pi| < \frac{\text{const}}{[kr]^{1/2-\delta}} \right\}$$

or

$$\omega \in \left\{ \omega :\ |\theta_A(\omega,\omega_0) - \pi| < \frac{\text{const}}{[kr]^{1/2-\delta}},\quad |\theta_B(\omega,\omega_0) - \pi| < \frac{\text{const}}{[kr]^{1/2-\delta}},\quad |\theta_i(\omega,\omega_0) - \pi| < \frac{\text{const}}{[kr]^{1/2-\delta}} \right\}.$$

The first zone corresponds to a small vicinity of the direction ω_* on the unit sphere which is connected with the tangency point of the fronts of the edge waves from A and B and that of the reflected wave with the spherical wave front, $\theta_r = \theta_A = \theta_B = \pi$ corresponds to ω_*. The singularies at θ_r, θ_A and θ_B are close to π (and due to parity $-\theta_r$, $-\theta_A$ and $-\theta_B$ are near $-\pi$). Exactly the point of intersection of two big circles on the sphere, shown in Fig. 4, specifies this direction. We consider the transition zone in a close vicinity of ω_* only, i.e. near the direction of the limiting ray 'reflected' from the vertex, whereas the second zone attributed to the limiting incident ray going through the vertex is studied very similarly. The second zone is attributed to the case of tangency of the incident wave front and those from the edges with the spherical front, i.e. in the forward direction. It is not considered herein.

The wave field can be then represented as

$$U_i(kr, \omega, \omega_0) + U(kr, \omega, \omega_0) = U_i(kr, \omega, \omega_0) + U_{csr}(kr, \omega, \omega_0) + \dots\,, \tag{32}$$

where the component $U_{csr}(kr, \omega, \omega_0)$ is attributed to the cylindrical, spherical and reflected waves which interfere in the transition zone. The dots in (32) means the other, for instance, secondary waves.

Exploiting the experience of the previous section (see also Section 7 of Lyalinov, 2013), introduce the notation

$$\Phi_*(\alpha,\omega,\omega_0)\log[\cos\alpha-\cos\theta_A(\omega,\omega_0)]+\Phi_{**}(\alpha,\omega,\omega_0)\log[\cos\alpha-\cos\theta_B(\omega,\omega_0)] =:$$

$$\left(\frac{\Phi(\alpha,\omega,\omega_0)}{-\sin\alpha}-\frac{A_0(\omega,\omega_0)}{\cos\alpha-\cos\theta_A(\omega,\omega_0)}-\frac{B_0(\omega,\omega_0)}{\cos\alpha-\cos\theta_B(\omega,\omega_0)}-\frac{\sqrt{\pi}}{4}\frac{R}{[\cos\alpha-\cos\theta_r(\omega,\omega_0)]^{3/2}}\right),$$

where $\Phi_*(\alpha,\omega,\omega_0)$ and $\Phi_{**}(\alpha,\omega,\omega_0)$ are holomorphic in a $O(1/[kr]^{1/2-\delta})$-vicinity of π.

We add and subtract the singular terms in the integrand $\Phi(\alpha,\omega,\omega_0)$ of the Sommerfeld integral corresponding to $\alpha\sim\theta_A$, $\alpha\sim\theta_B$ and $\alpha\sim\theta_r$, $\omega\sim\omega_*$. Consider the circle on α-plane of the radius of $O(1/[kr]^{1/2-\delta})$ with the centre at $\alpha=\pi$ (compare with Fig. 5, left). When computing the asymptotics of the Sommerfeld integral, we deform the the double-loop Sommerfeld contour in the SD paths that comprise the captured singularities including those $\alpha\sim\theta_A$, $\alpha\sim\theta_B$ and $\alpha\sim\theta_r$, located in the circle. The integration over the part γ_* of the SD paths, i.e. their parts located in this circle, is proved to give the principal asymptotic contribution as $kr\to\infty$. Remark that γ_* is actually similar to the the part of γ_π shown in Fig. 5. (left) and located inside the circle. We evaluate this principal contribution making the change of the integration variable and performing obvious asymptotic estimates.

We introduce the new variable[13] of integration $\zeta(\alpha)=\sqrt{\mathrm{i}(1+\cos\alpha)}=\mathrm{e}^{\mathrm{i}\pi/4}\sqrt{2}\cos(\alpha/2)$ with $2\zeta d\zeta=-\mathrm{i}\sin\alpha\mathrm{d}\alpha$, thus obtaining

$$U_{csr}(kr,\omega,\omega_0)=$$

$$\frac{1}{\pi\mathrm{i}}\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{-\mathrm{i}kr}}\left(\int_{\Gamma_*}\mathrm{d}\zeta\,\mathrm{e}^{-kr\zeta^2}\left(\frac{2\zeta A_0}{\zeta^2-\mathrm{i}(1+\cos\theta_A)}+\frac{2\zeta B_0}{\zeta^2-\mathrm{i}(1+\cos\theta_B)}+\frac{\sqrt{\pi}}{4}\frac{2\zeta R}{[\zeta^2-\mathrm{i}(1+\cos\theta_r)]^{3/2}}\right)+\right.$$

$$\int_{\Gamma_*}(-2\mathrm{i})\zeta\,\mathrm{e}^{-kr\zeta^2}\,\Phi_*(\alpha(\zeta),\omega,\omega_0)\,\log[\zeta^2/\mathrm{i}-(1+\cos\theta_A(\omega,\omega_0))]\,\mathrm{d}\zeta+$$

$$\left.\int_{\Gamma_*}(-2\mathrm{i})\zeta\,\mathrm{e}^{-kr\zeta^2}\,\Phi_{**}(\alpha(\zeta),\omega,\omega_0)\,\log[\zeta^2/\mathrm{i}-(1+\cos\theta_B(\omega,\omega_0))]\,\mathrm{d}\zeta+o(kr^{-\infty})\right) \tag{33}$$

with $\alpha(\zeta)=2\arccos(\frac{\zeta}{\sqrt{2\mathrm{i}}})$, where the contour γ_* is conformally mapped onto that Γ_*^a in Fig. 7 which is then supplemented by the semi-infinite rays $(-\infty,-a)$ and (a,∞) composing the contour $\Gamma_*=(\infty,a)\cup\Gamma_*^a\cup(-a,-\infty)$.[14] The singularities are mapped at $\zeta_A=\sqrt{\mathrm{i}(1+\cos\theta_A)}$, $\zeta_B=\sqrt{\mathrm{i}(1+\cos\theta_B)}$ and $\zeta_r=\sqrt{\mathrm{i}(1+\cos\theta_r)}$. It is useful to mention that $\arg\sqrt{\zeta-\zeta_r}\big|_{l_+}=-3\pi/8$, where l_+ is the right side of the branch cut in Fig. 7. This ensures that $[\cos\alpha-\cos\theta_r]^{-1/2}>0$ as $-\theta_r<\alpha<\theta_r$.

[13] The corresonding conformal mapping is shown in Fig. 5.

[14] The integration direction is opposite to the standard one.

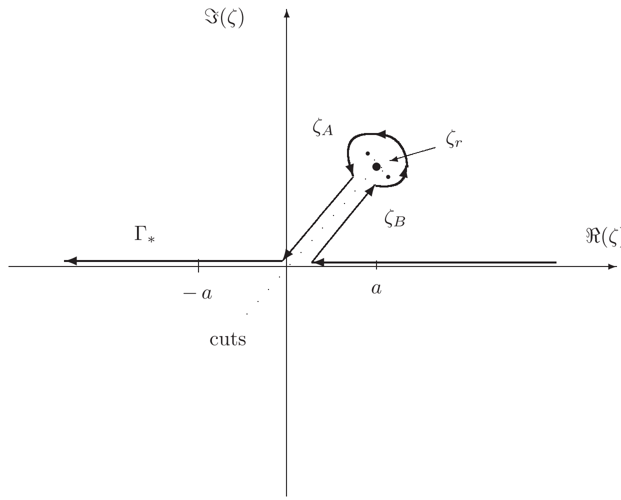

FIG. 7. The contour $\varGamma_*$ and the singularities

Finally, in (33) we introduce the variable $t = \sqrt{kr}\zeta$ and arrive at

$$U_{csr}(kr,\omega,\omega_0) =$$

$$\frac{1}{\pi \mathrm{i}} \frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{-\mathrm{i}kr}} \left(\int\limits_{\varGamma} \mathrm{d}t\ \mathrm{e}^{-t^2} \left\{ \frac{2tA_0}{t^2 - \mathrm{i}kr(1+\cos\theta_A)} + \frac{2tB_0}{t^2 - \mathrm{i}kr(1+\cos\theta_B)} + \frac{\sqrt{\pi}}{4} \frac{2t\sqrt{kr}\,R}{[t^2 - \mathrm{i}kr(1+\cos\theta_r)]^{3/2}} \right\} + \right.$$

$$\int\limits_{\varGamma} (-2\mathrm{i}) t\, \mathrm{e}^{-t^2} \frac{\varPhi_*\left(\alpha\left(\frac{t}{kr}\right),\omega,\omega_0\right)}{kr} \log\left[\frac{t^2}{\mathrm{i}kr} - (1+\cos\theta_A(\omega,\omega_0))\right] \mathrm{d}t +$$

$$\left. \int\limits_{\varGamma} (-2\mathrm{i}) t\, \mathrm{e}^{-t^2} \frac{\varPhi_{**}\left(\alpha\left(\frac{t}{kr}\right),\omega,\omega_0\right)}{kr} \log\left[\frac{t^2}{\mathrm{i}kr} - (1+\cos\theta_B(\omega,\omega_0))\right] \mathrm{d}t + o(kr^{-\infty})) \right), \quad (34)$$

where the contour $\varGamma$ is obtained by stretching from $\varGamma_*$ in accordance with $t = \sqrt{kr}\zeta$. The first integral in (34) is of the leading order $O(1)$ in comparison with the other two integrals that are estimated by $O(\log(kr)/kr)$ as ω is in the transition region. For the latter estimate one has to use regularity of $\varPhi_*$ and $\varPhi_{**}$ in a vicinity of $\alpha = \pi$. Remark that the first integral in (34) can be additionally modified to some canonical special functions like generalized Fresnel integrals. However, there is no actually need to do this because the integrand is an elementary function and the integral can be easily computed numerically. In the direction ω_* correponding to the ray 'reflected' from the vertex, in the leading terms, the far field has universal behaviour described by the first integral in (34). For this direction two poles and the branch point coalesce and the contour $\varGamma$ comprises them in the upper half-plane.

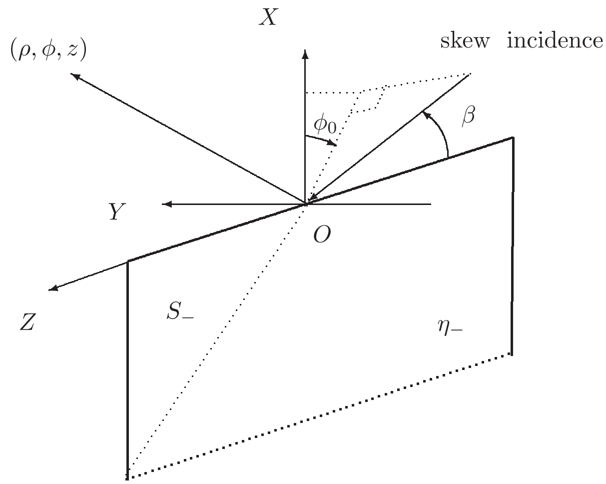

FIG. 8. Diffraction of a skew incident plane wave by an impedance half-plane. The edge A of the sector S coincides with OZ.

5. Conclusion

In this work we studied the asymptotics of the wave field scattered by an impedance sector illuminated by an acoustic plane wave. We made use of our recent results (Lyalinov, 2018), where the diffraction problem has been carefully studied and the Sommerfeld integral representation for the wave field has been derived. It has been shown that the Sommerfled transformant satisfies a problem for a 'hyperbolic' equation on the unit sphere with the cut. In the present work asymptotic evaluation of the Sommerfeld integral enabled us to deduce the desired asymptotics. To this end, we studied singularities of the transformants which were proved to be responsible for the different components of the far field. Some of the diffraction or excitation coefficients were found explicitly from the high frequency localization principle, whereas the others required the complete solution of the problem for the transformant. The wave field in the transition regions was also discussed.

It is ought to be noticed that numerical elaboration of our results requires further study of the problem at hand. In particular, efficient formulae for the analytic continuation of the Sommerfeld transformant are to be developed. These formulae will enable us to derive representations for the non-local diffraction or excitation coefficients, which are to be found numerically, in terms of solution of the problem on the unit sphere with the cut. It seems that tabulation of the transition functions in (34) and (31) is a simple task. They can be used for the uniform expressions (with respect to the observation direction) of the far field. These questions will be considered in our further studies.

A. Appendix

We use known expressions (see e.g. Lyalinov & Zhu, 2012, Section 1.5.7 for normal incidence) obtained for the skew incidence of an acoustic plane wave at the edge of an impedance half-plane $x \leqslant 0,\ y = 0,$ $x = \rho\cos\phi,\ \ y = \rho\sin\phi$. The angles specifying the direction of incidence are β and ϕ_0, see Fig. 7.

The incident plane wave is described by the expression

$$u_i(\rho, \phi, z) = \mathrm{e}^{\mathrm{i}k''z}\,\mathrm{e}^{-\mathrm{i}k'\rho\cos[\phi-\phi_0]},\quad k' = k\sin\beta,\quad k'' = k\cos\beta$$

then the total field is

$$u(\rho,\phi,z) = \mathrm{e}^{\mathrm{i}k''z}\, v(\rho,\phi)\,,$$

where the function $v(\rho,\phi)$ solves the Helmholtz equation

$$(\triangle_{\rho,\phi} + k'^2)v(\rho,\phi) = 0,$$

satisfies the boundary conditions on the faces $\phi = \pm\pi$

$$\pm\frac{1}{\rho}\frac{\partial v}{\partial\phi}\bigg|_{\phi=\pm\pi} - \mathrm{i}k'\frac{\eta_\pm}{\sin\beta}\, v\bigg|_{\phi=\pm\pi} = 0,$$

as well as the Meixner's and radiation conditions. The solution has been found by Malyuzhinets in the form of the Sommerfeld integral

$$v(\rho,\phi) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \mathrm{d}\alpha\, s(\alpha+\phi)\,\mathrm{e}^{-\mathrm{i}k'\rho\cos\alpha}\,,$$

where

$$s(z) = \frac{\Psi_0(z)}{\Psi_0(\phi_0)}\widetilde{s}(z)\,,$$

with

$$\widetilde{s}(z) = \frac{\frac{1}{2}\cos\frac{\phi_0}{2}}{\sin\frac{z}{2} - \sin\frac{\phi_0}{2}}\,, \qquad \Psi_0(z) \equiv \Psi(z,\zeta'_+)\,\Psi(z-2\pi,\zeta'_-),$$

$$\Psi(z,\zeta'_+) \equiv \psi_\pi(z + 3\pi/2 - \zeta'_+)\psi_\pi(z + \pi/2 + \zeta'_+)\,;$$

$\psi_\Phi(\cdot)$ is the Malyuzhinets function as $\Phi = \pi$, i.e. a specially normalized meromorphic solution of the functional equation

$$\frac{\psi_\Phi(\alpha + 2\Phi)}{\psi_\Phi(\alpha - 2\Phi)} = \cot(\alpha/2 + \pi/4).$$

We made use of the notations $\sin\zeta'_\pm = \frac{\eta_\pm}{\sin\beta}$.

Calculation of the asymptotics of the Sommerfeld integral as $k'\rho \to \infty$ leads to the expression of the cylindrical wave from the edge

$$u_c(\rho,\phi,z) = \frac{s(\phi-\pi) - s(\phi+\pi)}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}k''z}\,\frac{\mathrm{e}^{\mathrm{i}k'\rho+\mathrm{i}\pi/4}}{\sqrt{k'\rho}}\left(1 + O\left(\frac{1}{k'\rho}\right)\right). \tag{A.1}$$

For the surface wave propagating from the edge along the plane surface $\phi = \pi$ one has (see also Section 6.4 in Babich *et al.*, 2008)

$$u_s(\rho,\phi,z) = R_s^+\,\mathrm{e}^{\mathrm{i}k''z+\mathrm{i}k'\rho\cos[\pi+\zeta'_+-\phi]}\,, \tag{A.2}$$

where

$$R_s^+ = \frac{\widetilde{s}(2\pi+\zeta'_+)}{\Psi_0(\phi_0)}\,\Psi(\zeta'_+,\zeta'_-)\psi_\pi(5\pi/2 + 2\zeta'_+)\psi_\pi(-\pi/2)\,.$$

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