INCLUSIONS AMONG COMMUTATORS OF ELEMENTARY SUBGROUPS

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Abstract. In the present paper we continue the study of the elementary commutator subgroups $[E(n,A),E(n,B)]$, where A and B are two-sided ideals of an associative ring R, $n\geq 3$. First, we refine and expand a number of the auxiliary results, both classical ones, due to Bass, Stein, Mason, Stothers, Tits, Vaserstein, van der Kallen, Stepanov, as also some of the intermediate results in our joint works with Hazrat, and our own recent papers [40, 41]. The gimmick of the present paper is an explicit triple congruence for elementary commutators $[t_{ij}(ab),t_{ji}(c)]$, where a,b,c belong to three ideals A,B,C of R. In particular, it provides a sharper counterpart of the three subgroups lemma at the level of ideals. We derive some further striking corollaries thereof, such as a complete description of generic lattice of commutator subgroups $[E(n,I^r),E(n,I^s)]$, new inclusions among multiple elementary commutator subgroups, etc.

1. Introduction

Let $\mathrm{GL}(n,R)$ be the general linear group of degree $n\geq 3$ over an associative ring R with 1. For an ideal $A\trianglelefteq R$ we denote by $\mathrm{GL}(n,R,A)$ the principal congruence subgroup of level A and by $E(n,R,A)$ the corresponding relative elementary subgroup. The study of commutator subgroups

$$[\mathrm{GL}(n,R,A),\mathrm{GL}(n,R,B)],\quad [\mathrm{GL}(n,R,A),E(n,R,B)],\quad [E(n,R,A),E(n,R,B)]$$

and other related birelative groups has a venerable history. It goes back to the beginnings of algebraic K-theory in the works of Hyman Bass [4, 5, 6], and was then continued, at the stable level, by Alec Mason, Wilson Stothers [24, 23], and many others.

The next breakthrough, for rings satisfying commutativity conditions, came with the works by Andrei Suslin, Leonid Vaserstein, Zenon Borewicz and the first author, Anthony Bak, Alexei Stepanov, and many others [32, 33, 7, 2, 31]. However, these papers mostly addressed only the case where one of the ideals A or B was the ring R itself. These results depended on new powerful localisation methods introduced by Daniel Quillen and Suslin in connection with Serre's problem, and their off-springs, and also on remarkable geometric methods, see [3, 13] for a systematic description of that stage.

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The next stage started with our joint papers with Roozbeh Hazrat and Alexei Stepanov [36, 19, 37], where we addressed the general case, first for $\mathrm{GL}(n, R)$ over commutative rings, then over quasi-finite rings. Our works [14, 15, 20, 17, 30] address generalisations to other groups, such as Chevalley groups and Bak's unitary groups. These results are systematically described in our surveys and conference papers [10, 11, 12, 18].

One of the pivotal results of our theory, initially established in a somewhat weaker form by Hazrat and the second author in [20], and then stated in a more precise form in our joint papers [16, 18], is a description of generators of $[E(n,R,A), E(n,R,B)]$ as a subgroup. In those papers it was proven over quasi-finite rings, and involved three types of generators, the Stein—Tits–Vaserstein generators $t_{ji}(c)t_{ij}(ab)t_{ji}(-c)$ and $t_{ji}(c)t_{ij}(ba)t_{ji}(-c)$, the elementary commutators $[t_{ij}(a)t_{ji}(b)]$, where in both cases $i \neq j$, $a \in A$, $b \in B$, $c \in R$, and also a third type of generators.

In 2018 we observed that the generators of the third type were redundant, see, [34, 39]. Then in 2019 we noticed that even the elementary commutators should be only taken for one root, and finally that this generation result holds over *arbitrary* associative rings, see [40], Theorem 1 (which we reproduce below as Lemma 3). This proof is then repeated in a slightly more transparent form in [41], § 5, as part of the proof of the elementary multiple commutator formula

In turn, our proof of that result hinges on the following result, which is a stronger, and more precise version of [40], Lemma 5. In this form, it is proven as [41], Lemma 11. Here we denote by $A \circ B = AB + BA$ the symmetrised product of two-sided ideals A and B. For commutative rings, $A \circ B = AB = BA$ is the usual product of ideals A and B.

Theorem A. *Let R be an associative ring with 1, $n \geq 3$, and let A, B be two-sided ideals of R. Then for any $1 \leq i \neq j \leq n$, any $1 \leq k \neq l \leq n$, and all $a \in A$, $b \in B$, $c \in R$, one has*

$$y_{ij}(ac, b) \equiv y_{kl}(a, cb) \pmod{E(n, R, A \circ B)}.$$

During the talks "commutators of relative and unrelative elementary subgroups" presenting our papers [40]–[43], both at the algebraic group seminar at Chebyshev Lab on October 22 (see http://chebyshev.spbu.ru/en/schedule/?week=1571605200), and the at the Conference "Homological algebra, ring theory and Hochschild cohomology" at EIMI on October 29 (see http://www.pdmi.ras.ru/EIMI/2019/CR/index.html) the first author was invariably writing $a \in A$, $b \in B$, $c \in C$. During the second talk, Pavel Kolesnikov, who was keeping notes, asked, what was that C? After a little thinking, we decided that C should be $C \trianglelefteq R$ here, rather than R itself. Of course, modulo a smaller subgroup $E(n, R, ABC)$ the elementary commutator in the left hand side will be congruent to a product of *two* other elementary commutators, rather than to a single such commutator, as in the absolute case.

Theorem 1. *Let R be an associative ring with 1, $n \geq 3$, and let A, B, C be two-sided ideals of R. Then for any three distinct indices i, j, h such that $1 \leq i, j, h \leq n$, and all $a \in A$, $b \in B$, $c \in C$, one has*

$$y_{ij}(ab, c)y_{jh}(ca, b)y_{hi}(bc, a) \equiv 1 \pmod{E(n, R, ABC + BCA + CAB)}.$$

Observe, that the ideal defining the congruence module is precisely

$$ABC + BCA + CAB = A \circ BC + B \circ CA + C \circ AB.$$

This can be easily established in the same style as in our recent papers [40]–[43]. Morally, it is essentially the same computation by Mennicke [25] in the form given to it by Bass—Milnor—Serre in [6], Theorem 5.4. Subsequently, it was used in virtually each and every paper on bounded generation. Alternatively, one could use the Hall—Witt identity. However, everywhere before $C = R$ so that the last factor becomes trivial.

Theorem 1, and its immediate consequence Theorem 6, the three ideals lemma

$$[E(n, AB), E(n, C)] \leq [E(n, BC), E(n, A)] \cdot [E(n, CA), E(n, B)],$$

are the high points of the present paper, the rest are either preparations to their proof, or corollaries of the above *trirelative* congruence. However, a number of intermediate results generalise classical results and are decidedly interesting in themselves. Let us list other principal results of the present paper.

• Theorem 2: generators of partially relativised elementary subgroup $E(n, B, A)$, where $A, B \trianglelefteq R$, generalising a classical result by Stein—Tits—Vaserstein.

• Theorem 3: reduced set of generators of $E(n, R, A)$, in terms of the unipotent radicals of two opposite parabolic subgroups, generalising results by van der Kallen and Stepanov.

• Theorem 4: the three ideals lemma for partially relativised subgroups $E(n, A, BC)$.

• Theorem 5: a stable version of the standard commutator formula,

$$[\mathrm{GL}(n-1, R, A), E(n, R, B)] = [E(n, A), E(n, B)],$$

for arbitrary associative rings.

• Propositions 2 and 3: new inclusions for multiple elementary commutators.

• Theorem 7: a complete description of the generic lattice of inclusions among $[E(n, I^r), E(n, I^s)]$, for powers of one ideal $I \trianglelefteq R$.

• Proposition 5: inclusion $[E(n, A + B), E(n, A \cap B)] \leq [E(n, A), E(n, B)]$.

• Proposition 6: an explicit example, where $E(n, R, A \cap B)$ is strictly smaller than $E(n, R, A) \cap E(n, R, B)$.

The paper is organised as follows. In § 2 and § 3 we review some notation and briefly recall the requisite facts on elementary subgroups in $\mathrm{GL}(n, R)$. The next six sections are of a technical nature, they develop technical tools for the rest of the present paper. Namely, in § 4 we consider partially relativised elementary subgroups $E(n, B, A)$ and prove Theorem 2, which gives their sets of generators. In § 5 we recall some basic facts on intersections of parabolic subgroups with congruence subgroups, after which in § 6 we establish Theorem 3, which is a further strengthening of results by van der Kallen and Stepanov, generation of $E(n, R, A)$ in terms of unipotent radicals of two opposite parabolic subgroups. In § 7 we prove a toy version of our main results, the three ideals lemma for partially relativised elementary groups $E(n, C, AB)$, Theorem 4. In § 8 we establish Theorem 5, which is a stable version of the standard commutator formula, valid for all associative rings. In § 9 we discuss the important special case, behaviour of elementary commutators modulo relative elementary subgroups. The core of the present paper is § 10, where we prove Theorem 1 and using that derive an inclusion among birelative commutators, Theorem 6. This is kind of a three ideals lemma, to be used in all subsequent results. The balance of this paper is dedicated to applications. In § 11 we consider an application to the only outstanding case in our multiple elementary commutator paper [41], quadruple commutators in $\mathrm{GL}(3,R)$, and obtain some new inclusions among multiple elementary commutator subgroups. In § 12 we obtain definitive results for the crucial case of the powers of one ideal and prove Theorem 7. These results will be instrumental in the sequel of the present paper dedicated to the case of Dedekind rings. In § 13 we compare the commutator of two elementary subgroups of levels A and B with the commutator of elementary subgroups of levels $A\cap B$ and $A+B$. In § 14 we construct a counter-example concerning intersections of relative elementary subgroups. Finally, in § 15 we make some further related observations, and state some unsolved problems.

Initially, we planned to include in this paper also explicit computations over Dedekind rings. But then we realised that the topic is so extensive that it would be more appropriate to publish those results separately.

2. Notation

2.1. **Commutators.** Let G be a group. A subgroup $H\le G$ generated by a subset $X\subseteq G$ will be denoted by $H=\langle X\rangle$. For two elements $x,y\in G$ we denote by ${}^xy=xyx^{-1}$ and $y^x=x^{-1}yx$ the left and right conjugates of y by x, respectively. Further, we denote by

$$[x,y]=xyx^{-1}y^{-1}={}^xy\cdot y^{-1}=x\cdot {}^yx^{-1}$$

the left-normed commutator of x and y. Our multiple commutators are also left-normed. Thus, by default, $[x,y,z]$ denotes $[[x,y],z]$, we will use different notation for other arrangement of brackets. Throughout the present paper we repeatedly use the customary commutator identities, such as their multiplicativity with respect to the factors:

$$[x,yz]=[x,y]\cdot {}^y[x,z],\qquad [xy,z]={}^x[y,z]\cdot[x,z],$$

and a number of other similar identities, such as

$$[x,y]^{-1}=[y,x],\qquad {}^z[x,y]=[{}^zx,{}^zy],\qquad [x^{-1},y]=[y,x]^x,\qquad [x,y^{-1}]=[y,x]^y,$$

usually without any specific reference. Iterating multiplicativity we see that the commutator $[x_1\dots x_m,y]$ is the product of conjugates of the commutators $[x_i,y]$, $i=1,\dots,m$. Obviously, a similar claim holds also for $[x,y_1\dots y_m]$.

Further, for two subgroups $F,H\le G$ one denotes by $[F,H]$ their mutual commutator subgroup, spanned by all commutators $[f,h]$, where $f\in F$, $h\in H$. Clearly, $[F,H]=[H,F]$, and if $F,H\trianglelefteq G$ are normal in G, then $[F,H]\trianglelefteq G$ is also normal. Similarly, for $F,H,K\le G$ three subgroups of G their triple commutator $[F,H,K]$ is spanned by $[f,h,k]$, where $f\in F$, $h\in H$ and $k\in K$. We will use the following version of the three subgroups lemma.

Lemma 1. *If $F, H, K \le G$ be three subgroups of G. Assume that two of the subgroups $[F, H, K]$, $[H, K, F]$, $[K, F, H]$ are normal in G. Then the third of them is also normal and*

$$[F, H, K] \le [H, K, F] \cdot [K, F, H].$$

Often times, elementary textbooks needlessly assume that the subgroups F, H, K themselves are normal in G. This depends, of course, on the exact form of the Hall—Witt identity one is using. In the correct form, the only conjugations occur outside of the commutators, one such form is

$$[x, y^{-1}, z^{-1}]^x \cdot [z, x^{-1}, y^{-1}]^z \cdot [y, z^{-1}, x^{-1}]^y = 1.$$

2.2. **General linear group.** Let R be an associative ring with 1, R^* be the multiplicative group of the ring R. For two natural numbers m, n we denote by $M(m, n, R)$ the additive group of $m \times n$-matrices with entries in R. By $M(n, R) = M(n, n, R)$ we denote the full matrix ring of degree n over R.

Let $G = \mathrm{GL}(n, R) = M(n, R)^*$ be the general linear group of degree n over R. In the sequel for a matrix $g \in G$ we denote by g_{ij} its matrix entry in the position (i, j), so that $g = (g_{ij})$, $1 \le i, j \le n$. The inverse of g will be denoted by $g^{-1} = (g'_{ij})$, $1 \le i, j \le n$.

As usual we denote by e the identity matrix of degree n and by e_{ij} a standard matrix unit, i. e., the matrix that has 1 in the position (i, j) and zeros elsewhere. An elementary transvection $t_{ij}(\xi)$ is a matrix of the form $t_{ij}(c) = e + ce_{ij}$, $1 \le i \ne j \le n$, $c \in R$.

Further, let A be a two-sided of R. We consider the corresponding reduction homomorphism

$$\rho_A : \mathrm{GL}(n, R) \longrightarrow \mathrm{GL}(n, R/A), \quad (g_{ij}) \mapsto (g_{ij} + A).$$

Now, the *principal congruence subgroup* $\mathrm{GL}(n, R, A)$ of level A is the kernel ρ_A,

For a commutative ring R we denote by $\mathrm{SL}(n, R)$ the corresponding general linear group. All other subgroups are interpreted similarly. Thus, for instance, the principal congruence subgroup $\mathrm{SL}(n, R, A)$ is defined as $\mathrm{SL}(n, R, A) = \mathrm{GL}(n, R, A) \cap \mathrm{SL}(n, R)$.

3. Generation of relative elementary subgroups

The *unrelative elementary subgroup* $E(n, A)$ of level A in $\mathrm{GL}(n, R)$ is generated by all elementary matrices of level A. In other words,

$$E(n, A) = \langle e_{ij}(a),\ 1 \le i \ne j \le n,\ a \in A\rangle.$$

In general $E(n, A)$ has little chances to be normal in $\mathrm{GL}(n, R)$. The *relative elementary subgroup* $E(n, R, A)$ of level A is defined as the normal closure of $E(n, A)$ in the absolute elementary subgroup $E(n, R)$:

$$E(n, R, A) = \langle e_{ij}(a),\ 1 \le i \ne j \le n,\ a \in A\rangle^{E(n,R)}.$$

The following lemma on generation of relative elementary subgroups $E(n, R, A)$ is a classical result discovered in various contexts by Stein, Tits and Vaserstein, see, for instance, [33] (or [18], Lemma 3, for a complete elementary proof). It is stated in terms of the *Stein—Tits—Vaserstein generators*):

$$z_{ij}(a,c)=t_{ij}(c)t_{ji}(a)t_{ij}(-c),\qquad 1\le i\neq j\le n,\quad a\in A,\quad c\in R.$$

Lemma 2. *Let R be an associative ring with 1, $n\geq 3$, and let A be a two-sided ideal of R. Then as a subgroup $E(n,R,A)$ is generated by $z_{ij}(a,c)$, for all $1\le i\neq j\le n$, $a\in A$, $c\in R$.*

The following result is a generalisation of Lemma 2 to mutual commutator subgroups $[E(n,R,A),E(n,R,B)]$ of relative elementary subgroups. a further type of generators occur, the *elementary commutators*:

$$y_{ij}(a,b)=[t_{ij}(a),t_{ji}(b)],\qquad 1\le i\neq j\le n,\quad a\in A,\quad b\in B.$$

In slightly less precise forms, Theorem A was discovered by Roozbeh Hazrat and the second author, see [20], Lemma 12 and then in our joint paper with Hazrat [18], Theorem 3A. The strong form reproduced above was only established in our paper [40], Theorem 1, as an aftermath of our papers [34, 39].

Lemma 3. *Let R be any associative ring with 1, let $n\geq 3$, and let A,B be two-sided ideals of R. Then the mixed commutator subgroup $[E(n,R,A),E(n,R,B)]$ is generated as a group by the elements of the form*

- $z_{ij}(ab,c)=t_{ij}(c)t_{ji}(ab)t_{ij}(-c)$ *and* $z_{ij}(ba,c)=t_{ij}(c)t_{ji}(ba)t_{ij}(-c)$,
- $y_{ij}(a,b)=[t_{ij}(a),t_{ji}(b)]$,

where $1\le i\neq j\le n$, $a\in A$, $b\in B$, $c\in R$. Moreover, for the second type of generators, it suffices to fix one pair of indices (i,j).

Since all generators listed in Lemma 3 belong already to the commutator subgroup of unrelative elementary subgroups, we get the following corollary, [40], Theorem 2.

Lemma 4. *Let R be any associative ring with 1, let $n\geq 3$, and let A,B be two-sided ideals of R. Then one has*

$$\big[E(n,R,A),E(n,R,B)\big]=\big[E(n,R,A),E(n,B)\big]=\big[E(n,A),E(n,B)\big].$$

Let us state also some subsidiary results we use in our proofs. The following level computation is standard, see, for instance, [36, 37, 18], and references there.

Lemma 5. *R be an associative ring with 1, $n\geq 3$, and let A and B be two-sided ideals of R. Then*

$$E(n,R,A\circ B)\le\big[E(n,A),E(n,B)\big]\le\big[E(n,R,A),E(n,R,B)\big]\le \mathrm{GL}(n,R,A\circ B).$$

However, when applying this lemma to multiple commutators, one should bear in mind that the symmetrised product is not associative. Thus, when writing something like $A\circ B\circ C$, we have to specify the order in which products are formed. Of course, for commutative rings this dependence on the original bracketing disappears.

For quasi-finite rings the following result is [37], Theorem 5 and [18], Theorem 2A, but for arbitrary associative rings it was only established in [41], Theorem 2.

Lemma 6. *Let R be any associative ring with* 1*, let $n \geq 3$, and let A and B be two-sided ideals of R. If A and B are comaximal, $A + B = R$, then*

$$[E(n, A), E(n, B)] = E(n, R, A \circ B).$$

4. Partially relativised elementary subgroups

Actually, the recent work by Alexei Stepanov [28, 29, 30, 1] makes apparent that in many contexts it is very useful to consider *partially* relativised subgroups. One such context is relative localisation, as introduced in the papers by Roozbeh Hazrat and the second author [19, 20], then expanded and developed in a series of our joint papers with Hazrat, and reconsidered by Stepanov, see [14, 15, 17, 10, 11, 12, 18, 29, 30].

Namely, for two ideals $A, B \trianglelefteq R$ we denote by $E(n, B, A)$ the smallest subgroup containing $E(n, A)$ and normalised by $E(n, B)$:

$$E(n, B, A) = E(n, A)^{E(n,B)}.$$

In particular, when $B = R$ we get the usual relative group $E(n, R, A)$, as defined above. Clearly, if $B \leq C$, then $E(n, B, A) \leq E(n, C, A)$. It follows that

$$E(n, A) = E(n, 0, A) \leq E(n, B, A) \leq E(n, R, A)$$

On the other hand,

$$\big[E(n, A), E(n, B)\big] \leq E(n, B, A) \cap E(n, A, B).$$

Thus, Lemma 5 implies the following inclusion, which is a broad generalisation of [1], Lemma 4.1, in the linear case.

Proposition 1. *Let R be any associative ring with* 1*, let $n \geq 3$, and let A, B be two-sided ideals of R. Then one has*

$$E(n, R, A \circ B) \leq E(n, B, A) \cap E(n, A, B).$$

Now, we start working towards a partially relativised generalisation of Lemma 2.

Lemma 7. *Let R be any associative ring with* 1*, let $n \geq 2$, and let A, B be two-sided ideals of R. Then one has*

$$\big\langle E(n, A), E(n, B)\big\rangle = E(n, A + B).$$

Proof. Clearly, the left hand side is contained in the right hand side. On the other hand $E(n, A + B)$ is generated by the elementary transvections $t_{ij}(a + b)$, where $1 \leq i \neq j \leq n$, $a \in A$, $b \in B$. But every $t_{ij}(a+b) = t_{ij}(a)t_{ij}(b) \in E(n, A)E(n, B)$. □

In particular, even $E(n, A)E(n, B) = E(n, A + B)$, if the left hand side is a subgroup. But this is easy to remedy. Indeed, the above lemma implies that in the definition of partially relativised subgroups one can assume that $A \leq B$.

Corollary 1. *Let R be any associative ring with* 1*, let $n \geq 2$, and let A, B be two-sided ideals of R. Then $E(n, B, A) = E(n, A + B, A)$.*

But since $E(n, B, A)$ is normalised by both $E(n, A)$ and $E(n, B)$, it is normal in $E(n, A + B)$.

Corollary 2. *Let R be any associative ring with 1, let $n \geq 2$, and let A, B be two-sided ideals of R. Then*

$$E(n, B, A)E(n, B) = E(n, A + B).$$

Passing to the normal closures in $E(n, R)$ we get the familiar equality, see, in particular, [36], Lemma 1.

Corollary 3. *Let R be any associative ring with 1, let $n \geq 2$, and let A, B be two-sided ideals of R. Then*

$$E(n, R, A)E(n, R, B) = E(n, R, A + B).$$

The following result is a generalisation of a classical result on generation of relative elementary subgroups $E(n, R, A)$, discovered in various contexts by Stein, Tits and Vaserstein, see, for instance, [33]. It is stated in terms of the *Stein—Tits—Vaserstein generators*):

$$z_{ij}(a, c) = t_{ij}(c)t_{ji}(a)t_{ij}(-c), \qquad 1 \leq i \neq j \leq n, \quad a \in A, \quad c \in R.$$

Essentially, its proof follows the proofs of Lemma 2 (as reproduced in [38], Theorem 1 or [18], Lemma 3, for instance). But of course a posteriori we can take advantage of the simplifications that result from Lemma 3.

Theorem 2. *Let R be an associative ring with identity 1, $n \geq 3$, and let A and B be two-sided ideals of R. Then the partially relativised elementary subgroup $E(n, B, A)$ is generated the elements $z_{ij}(a, b)$, for all $1 \leq i \neq j \leq n$, $a \in A$, $b \in B$.*

Proof. Clearly, $E(n, B, A)$ is generated by ${}^{y}x$ with $x \in E(n, A)$ and $y \in E(n, B)$. Since ${}^{y}x = [y, x] \cdot x$ we get $[y, x] \in [E(n, A), E(n, B)]$. By definition, the factor x is a product of elementary matrices $t_{ij}(a)$ with $i \neq j$ and $a \in A$. By Lemma 3 the commutator subgroup $[E(n, A), E(n, B)]$ is generated by $E(n, R, A \circ B)$ together with the elementary commutators

$$y_{ij}(a, b) = [t_{ij}(a), t_{ji}(b)] = t_{ij}(a) \cdot {}^{t_{ji}(b)}t_{ij}(-a) = z_{ij}(a, 0)z_{ij}(a, b),$$

where $1 \leq i \neq j \leq n$, $a \in A$ and $b \in B$.

Thus, it only remains to show that the generators $z_{ij}(ab, c) = {}^{t_{ji}(c)}t_{ij}(ab)$ and $z_{ij}(ba, c) = {}^{t_{ji}(c)}t_{ij}(ba)$ of the relative elementary group $E(n, R, A \circ B)$ are products of generators listed in the statement of the lemma. Here, as above, $1 \leq i \neq j \leq n$, $a \in A$, $b \in B$ and $c \in C$.

We choose a $h \neq i, j$, then

$$\begin{aligned} {}^{t_{ji}(c)}t_{ij}(ab) &= {}^{t_{ji}(c)}[t_{ih}(a), t_{hj}(b)] = [{}^{t_{ji}(c)}t_{ih}(a), {}^{t_{ji}(c)}t_{hj}(b)] = \\ &\big[[t_{ji}(c), t_{ih}(a)]t_{ih}(a), [t_{ji}(c), t_{hj}(b)]t_{hj}(b)\big] = \big[t_{jh}(ca)t_{ih}(a), t_{hi}(-bc)t_{hj}(b)\big] = \\ &{}^{t_{jh}(ca)}[t_{ih}(a), t_{hi}(-bc)t_{hj}(b)] \cdot [t_{jh}(ca), t_{hi}(-bc)t_{hj}(b)]. \end{aligned}$$

To finish the proof, we consider the two above commutators separately

$$u = [t_{ih}(a), t_{hi}(-bc)t_{hj}(b)], \qquad v = [t_{jh}(ca), t_{hi}(-bc)t_{hj}(b)]$$

The following computation shows that u is a products of generators listed in the statement of the lemma.

$$u = [t_{ih}(a), t_{hi}(-bc)t_{kj}(b)] = [t_{ih}(a), t_{hi}(-bc)] \cdot {}^{t_{hi}(-bc)}[t_{ih}(a), t_{hj}(b)] =$$
$$[t_{ih}(a), t_{hi}(-bc)] \cdot {}^{t_{hi}(-bc)}t_{ij}(ab) = [t_{ih}(a), t_{hi}(-bc)] \cdot [t_{hi}(-bc), t_{ij}(ab)]t_{ij}(ab) =$$
$$[t_{ih}(a), t_{hi}(-bc)] \cdot t_{hj}(-bcab)t_{ij}(ab) = t_{ih}(a) \cdot {}^{t_{hi}(-bc)}t_{ih}(-a)t_{hj}(-bcab)t_{ij}(ab),$$

A similar computation shows that the same holds also for v, which finishes the proof. □

Lemma 8. *Let R be an associative ring with identity 1, and let A, B, C and D be its two-sided ideals. Then we have the following commutator formula for partially relativised elementary subgroups*

$$[E(n, B, A), E(n, D, C)] = [E(n, A), E(n, C)].$$

Proof. Combining the inclusions among partially relativised subgroups with Lemma 4, we get

$$[E(n, A), E(n, C)] = [E(n, 0, A), E(n, 0, C]] \leq [E(n, B, A), E(n, D, C)] \leq$$
$$[E(n, R, A), E(n, R, C)] = [E(n, A), E(n, C)].$$

□

5. Parabolic subgroups

In this section and the next one we collect some results on generation of $E(n, R, A)$ by the elements in unipotent radicals, or their conjugates. We start with the absolute case.

5.1. **Standard parabolic subgroups.** Denote by R^n the free right R-module consisting of columns of height n with components from R. Similarly, ${}^nR$ denotes the free left R-module, consisting of rows of length n with components from R. The module ${}^nR$ is dual to R^n, with the pairing of ${}^nR$ and R^n defined by the multiplication of a row by a column, ${}^nR \times R^n \longrightarrow R$, $(v, u) \mapsto vu \in R$. The standard based of R^n and ${}^nR$ will be denoted by $e_1, \dots, e_n$ and $f_1, \dots, f_n$, respectively. Recall that e_i is the column of height n, whose i-th component equals 1, while all other components are zeroes. Similarly, f_i is the row of length n, whose i-th component equals 1, while all other components are zeroes. The base $f_1, \dots, f_n$ is dual to $e_1, \dots, e_n$, with respect to the above pairing. The group $G = \mathrm{GL}(n, R)$ acts on R^n on the left, by multiplication of a column $u \in R^n$ by a matrix $g \in G$, $(g, u) \mapsto gu$. By the same token, the group G acts on ${}^nR$ on the right by multiplication: $(v, g) \mapsto vg$ for $v \in {}^nR$, $g \in G$.

Denote by P_m the m-th standard *maximal parabolic subgroup* in $G = \mathrm{GL}(n, R)$. From a geometric viewpoint the subgroup P_i, $m = 1, \dots, n-1$, is precisely the stabiliser of the submodule V_m in V, generated by $e_1, \dots, e_m$. In matrix form P_m can be thought of as the group of upper block triangular matrices

$$P_m = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x \in \mathrm{GL}(m, R), y \in M(m, n-m, R), z \in \mathrm{GL}(n-m, R) \right\}.$$

Simultaneously we consider the *opposite* maximal parabolic subgroup P_m^-

$$P_m^- = \left\{ \begin{pmatrix} x & 0 \\ w & z \end{pmatrix} \mid x \in \mathrm{GL}(m,R), w \in M(n-m,m,R), z \in \mathrm{GL}(n-m,R) \right\}.$$

These subgroups admit Levi decompositions $P_m = L_m \leftthreetimes U_m$ and $P_m^- = L_m \leftthreetimes U_m^-$ with common Levi subgroup

$$L_m = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x \in \mathrm{GL}(i,R), z \in \mathrm{GL}(n-m,R) \right\},$$

and opposite unipotent radicals

$$U_m = \left\{ \begin{pmatrix} e & y \\ 0 & e \end{pmatrix} \mid y \in M(m,n-m,R) \right\}, \quad U_m^- = \left\{ \begin{pmatrix} e & 0 \\ w & e \end{pmatrix} \mid w \in M(n-m,m,R) \right\}.$$

In particular, L_m, and thus all of its subgroups, normalise both U_m and U_m^-.

5.2. **Generation by two opposite unipotent radicals.** The following result asserts that $E(n,R)$ is generated by the unipotent radicals of two standard parabolic subgroups. It is obvious from the Chevalley commutator formula, and well known. Actually, in more general settings this is the *definition* of elementary subgroups, see the paper by Victor Petrov and Anastasia Stavrova [26] and references there[1].

Lemma 9. *Let R be an associative ring with 1, $n \geq 3$, and $1 \leq m \leq n-1$. Then*

$$E(n,R) = \langle U_m, U_m^- \rangle.$$

What is important, is that both generators here are normalised by the Levi subgroup L_m and all of its subgroups.

As usual, for $m < n$ we consider the stability embedding

$$\mathrm{GL}(m,R) \longrightarrow \mathrm{GL}(n,R), \qquad g \mapsto \begin{pmatrix} g & 0 \\ 0 & e \end{pmatrix}.$$

This embedding is compatible with elementary subgroups, congruence subgroups, relative elementary subgroups, etc. When we consider $\mathrm{GL}(m,R,A)$, etc., as a subgroup of $\mathrm{GL}(n,R)$ we always mean its image under this embedding.

5.3. **Unipotent radicals of level A.** Now, let $A \trianglelefteq R$ be an ideal of R. We denote by $U_m(A)$ and $U_n^-(A)$ the intersections of U_m and U_m^- with $\mathrm{GL}(n,R,A)$:

$$U_m(A) = \left\{ \begin{pmatrix} e & y \\ 0 & e \end{pmatrix} \mid y \in M(m,n-m,A) \right\},$$

$$U_m^-(A) = \left\{ \begin{pmatrix} e & 0 \\ w & e \end{pmatrix} \mid w \in M(n-m,m,A) \right\}.$$

The following lemma is a direct corollary of the Levi decomposition for P_m and its opposite P_m^-.

[1]Of course, with this advanced approach one has to prove that this definition is correct, in other words that various parabolic subgroups lead to the same elementary subgroup. This is precisely what is accomplished in [26].

Lemma 10. *Let R be an associative ring with 1, $n\geq 3$, $1\leq m\leq n-1$, and let A,C be two-sided ideals of R. Then one has*

$$[\mathrm{GL}(m,R,A),U_m(C)]\leq U_m(AC),\qquad [\mathrm{GL}(m,R,A),U_m^-(C)]\leq U_m^-(CA).$$

Proof. Let $g\in\mathrm{GL}(m,R,A)$, $y\in M(m,n-m,C)$, and $w\in M(n-m,m,C)$. Then, clearly,

$$\left[\begin{pmatrix} g & 0\\ 0 & e\end{pmatrix},\begin{pmatrix} e & y\\ 0 & e\end{pmatrix}\right]=\begin{pmatrix} e & (g-e)y\\ 0 & e\end{pmatrix},\quad \left[\begin{pmatrix} g & 0\\ 0 & e\end{pmatrix},\begin{pmatrix} e & 0\\ w & e\end{pmatrix}\right]=\begin{pmatrix} e & 0\\ w(g^{-1}-e) & e\end{pmatrix},$$

where $y\equiv 0 \pmod C$, $w\equiv 0\pmod C$ and $g\equiv g^{-1}\equiv e\pmod A$. Then all outer-diagonal entries of the matrices in the right hand sides are congruent to 0 modulo AC, as claimed. □

6. Limiting the set of generators for $E(n,R,A)$

Let us state a result by Wilberd van der Kallen, [21], Lemma 2.2. Morally, it is a trickier and mightier version of Lemma 1, with a smaller set of generators.

Lemma 11. *Let $A\trianglelefteq R$ be an ideal of an associative ring, $n\geq 3$. Then as a subgroup $E(n,R,A)$ is generated by $E(n,A)$ and $z_{in}(a,d)$, for all $1\leq i\leq n-1$, $a\in A$, $d\in R$*

Generalisations of this result to unipotent radicals of parabolics in arbitrary Chevalley groups were obtained by Alexei Stepanov in [28, 29, 30], (of course, in these papers R was assumed commutative).

Below we extract the rationale behind these results by van der Kallen and Stepanov and prove a still stronger version of their results, for the case of $\mathrm{GL}(n,R)$. Formally, it is not necessary for the rest of the paper, Lemma 11 would suffice. Yet, it is so natural in itself, that it will be certainly prove useful for future applications. Also, it is a midget version of our main result, Theorem 1, with one ideal instead of three, and with $y_{ij}(ab,c)$ replaced by $z_{ij}(a,d)$. Actually, in the next section we breed it up to a life-size toy version of our main results. Eventually, we believe it should be taken as the correct definition of relative elementary groups in more general settings, viz., for isotropic reductive groups, see [26, 27].

First, we reproduce the proof of Lemma 11 for $\mathrm{GL}(3,R)$.

Lemma 12. *Let $A\trianglelefteq R$ be an ideal of an associative ring, $n\geq 3$, and let i,j,h be three pair-wise distinct indices. Then if a subgroup $E(n,A)\leq H\leq \mathrm{GL}(n,R)$ contains*

- *either $z_{ih}(a,d)$ and $z_{jh}(a,d)$,*
- *or $z_{hi}(a,d)$ and $z_{hj}(a,d)$,*

for all $a\in A$, $d\in R$, then it also contains $z_{ij}(a,d)$ and $z_{ji}(a,d)$, for all such a and d.

Proof. We prove the desired inclusions for the first case, the second one is treated similarly. Indeed,

$$z_{ij}(a,d)={}^{t_{ji}(d)}t_{ij}(a)={}^{t_{ji}(d)}[t_{ih}(a),t_{hj}(1)]=[t_{ih}(a)t_{jh}(da),t_{hj}(1)t_{hi}(-d)].$$

Expanding the commutator with respect to the second factor, we see that

$$z_{ij}(a,d)=[t_{ih}(a)t_{jh}(da),t_{hj}(1)]\cdot {}^{t_{hj}(1)}[t_{ih}(a)t_{jh}(da),t_{hi}(-d)].$$

Now, expanding both commutators with respect to the first factor, we see that

$$z_{ij}(a,d) = {}^{t_{jh}(da)}[t_{ih}(a), t_{hj}(1)] \cdot [t_{jh}(da), t_{hj}(1)] \cdot \\ {}^{t_{hj}(1)t_{ih}(a)}[t_{jh}(da), t_{hi}(-d)] \cdot {}^{t_{hj}(1)}[t_{ih}(a), t_{hi}(-d)].$$

Consider the four factors on the right hand side individually

- ${}^{t_{jh}(da)}[t_{ih}(a), t_{hj}(1)] = t_{ij}(a)t_{ih}(-ada)$,
- $[t_{jh}(da), t_{hj}(1)] = t_{jh}(da) \cdot z_{jh}(-da, 1)$,
- ${}^{t_{hj}(1)t_{ih}(a)}[t_{jh}(da), t_{hi}(-d)] = t_{ji}(-dad)t_{hi}(-dad) \cdot z_{jh}(dada, 1)$,
- ${}^{t_{hj}(1)}[t_{ih}(a), t_{hi}(-d)] = t_{ih}(a)t_{ij}(-a) \cdot {}^{t_{hj}(1)}z_{ih}(-a, -d)$, but by Theorem A the last factor only differs from $z_{ih}(-a, -d) \in \mathrm{GL}(2, R, A)$ by a factor from $E(n, A)$.

We see that all factors only involve matrices from $E(n, A)$ and elementary conjugates of the form $z_{ih}(b, c)$ and $z_{jh}(b, c)$, for some $b \in A$, $c \in R$, as claimed. □

Theorem 3. *Let R be an associative ring with identity 1, $n \geq 3$, and let A be a two-sided ideal of R. Fix an m, $1 \leq m \leq n-1$. Then the relative elementary subgroup $E(n, R, A)$ is generated by*

$$U_m^-(A) \qquad \textit{and} \qquad uvu^{-1}, \quad \textit{where} \quad v \in U_m(A), u \in U_m^-.$$

Proof. Let H be the group generated by the above elements. First, observe that $E(n, A) \leq H$. Indeed, the following case analysis shows that H contains all generators of $E(n, A)$:

- When $i \leq m$ and $j \geq m+1$, or when $i \geq m+1$ and $j \leq m$, the generator $t_{ij}(a) = z_{ij}(a, 0)$ belongs to H by assumption.
- When $i, j \leq m$ take any $h \geq m+1$. Then $t_{ij}(a) = t_{ih}(a) \cdot {}^{t_{hj}(1)}t_{ih}(-a) \in H$.
- When $i, j \geq m+1$ take any $h \leq m$. Then $t_{ij}(a) = {}^{t_{ih}(1)}t_{hj}(a) \cdot t_{hj}(-a) \in H$.

Now, we are done by repeated application of Lemma 12. Indeed, $z_{ij}(a, d) \in H$ by assumption when $i \leq m$, $j \geq m+1$.

- When $i, j \leq m$ take any $h \geq m+1$. Then $z_{ih}(a, d), z_{jh}(a, d) \in H$ and thus $z_{ij}(a, d), z_{ji}(a, d) \in H$ by the first item of Lemma 12.
- When $i, j \geq m+1$ take any $h \leq m$. Then $z_{hi}(a, d), z_{hj}(a, d) \in H$ and thus $z_{ij}(a, d), z_{ji}(a, d) \in H$ by the second item of Lemma 12.

Finally, when $i \geq m+1$ and $j \leq m$, one has to distinguish two cases.

- If $m \geq 2$, one can choose $h \leq m$, $h \neq j$. Then $z_{hj}(a, d) \in H$ by assumption, whereas $z_{hi}(a, d) \in H$ by the first item above. Thus, $z_{ij}(a, d) \in H$ by the second item of Lemma 12.
- If $m \leq n-2$, one can choose $h \geq m+1$, $h \neq j$. Then $z_{ih}(a, d) \in H$ by assumption, whereas $z_{jh}(a, d) \in H$ by the second item above. Thus, $z_{ij}(a, d) \in H$ by the first item of Lemma 12.

It remains only to refer to Lemma 2 — or Theorem 2, for that matter. □

Corollary. *Let R be an associative ring with identity 1, $n\geq 3$, and let A be a two-sided ideal of R. Fix an m, $1\leq m\leq n-1$. Then the relative elementary subgroup $E(n,R,A)$ is generated by the group $E(n,A)$ and the elements $z_{ij}(a,d)$, for all $i\neq j$, $1\leq i\leq m$, $m+1\leq j\leq n$. $a\in A$, $d\in R$.*

7. Three ideals lemma for $E(n,C,AB)$

It is natural to ask, whether Theorem 2 admits a similar stronger version. Unfortunately, the above proof of Theorem 3 does not generalise immediately to the partially relativised case.

Problem 1. *Let R be an associative ring with identity 1, $n\geq 3$, and let A, B be a two-sided ideals of R. Fix an m, $1\leq m\leq n-1$. Is the partially relativised elementary subgroup $E(n,B,A)$ generated by*

$$U_m^-(A)\qquad\textit{and}\qquad uvu^{-1},\quad\textit{where}\quad v\in U_m(A),u\in U_m^-(B)?$$

Instead, we prove the following refinement of Lemma 12, which gives a full scale generalisation of Proposition 1, and a toy version of our Theorems 1 and 6.

Theorem 4. *Let R be an associative ring with identity 1, $n\geq 3$, and let A,B,C be a two-sided ideals of R. Then $E(n,C,AB)$ is contained in any of the following three spans:*

$$\big\langle E(n,BC,A),E(n,B,CA)\big\rangle,\qquad \big\langle E(n,A,BC),E(n,CA,B)\big\rangle,$$
$$\big\langle E(n,BC,A),E(n,CA,B)\big\rangle.$$

Proof. By Theorem 2 the group $E(n,C,AB)$ is generated by the elementary commutators $z_{ij}(ab,c)$, where $a\in A$, $b\in B$, $c\in C$. Now we imitate the proof of Lemma 12, but now monitor the levels of the occurring parameters of the z_{rs}'s in the right hand side, rather than their positions. As in Lemma 12 we take and $h\neq i,j$ and rewrite the generator $z_{ij}(c,ab)$ as a commutator:

$$z_{ij}(ab,c)={}^{t_{ji}(c)}t_{ij}(ab)={}^{t_{ji}(c)}[t_{ih}(a),t_{hj}(b)]=[t_{ih}(a)t_{jh}(ca),t_{hj}(b)t_{hi}(-bc)].$$

Expanding the commutator with respect to the second factor, we see that

$$z_{ij}(ab,c)=[t_{ih}(a)t_{jh}(ca),t_{hj}(b)]\cdot{}^{t_{hj}(b)}[t_{ih}(a)t_{jh}(ca),t_{hi}(-bc)].$$

Now, expanding both commutators with respect to the first factor, we see that

$$z_{ij}(ab,c)={}^{t_{jh}(ca)}[t_{ih}(a),t_{hj}(b)]\cdot[t_{jh}(ca),t_{hj}(b)]\cdot$$
$${}^{t_{hj}(b)t_{ih}(a)}[t_{jh}(ca),t_{hi}(-bc)]\cdot{}^{t_{hj}(b)}[t_{ih}(a),t_{hi}(-bc)].$$

Consider the four factors on the right hand side individually. Observe that by Lemma 4 the commutator subgroup $[E(n,A),E(n,B)]$ and other such double commutators are normal in $E(n,R)$, so that we can ignore all occurring elementary conjugations. Amazingly, the only problematic factor is the first one!

- Clearly, ${}^{t_{jh}(ca)}[t_{ih}(a),t_{hj}(b)]=t_{ij}(ab)t_{ih}(-abca)$ belongs to

$$E(n,AB)\leq E(n,A)\cap E(n,B)\leq E(n,BC,A)\cap E(n,CA,B).$$

- Further, $[t_{jh}(ca), t_{hj}(b)]$ belongs to
$$[E(n, CA), E(n, B)] \le E(n, CA, B) \cap E(n, B, CA).$$
- Next, ${}^{t_{hj}(b)t_{ih}(a)}[t_{jh}(ca), t_{hi}(-bc)]$ belongs to
$$[E(n, CA), E(n, BC)] \le E(n, CA, BC) \cap E(n, BC, CA) \le \\ E(n, A, BC) \cap E(n, B, CA) \cap E(n, BC, A) \cap E(n, CA, B).$$
- Finally, ${}^{t_{hj}(b)}[t_{ih}(a), t_{hi}(-bc)]$ belongs to
$$[E(n, A), E(n, BC)] \le E(n, A, BC) \cap E(n, BC, A).$$

We see that the third factor belongs to all four subgroups, and can be discarded, whereas the other three factors are contained in two of the subgroups $E(n, BC, A)$, $E(n, B, CA)$, $E(n, A, BC)$, $E(n, CA, B)$, each. Inspecting the cases listed in the statement, we see that all of them contain all three factors. □

The other three possible combinations of the subgroups $E(n, BC, A)$, $E(n, B, CA)$, $E(n, A, BC)$, $E(n, CA, B)$, do not seem to work in general. Thus, for instance, $\big\langle E(n, A, BC), E(n, B, CA)\big\rangle$ does not contain the first factor, and so on.

8. Stable version of the standard commutator formula

We start with a slightly more general form of [40], Lemma 3 and [41]. Lemma 9. Essentially, it is a classical corollary of surjective stability for K_1, but again we need a birelative version.

The following lemma is what stays behind [40], Lemma 3, and [41], Lemma 9. Our argument here is both much more general, and much easier, since it avoids all explicit computations.

Lemma 13. *Let R be an associative ring with 1, $n \ge 3$, and let A be a two-sided ideal of R. Then for any $g \in \mathrm{GL}(n-1, R, A)$ and any $x \in E(n, R)$ one has*
$${}^x g \equiv g \pmod{E(n, R, A)}.$$

Proof. By Lemma 9 any $x \in E(n, R)$ can be expressed as a product $x = y_1 \dots y_m$, where y_i alternatively belong to U_{n-1} or U^-_{n-1}. Consider a shorts such product. We argue by induction on m.

Let $x = yz$, where $y \in E(n, R)$ is shorter than x, whereas $z \in U_{n-1}$ or $z \in U^-_{n-1}$. By Lemma 10 $[g, z] \in U_{n-1}(A)$ in the first case, and $[g, z] \in U^-_{n-1}(A)$ in the first case. Since $U_{n-1}(A), U^-_{n-1}(A) \le E(n, A) \le E(n, R, A)$, this means that ${}^z g \equiv g \pmod{E(n, R, A)}$. This means that ${}^x g \equiv {}^y g \pmod{E(n, R, A)}$. But ${}^y g \equiv g \pmod{E(n, R, A)}$ by induction hypothesis. □

Of course, one would love to have a similar *birelative* lemma, asserting that for any $g \in \mathrm{GL}(n-1, R, A)$ and any $x \in E(n, R, C)$ one has ${}^x g \equiv g \pmod{E(n, R, A)}$. This would give plenty of leverage, to establish very strong results, including Theorem 1, with minimum direct calculations.

Unfortunately, it seems that such a lemma does not hold. What we can see easily, is only the weaker congruence ${}^x g \equiv g \pmod{[E(n, A), E(n, C)]}$. The following result is a version of the standard commutator formula that survives for arbitrary associative rings. Various forms of this result are known for decades, since the groundbreaking paper by Hyman Bass [4], and the refinements by Alec Mason and Wilson Stothers [24], see our exposition in [18]. However, the proofs proceeded as follows. First, one established a more sophisticated double relative version of Whitehead lemma, and then invoked deep results, such as Bass—Vaserstein injective stability for K_1. Our proof below is entirely elementary, works for all associative rings, and only uses the sharp generation results obtained in the previous sections.

Theorem 5. *Let R be an associative ring with 1, $n \geq 3$, and let A and C be two-sided ideals of R. Then*

$$[\mathrm{GL}(n-1, R, A), E(n, R, C)] = [E(n, A), E(n, C)].$$

Proof. Indeed, by Theorem 3 the group $E(n, R, C)$ is generated by $w \in U^-_{n-1}(C)$ and by uvu^{-1}, where $v \in U_{n-1}(C)$, $u \in U^-_{n-1}$. Take an arbitrary $g \in \mathrm{GL}(n-1, R, A)$. Then $[g, w] \in E(n, CA)$ by Lemma 10. On the other hand, for the other type of generators one has

$$[g, uvu^{-1}] = [g, u] \cdot {}^u[g, v] \cdot {}^{uv}[g, u^{-1}] = [g, u] \cdot {}^u[g, v] \cdot {}^u[v, [g, u^{-1}]] \cdot {}^u[g, u^{-1}].$$

Now, by Lemma 11 one has $[g, v] \in E(n, AC)$, so that ${}^u[g, v] \in E(n, R, AC)$. Similarly, $[g, u^{-1}] \in E(n, A)$, so that $[v, [g, u^{-1}]] \in [E(n, A), E(n, C)]$. It follows from Lemma 4 (but was known before, in fact), that $[E(n, A), E(n, C)]$ is normal in $E(n, R)$. Thus, ${}^u[v, [g, u^{-1}]] \in [E(n, A), E(n, C)]$.

By Lemma 5, one has $E(n, R, AC) \leq [E(n, A), E(n, C)]$ so that both central factors belong to $[E(n, A), E(n, C)]$. On the other hand, ${}^u[g, u^{-1}] = [g, u]^{-1}$. Again invoking the fact that $[E(n, A), E(n, C)]$ is normal in $E(n, R)$ we see that the commutator $[g, uvu^{-1}]$ belongs to $[E(n, A), E(n, C)]$.

Since the elements uvu^{-1} generate $E(n, R, C)$ and are themselves elementary, the left hand side of the equality in the statement of the theorem is contained in the right hand side. The other inclusion is obvious. □

Of course, when surjective stability holds for $\mathrm{K}_1(n-1, R, A)$, one has

$$\mathrm{GL}(n, R, A) = \mathrm{GL}(n-1, R, A)E(n, R, A),$$

so that Theorem 4 implies the usual standard commutator formula

$$[\mathrm{GL}(n, R, A), E(n, R, C)] = [E(n, A), E(n, C)].$$

Otherwise, we use Theorem 4 in the following form.

Corollary. *Let R be an associative ring with 1, $n \geq 3$, and let A and C be two-sided ideals of R. Then for any $g \in \mathrm{GL}(n-1, R, A)$ and any $x \in E(n, R, C)$ one has*

$${}^x g \equiv g \pmod{[E(n, A), E(n, C)]}.$$

9. Elementary commutators modulo $E(n, R, A \circ B)$

In the present section we collect special cases of the previous results concerning the behaviour of elementary commutators modulo the level.

Since the elementary commutator $y_{ij}(a, b)$, where $1 \leq i \neq j \leq n$, $a \in A$, $b \in B$, has level $A \circ B$, we get the following result, which is [40], Lemma 3, and [41], Lemma 9.

Lemma 14. *Let R be an associative ring with 1, $n \geq 3$, and let A, B be two-sided ideals of R. Then for any $1 \leq i \neq j \leq n$, $a \in A$, $b \in B$, and any $x \in E(n, R)$ one has*

$$ {}^{x}y_{ij}(a, b) \equiv y_{ij}(a, b) \pmod{E(n, R, A \circ B)}. $$

It is well known that the absolute elementary group $E(n, R)$ contains all permutation matrices, maybe after correcting the sign of one entry. Thus, already this lemma implies that elementary commutators $y_{ij}(a, b)$ and $y_{hk}(a, b)$ are congruent modulo $E(n, R, A \circ B)$. Of course, we still need Theorem A, since we need to move around not only the indices, but also the parameters.

The following result is [41], Lemmas 10 and 12.

Lemma 15. *Let R be an associative ring with 1, $n \geq 3$, and let A, B be two-sided ideals of R. Then for any $1 \leq i \neq j \leq n$, $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$ one has*

$$ \begin{aligned} &y_{ij}(a_1 + a_2, b) \equiv y_{ij}(a_1, b) \cdot y_{ij}(a_1, b) \pmod{E(n, R, A \circ B)}, \\ &y_{ij}(a, b_1 + b_2) \equiv y_{ij}(a, b_1) \cdot y_{ij}(a, b_2) \pmod{E(n, R, A \circ B)}, \\ &y_{ij}(a, b)^{-1} \equiv y_{ij}(-a, b) \equiv y_{ij}(a, -b) \pmod{E(n, R, A \circ B)}, \\ &y_{ij}(ab_1, b_2) \equiv y_{ij}(a_1, a_2 b) \equiv e \pmod{E(n, R, A \circ B)}, \\ &y_{ij}(a_1 a_2, b) \equiv y_{ij}(a, b_1 b_2) \equiv e \pmod{E(n, R, A \circ B)}. \end{aligned} $$

Together with Theorem A this lemma asserts that modulo $E(n, R, A \circ B)$ the elementary commutators $y_{ij}(a, b)$ do not depend on the choice of a pair (i, j), $i \neq j$, and can be considered as symbols

$$ \sigma : A/A(A+B) \otimes_R B/B(A+B) \longrightarrow [E(n, A), E(n, B)]/E(n, R, A \circ B), \qquad (1) $$

$$ (a + A(A+B)) \otimes (b + B(A+B)) \mapsto y_{12}(a, b) \pmod{E(n, R, A \circ B)}. \qquad (2) $$

Let us reiterate [40], Problem 1, and [41], Problem 2.

Problem 2. *Give a presentation of*

$$ \big[E(n, A), E(n, B)\big]/E(n, R, A \circ B) $$

by generators and relations. Does this presentation depend on $n \geq 3$?

The following lemma is classically known and obvious. It follows from the fact that in a Dedekind ring R for any two ideals A and B there exists an ideal $C \cong A$ such that $C + B = R$.

Lemma 16. *Let R be a Dedekind ring, A and B be ideals of R. Then*

$$ A/A(A+B) \cong B/B(A+B) \cong R/(A+B). $$

Thus, in this case the above symbols σ can be considered as symbols

$$\sigma : R/(A+B) \otimes_R R/(A+B) \longrightarrow [E(n,A), E(n,B)]/E(n,R,A\circ B),$$

closely related to the usual Mennicke symbols. We intend to address Problem 2 for Dedekind rings in a subsequent paper.

10. Proof of Theorem 1

Now, we are all set to prove the technical heart of the present paper, Theorem 1.

Proof. We take any $h \neq i,j$ and rewrite the elementary commutator $y_{ij}(ab,c) = \big[t_{ij}(ab), t_{ji}(c)\big]$ as

$$y_{ij}(ab,c) = t_{ij}(ab) \cdot {}^{t_{ji}(c)}t_{ij}(-ab) = t_{ij}(ab) \cdot {}^{t_{ji}(c)}\big[t_{ih}(a), t_{hj}(-b)\big].$$

Expanding the conjugation by $t_{ji}(b)$, we see that

$$y_{ij}(ab,c) = t_{ij}(ab) \cdot \big[{}^{t_{ji}(c)}t_{ih}(a), {}^{t_{ji}(c)}t_{hj}(-b)\big] = t_{ij}(ab) \cdot \big[t_{jh}(ca)t_{ih}(a), t_{hj}(-b)t_{hi}(bc)\big].$$

Expanding the commutator in the right hand side, using multiplicativity of the commutator w.r.t. the second argument, we get

$$y_{ij}(ab,c) = t_{ij}(ab) \cdot \big[t_{jh}(ca)t_{ih}(a), t_{hj}(-b)\big] \cdot {}^{t_{hj}(-b)}\big[t_{jh}(ca)t_{ih}(a), t_{hi}(bc)\big].$$

Expanding the first commutator in the right hand side, and using multiplicativity of the commutator w.r.t. the first argument, we get

$$\big[t_{jh}(ca)t_{ih}(a), t_{hj}(-b)\big] = {}^{t_{jh}(ca)}\big[t_{ih}(a), t_{hj}(-b)\big] \cdot \big[t_{jh}(ca), t_{hj}(-b)\big] = t_{ij}(-ab) \cdot t_{ih}(abca) \cdot y_{jh}(ca,-b)$$

Now, the first factor cancels with $t_{ij}(ab)$, the second factor belongs to $E(n,ABC)$, and can be discarded, so that the first commutator is congruent modulo $E(n,R,ABC)$ to $y_{jh}(ca,-b)$. By Lemma 14 one has

$$y_{jh}(ca,-b) \equiv y_{jh}(ca,b)^{-1} \pmod{E(n,R,CA\circ B)}.$$

Next, we look at the second commutator in the right hand side of the formula for $y_{ij}(ab,c)$, and using multiplicativity of the commutator w.r.t. the first argument, we get

$${}^{t_{hj}(-b)}\big[t_{jh}(ca)t_{ih}(a), t_{hi}(bc)\big] = {}^{t_{hj}(-b)t_{jh}(ca)}\big[t_{ih}(a), t_{hi}(bc)\big] \cdot {}^{t_{hj}(-b)}\big[t_{jh}(ca), t_{hi}(bc)\big] = {}^{t_{hj}(-b)t_{jh}(ca)}y_{ih}(a,bc) \cdot {}^{t_{hj}(-b)}t_{ji}(cabc).$$

Now, the second factor belongs to $E(n,ABC)$, and stays there after an elementary conjugation, so it can be discarded. The first factor is congruent to $y_{ih}(a,bc)$ modulo $E(n,R,A\circ BC)$ by Lemma 14. Again by Lemma 14 one has

$$y_{ih}(a,bc) \equiv y_{hi}(bc,-a) \equiv y_{hi}(bc,a)^{-1} \pmod{E(n,R,A\circ BC)}.$$

Summarising the above, we see that

$$y_{ij}(ab,c)y_{jh}(ca,b)y_{hi}(bc,a) \equiv 1 \pmod{E(n,R,ABC+BCA+CAB)},$$

as claimed. Observe that since all factors are central in $E(n,R)$ modulo the normal subgroup $E(n,R,ABC+BCA+CAB)$, which equals

$$E(n,R,A\circ BC)\cdot E(n,R,B\circ CA)\cdot E(n,R,C\circ AB),$$

their order does not matter. □

By the last remark in the proof of Theorem 1, the levels of all three commutators in the next result are contained in the normal subgroup $E(n,R,ABC+BCA+CAB)$. Thus, Theorem 1 immediately implies the following result that can be interpreted as a *three ideals lemma*.

Theorem 6. *Let R be an associative ring with 1, $n\geq 3$, and let A,B,C be two-sided ideals of R. Then*

$$[E(n,AB),E(n,C)]\leq [E(n,BC),E(n,A)]\cdot [E(n,CA),E(n,B)].$$

Modulo Lemma 14 on triple commutator subgroups, it becomes a special case of the three subgroups lemma, at least in the *commutative* case. However, the proof of Lemma 14 itself crucially depends on a version of Theorem A, Theorem 1, or a similar calculation.

11. Applications to multiple commutators

Our proof of elementary multiple commutator formulas in [41] is an easy induction that proceeds from the following two special cases, triple commutators, and quadruple commutators. The following results are [41], Lemma 7 and Lemma 8, respectively.

Lemma 17. *Let R be an associative ring with 1, $n\geq 3$, and let A,B,C be two-sided ideals of R. Then*

$$\big[\big[E(n,A),E(n,B)\big],E(n,C)\big]=\big[E(n,A\circ B),E(n,C)\big].$$

Lemma 18. *Let R be an associative ring with 1, $n\geq 4$, and let A,B,C,D be two-sided ideals of R. Then*

$$\big[\big[E(n,A),E(n,B)\big],\big[E(n,C),E(n,D)\big]\big]=\big[E(n,A\circ B),E(n,C\circ D)\big].$$

These results were first proven for quasi-finite rings by Roozbeh Hazrat and the second author, under assumption $n\geq 3$, see [20]. However, in that paper the proof was based on (a weaker version of) Theorem A and the usual (commutative!) localisation, so that there is no chance to make it work over arbitrary associative rings. Here, for quadruple commutators we assume that $n\geq 4$. The reason was that in [41] the proof of Lemma proceeds as follows. By Theorem A and Lemma 3 one only has to prove that

$$[y_{ij}(a,b),y_{hk}(c,d)]\in \big[E(n,A\circ B),E(n,C\circ D)\big],$$

for $1\leq i\neq j\leq n$, $1\leq h\neq k\leq n$, $a\in A$, $b\in B$, $c\in C$, $d\in D$. By Lemma 13 conjugations by elements $x\in E(n,R)$ do not matter, since they amount to extra factors from the above triple commutators, which are already accounted for. Now, for $n\geq 4$ this finishes the proof, since in this case modulo $E(n,R,C\circ D)$ we can move $y_{hk}(c,d)$ to a position, where it commutes with $y_{ij}(a,b)$, by Lemma 13 or Theorem A.

Problem 3. *Prove that Lemma* 18 *holds also for* $n = 3$*, or construct a counter-example.*

Actually, it seems to us that either way it will be non-trivial. To prove the lemma one will have to verify that the commutator $[y_{ij}(a,b), y_{ih}(c,d)]$ of two interlaced elementary commutators belongs where it should, and that's a non-trivial calculation. On the other hand, since Lemma 18 holds for quasi-finite rings, none of the usual counter-examples will work, so that one will have to construct a truly non-commutative counter-example. But to imitate Gerasimov's universal counter-example would be an extremely troublesome business.

Observe that in fact the three subgroups lemma and Lemma 17 imply the following poor man's version of our Theorem 1. In the commutative case it is essentially a slight generalisation of a result by Himanee Apte and Alexei Stepanov [1], Lemma 3.4.

Proposition 2. *Let* R *be an associative ring with identity* 1*,* $n \geq 3$*, and let* A*,* B *and* C *be its two-sided ideals. Then*

$$[E(n, A \circ B), E(n, C)] \leq [E(n, A \circ C), E(n, B)] \cdot [E(n, A), E(n, B \circ C)].$$

Proof. By the triple commutator formula of elementary subgroups

$$[E(n, A \circ B), E(n, C)] = \big[[E(n, A), E(n, B)], E(n, C)\big].$$

By the three subgroups lemma

$$\big[[E(n, A), E(n, B)], E(n, C)\big] \leq \big[[E(n, A), E(n, C)], E(n, B)\big] \cdot \big[E(n, A), [E(n, B), E(n, C)]\big].$$

Now, applying the triple commutator formula of elementary subgroups to the factors in the right hand side, we get

$$\big[[E(n, A), E(n, C)], E(n, B)\big] = \big[E(n, A \circ C), E(n, B)\big],$$
$$\big[E(n, A), [E(n, B), E(n, C)]\big] = \big[E(n, A), E(n, B \circ C)\big].$$

□

By analogy, we can do the same applying the three subgroup lemma to a quadruple commutator, and then combine it with Lemma 15 again. Of course, this might be interesting only for the case $n = 3$.

Proposition 3. *Let* R *be an associative ring with* 1*,* $n \geq 3$*, and let* A, B, C, D *be two-sided ideals of* R*. Then*

$$\big[[E(n, A), E(n, B)], [E(n, C), E(n, D)]\big] \leq [E(n, (A \circ B) \circ C), E(n, D)] \cdot [E(n, (A \circ B) \circ D), E(n, C)].$$

Proof. Indeed, by the three subgroups lemma one has

$$\big[[E(n, A), E(n, B)], [E(n, C), E(n, D)]\big] \leq$$
$$\big[\big[[E(n, A), E(n, B)], E(n, C)\big], E(n, D)\big] \cdot \big[\big[[E(n, A), E(n, B)], E(n, D)\big], E(n, D)\big].$$

It remains to twice apply Lemma 15 to each factor. □

However, it is not feasible to prove Lemma 16 using calculations at the level of subgroups. Rather, one should go to the level of individual elements. We have a blueprint how to do that by first applying the Hall—Witt identity, and then the identity in Theorem 1 to both resulting factors. However, the calculations seem to be formidable, and as of today we have not succeeded.

12. Inclusions among commutators for powers of one ideal

Let us state the following exciting special case of Theorem 6.

Proposition 4. *Let I be an ideal of an associative ring R. Then*

$$[E(n, I^{r+s}), E(n, I^t)] \leq [E(n, I^r), E(n, I^{s+t})] \cdot [E(n, I^s), E(n, I^{r+t})].$$

In the case of an even exponent this gives an inclusion among two such commutators.

Corollary. *Let I be an ideal of an associative ring R. Then*

$$[E(n, I^{2r}), E(n, I^t)] \leq [E(n, I^r), E(n, I^{r+t})].$$

Iterated application of this proposition allows to establish all inclusions among the commutator subgroups $[E(n, I^r), E(n, I^s)]$ of a given level. The following remarkably easy argument was suggested to the authors by Fedor Petrov. In the following theorem we call inclusions that result from Proposition 4, generic. It is not only interesting in itself, but also very relevant to obtain definitive results for Dedekind rings. These inclusions hold for arbitrary associative rings. Of course, for a specific ring some of them may become equalities.

Theorem 7. *Let I be an ideal of an associative ring R, $m \geq 1$. Then the generic lattice of elementary commutator subgroups*

$$H(r) = [E(n, I^r), E(n, I^{m-r})] \leq E(n, R, I^m), \qquad 0 \leq r \leq m,$$

of level I^m is isomorphic to the lattice of divisors of m. In other words, generically,

$$[E(n, I^r), E(n, I^{m-r})] \leq [E(n, I^s), E(n, I^{m-s})] \iff \gcd(s, m) | \gcd(r, m).$$

Proof. Let us understand r in the definition of $H(r)$ modulo m. Then, clearly, one has $H(r) = H(m - r) = H(-r)$ and $H(r + s) \leq H(r)H(s)$. Indeed, for $r, s \leq m/2$ this is precisely Proposition 4. When $r > m/2$ or/and $s > m/2$, we replace one or both of them by $m - r$ or/and $m - s$, and then apply Proposition 4.

In particular, this means that $H(kr) \leq H(r)$ for all $k \in \mathbb{Z}/m\mathbb{Z}$. Indeed, by induction $H(kr) \leq H((k - 1)r)H(r) \leq H(r)$. This establishes the second, and thus also the first claim of the Theorem. □

- In particular, this theorem implies that

$$H(r) = H(s) \iff \gcd(r, m) = \gcd(s, m)$$

and that

$$H(r) \leq H(s_1) \dots H(s_l) \iff \gcd(\gcd(s_1, m), \dots, \gcd(s_l, m)) | \gcd(r, m).$$

• At level I^p, where p is a prime, all non-trivial double commutators $[E(n,I^r),E(n,I^s)]$, $r+s=p$, are equal.

• At level I^4 one has
$$E(n,R,I^4)\leq[E(n,I^2),E(n,I^2)]\leq[E(n,I^3),E(n,I)].$$
The second claim of [24], Theorem 5.4 asserts that the second inclusion may be strict! Actually, it is strict already in the simplest example, where $R=\mathbb{Z}[i]$ is the ring of Gaussian integers, and $I=\mathfrak{p}=(1+i)\mathbb{Z}[i]$ is the prime divisor of 2,
$$[E(n,\mathbb{Z}[i],\mathfrak{p}^2),E(n,\mathbb{Z}[i],\mathfrak{p}^2)]<[E(n,\mathbb{Z}[i],\mathfrak{p}^3),E(n,\mathbb{Z}[i],\mathfrak{p})],$$
of index 2. In other words,
$$[E(n,\mathbb{Z}[i],\mathfrak{p}^2),E(n,\mathbb{Z}[i],\mathfrak{p}^2)]=E(n,\mathbb{Z}[i],\mathfrak{p}^4),$$
whereas
$$[E(n,\mathbb{Z}[i],\mathfrak{p}^3),E(n,\mathbb{Z}[i],\mathfrak{p})]=\mathrm{SL}(n,\mathbb{Z}[i],\mathfrak{p}^4).$$

• At level I^6 one has
$$E(n,R,I^6)\leq[E(n,I^3),E(n,I^3)],[E(n,I^4),E(n,I^2)]\leq[E(n,I^5),E(n,I)],$$
and there are no obvious inclusions between $[E(n,I^3),E(n,I^3)]$ and $[E(n,I^4),E(n,I^2)]$. However, the third claim of [24], Theorem 5.4 asserts in the above example of Gaussian integers one has
$$[E(n,\mathbb{Z}[i],\mathfrak{p}^4),E(n,\mathbb{Z}[i],\mathfrak{p}^2)]=E(n,\mathbb{Z}[i],\mathfrak{p}^6),$$
whereas
$$[E(n,\mathbb{Z}[i],\mathfrak{p}^3),E(n,\mathbb{Z}[i],\mathfrak{p}^3)]=[E(n,\mathbb{Z}[i],\mathfrak{p}^5),E(n,\mathbb{Z}[i],\mathfrak{p})],$$
is strictly larger, being a proper intermediate subgroup between $E(n,\mathbb{Z}[i],\mathfrak{p}^6)$ and $\mathrm{SL}(n,\mathbb{Z}[i],\mathfrak{p}^6)$, both indices being equal to 2.

• At level I^{30}, any of the three subgroups
$$[E(n,I^6),E(n,I^{10})],\quad[E(n,I^6),E(n,I^{15})],\quad[E(n,I^{10}),E(n,I^{15})]$$
is contained in the product of the other two.

13. When $[E(n,A+B),E(n,A\cap B)]=[E(n,A),E(n,B)]$?

For the ideals themselves, one has an obvious inclusion
$$(A+B)\circ(A\cap B)=(A+B)(A\cap B)+(A\cap B)(A+B)\leq AB+BA=A\circ B.$$
Only very rarely this inclusion is always an equality. In fact, it is classically known that among commutative integral domains $(A+B)(A\cap B)=AB$ characterises Prfer domains.

A Noetherian Prüfer domain is a Dedekind domain, so any Noetherian domain that is not Dedekind provides a counterexample to the equality. Let, for instance, $R=K[x,y]$, $A=xR$, $B=yR$. Then $A+B=xR+yR$, whereas $A\cap B=AB=xyR$, since R is factorial and x and y are coprime. Thus,
$$(A+B)(A\cap B)=x^2yR+xy^2R<AB.$$

The following inclusion can be verified in the style of the original proof of Lemma 7, see [37]. But it is also an immediate corollary of Lemmas 2–6.

Proposition 5. *For any two ideals $A, B \trianglelefteq R$, $n \geq 3$, one has*

$$[E(n, A+B), E(n, A\cap B)] \leq [E(n, A), E(n, B)]$$

Proof. Lemma 3 and the formula at the beginning of this section show that the level of the left hand side is contained in the level of the right hand side,

$$E\big(n, R, (A+B)\circ(A\cap B)\big) \leq E(n, R, A\circ B).$$

Thus, it only remains to prove that the elementary commutators $y_{ij}(a+b, c)$, where $a \in A$, $b \in B$, $c \in A\cap B$, in the left hand side belong to the right hand side.

By Lemma 5, one has

$$y_{ij}(a+b, c) \equiv y_{ij}(a, c)\cdot y_{ij}(b, c) \pmod{E\big(n, R, (A+B)\circ(A\cap B)\big)}.$$

Thus, this congruence holds also modulo the larger subgroup $E(n, R, A\circ B)$.

On the other hand, Lemma 4 implies that

$$y_{ij}(b, c) \equiv y_{ij}(c, -b) \pmod{E(n, R, A\circ B)}.$$

Combining the above congruences, we see that

$$y_{ij}(a+b, c) \equiv y_{ij}(a, c)\cdot y_{ij}(c, -b) \pmod{E(n, R, A\circ B)},$$

where both commutators in the right hand side belong to $[E(n, A), E(n, B)]$, which proves the desired inclusion. ☐

Of course, when A and B are comaximal, by Lemma 7 one has

$$[E(n, A+B), E(n, A\cap B)] = [E(n, A), E(n, B)].$$

Indeed, in this case $A\cap B = AB$ so that both sides are equal to $E(n, R, AB)$. This is also true in the opposite case, when $A = B$, as in all counter-examples listed in [41]. But, as we've seen, in general it may break already as regards the levels of these subgroups, since the level of the left hand side may be strictly smaller, than the level of the right hand side. Thus, the question remains

Problem 4. *When*

$$[E(n, A+B), E(n, A\cap B)] = [E(n, A), E(n, B)]?$$

In a subsequent paper we will show that this equality, and in fact much more general statements, hold for Dedekind rings.

14. Intersections of elementary subgroups

Let A and B be two ideals of a commutative ring R, $n \geq 3$. Clearly,

$$[E(n, A), E(n, B)] \leq E(n, R, A)\cap E(n, R, B)].$$

There is no obvious counter-example to the following stronger claim.

Problem 5. *Is it true that for all ideals $A, B \trianglelefteq R$, $n \geq 3$, one has*

$$[E(n, A), E(n, B)] \leq E(n, R, A\cap B)].$$

This is obviously true when $[E(n,A),E(n,B)] = E(n,R,AB)$. But in all examples where $[E(n,A),E(n,B)] > E(n,R,AB)$ we are aware of, one still has

$$[E(n,A),E(n,B)] \leq E(n,R,A\cap B).$$

In these examples usually $A = B$, when the above inclusion is obvious.

Dually to the Corollary 3 of Lemma 3 one has

$$\mathrm{GL}(n,R,A)\cap \mathrm{GL}(n,R,B) = \mathrm{GL}(n,R,A\cap B),$$

this equality is classically known, and obvious. The same holds also for congruence subgroups in $\mathrm{SL}(n,R)$.

However, a similar statement for sums of ideals is obviously false for $\mathrm{GL}(n,R,A)$, already in the case $R = \mathbb{Z}$. In other words, in general $\mathrm{GL}(n,R,A)\,\mathrm{GL}(n,R,B)$ is strictly smaller than $\mathrm{GL}(n,R,A+B)$, even for comaximal A and B. The trivial reason is that R may have more invertible elements than just those that can be expressed as products of invertible elements congruent to 1 modulo A or modulo B.

In fact, even for the easier case of $\mathrm{SL}(n,R)$ the equality

$$\mathrm{SL}(n,R,A)\,\mathrm{SL}(n,R,B) = \mathrm{SL}(n,R,A+B)$$

only holds under some very strong assumptions, such as one of the factor-rings R/A or R/B being semi-local[2], see [5], Corollary 9.3, p. 267, or [24], Theorem 2.2.

Also the corresponding property for intersections of elementary subgroups fails in general.

Proposition 6. *For two ideals $A,B\trianglelefteq R$ the group $E(n,R,A\cap B)$ can be strictly smaller than $E(n,R,A)\cap E(n,R,B)$.*

Proof. Here is the smallest such example. Let R be the ring of integers of the imaginary quadratic field $\mathbb{Q}\left(\sqrt{-7}\right)$. Then $R = \mathbb{Z}[\zeta]$, where $\zeta = \dfrac{1+i\sqrt{7}}{2}$ and $R^* = \mu(R) = \{\pm 1\}$.

Now, set $\mathfrak{p}_1 = \zeta R$, and $\mathfrak{p}_2 = \overline{\mathfrak{p}}_1 = \overline{\zeta}R$. Since $\zeta+\overline{\zeta} = 1$, the ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are coprime (= comaximal, in this case). And since $\zeta\cdot\overline{\zeta} = 2$, one has $2 = \mathfrak{p}_1\mathfrak{p}_2$ so that the prime 2 completely decomposes in R.

Recall the formula of Bass–Milnor–Serre, [6] for the exponent of the p-part of the order of $\mathrm{SK}_1(R,I)$:

$$v_p\big(|\,\mathrm{SK}_1(R,I)|\big) = \min_{\mathfrak{p}|p}\left[\frac{v_{\mathfrak{p}}(I)}{v_{\mathfrak{p}}(p)} - \frac{1}{p-1}\right]_{[0,v_p(\mu(R))]}.$$

Here the minimum is taken over all prime divisors of p in R, while $[x]_{[0,m]}$ is the closest integer in the interval $[0,m]$ to the integer part $[x]$ of x.

Now, set $A = \mathfrak{p}_1^2$, $B = \mathfrak{p}_2^2$. The ideals A and B are still comaximal, $A+B = R$. In particular, $A\cap B = AB$.

[2]This is automatically the case, for instance, for non-zero ideals in Noetherian integral domains of dimension 1. Say, for Dedekind rings.

Now, this formula implies that $\mathrm{SK}_1(R,A)=\mathrm{SK}_1(R,B)=1$, in other words,

$$E(n,R,A)=\mathrm{SL}(n,R,A),\qquad E(n,R,B)=\mathrm{SL}(n,R,B),$$

and thus

$$E(n,R,A)\cap E(n,R,B)=\mathrm{SL}(n,R,A)\cap\mathrm{SL}(n,R,B)=\mathrm{SL}(n,R,AB).$$

On the other hand, $\mathrm{SK}_1(R,A\cap B)=\mathrm{SK}_1(R,AB)=\{\pm 1\}$, so that the subgroup $E(n,R,A\cap B)=E(n,R,AB)$ has index 2 in $E(n,R,A)\cap E(n,R,B)$. □

Of course, since A and B are comaximal, by Lemma 7 one has

$$[E(n,A),E(n,B)]=E(n,R,AB)=E(n,R,A\cap B),$$

so that we do not get a counter-example to Problem 3. By the same token, there are no such counter-examples for imaginary quadratic rings. On the other hand, Lemma 9 the last equality always holds for Dedekind rings of arithmetic type with *infinite* multiplicative group, so that in this case there are no counter-examples to Problem 3 either.

15. Final remarks

It would be natural to generalise results of the present paper to more general contexts.

Problem 6. *Generalise Theorem* 1 *and other results of the present paper to Chevalley groups.*

We do not see any difficulties in treating the simply laced case. However, for doubly laced systems and for type G_2 one might get longer and fancier formulas, than those in Theorem 1.

Problem 7. *Generalise Theorems* 1 *and other results of the present paper to Bak's unitary groups.*

It was a great experience to collaborate in this field with Roozbeh Hazrat and Alexei Stepanov over the last decades. Also, we are very grateful to Pavel Kolesnikov for his questions during our talk, and to Fedor Petrov for suggesting the above proof of Theorem 6.

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