Elliptic Venttsel problems with VMO coefficients*

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Abstract

We announce new results about strong solvability of linear and quasilinear Venttsel boundary value problems with discontinuous principal coefficients.

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1. Introduction

The aim of this note is to announce very recent results [4] regarding the regularity and solvability of the Venttsel boundary value problem for second order elliptic equations with discontinuous principal coefficients. Precisely, we deal with the *linear* problem

$$\begin{cases} \mathcal{L}u := -a^{ij}(x)D_iD_ju + b^i(x)D_iu + c(x)u = f(x) & \text{a.e. in } \Omega, \\ \mathcal{B}u := -\alpha^{ij}(x)d_id_ju + \beta^i(x)D_iu + \gamma(x)u = g(x) & \text{a.e. on } \partial\Omega \end{cases} \tag{1}$$

and with the *quasilinear* one

$$\begin{cases} -a^{ij}(x,u)D_iD_ju + a(x,u,Du) = 0 & \text{a.e. in } \Omega, \\ -\alpha^{ij}(x,u)d_id_ju + \alpha(x,u,Du) = 0 & \text{a.e. on } \partial\Omega. \end{cases} \tag{2}$$

Here $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded domain with $\mathcal{C}^{1,1}$-smooth boundary $\partial\Omega$. The symbol D_i stands for the operator of weak differentiation w.r.t. x_i and $Du = (D_1u, \ldots, D_nu)$

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is the gradient of a function u. Next, d_i denotes tangential differentiation on $\partial\Omega$, i.e., $d_i = D_i - \mathbf{n}_i\mathbf{n}_j D_j$ where $\mathbf{n} = \mathbf{n}(x)$ is the unit outward normal to $\partial\Omega$ at $x \in \partial\Omega$, and $du = (d_1u, \dots, d_nu)$. The repeated indices mean summation, and we use usual notation for classical functional spaces.

We study *strong* solutions of (1) and (2) which belong to the space $V_{p,q}(\Omega)$ of all functions $u \in W^2_p(\Omega)$ with traces in $W^2_q(\partial\Omega)$, normed by

$$\|u\|_{V_{p,q}(\Omega)} = \|u\|_{W^2_p(\Omega)} + \|u\|_{W^2_q(\partial\Omega)}.$$

The history of the linear problem (1) goes back to the pioneering work [15] where, given an elliptic operator $\mathcal{L}$ in Ω, A.D. Venttsel found the most general admissible boundary conditions which restrict $\mathcal{L}$ to an infinitesimal generator of a Markov process in Ω. These conditions are given in terms of a second order integro-differential operator, and include as particular cases the Dirichlet, Neumann, and mixed boundary conditions. In a narrower sense, the Venttsel condition is given by a combination of the second order tangential differential operator $-\alpha^{ij}d_id_ju + \beta'^i d_iu$ corresponding to the diffusion with drift on $\partial\Omega$, the normal derivative term $\beta_0\partial_{\mathbf{n}}u$ describing reflection phenomena, and the absorption term γu.

The Venttsel problems arise in various fields of science and industry, e.g. in water wave theory, electromagnetic and phase-transition phenomena, elasticity theory problems, engineering problems of hydraulic fracturing, models of fluid diffusion and some climate models or non-isothermal phase separation, as well as in financial mathematics (see [5, 7, 9, 12, 13, 14]).

Linear and quasilinear Venttsel problems were deeply studied by many authors, see, e.g., the survey [3]. However, our results are completely new, being the first to treat Venttsel problems with *discontinuous* principal coefficients. Under suitable hypotheses on data, which can be viewed as optimal, we derive a $V_{p,q}(\Omega)$-a priori estimate for the strong solutions of (1), and additional assumptions lead to unique strong solvability of the problem (1). The linear results are then combined with the Leray–Schauder fixed point principle to get solvability of the quasilinear Venttsel problem (2) with discontinuous data.

Apart from the interest to the general theory of PDEs, we believe that our results could give rise to a priori error estimates (e.g. [10]) for the Finite-Element-Method implementation of Venttsel problems.

2. Main results

Dealing with the linear Venttsel problem, we suppose that the second order operators appearing in (1) are *uniformly elliptic,* that is, there exists a constant $\nu > 0$ such that

$$\nu|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \nu^{-1}|\xi|^2 \quad \text{a.e. } x \in \Omega,\ \forall\xi \in \mathbb{R}^n, \qquad a^{ij}(x) = a^{ji}(x), \tag{3}$$

$$\nu|\xi'|^2 \le \alpha^{ij}(x)\xi'_i\xi'_j \le \nu^{-1}|\xi'|^2 \quad \text{a.e. } x \in \partial\Omega,\ \forall\xi' \in \mathbb{R}^n,\ \xi' \perp \mathbf{n}, \qquad \alpha^{ij}(x) = \alpha^{ji}(x). \tag{4}$$

The principal coefficients of the operators $\mathcal{L}$ and $\mathcal{B}$ are supposed to be functions of *vanishing mean oscillation,* $a^{ij} \in VMO(\Omega)$ and $\alpha^{ij} \in VMO(\partial\Omega)$. Referring the reader to [8] for more details about these spaces, let us only mention that VMO contains as proper subsets $\mathcal{C}^0(\overline{\Omega})$, $W^1_n(\Omega)$ and the fractional Sobolev space $W^\theta_{n/\theta}(\Omega)$ with $\theta \in (0,1)$.

For the lower order coefficients, we impose minimal integrability requirements as follows

$$b^i \in L^{\max\{p,n\}}(\Omega) \ \text{ if } \ p \neq n, \qquad b^i\big(\log(1+|b^i|)\big)^{1-\frac{1}{n}} \in L^n(\Omega) \ \text{ if } \ p=n;$$
$$c \in L^{\max\{p,\frac{n}{2}\}}(\Omega) \ \text{ if } \ p \neq \frac{n}{2}, \qquad c\big(\log(1+|c|)\big)^{1-\frac{1}{n}} \in L^{\frac{n}{2}}(\Omega) \ \text{ if } \ p=\frac{n}{2}.$$

As for the lower order coefficients of the operator $\mathcal{B}$, we decompose $\beta^i(x)$ into sum of normal and tangential components, $\beta^i(x) = \beta_0(x)\mathbf{n}_i + \beta'^i$ with $\beta_0(x) = \beta^j(x)\mathbf{n}_j$ and assume that $\beta_0 \in L^q(\partial\Omega)$ if $p>n$, $\beta_0 \in L^{\frac{qp^*}{p^*-\frac{qn}{n-1}}}(\partial\Omega)$ if $p<n$, $\beta_0\,(\log(1+|\beta_0|))^{1-\frac{1}{n}} \in L^q(\partial\Omega)$ if $p=n$;

$$\beta'^i \in L^{\max\{q,n-1\}}(\partial\Omega) \ \text{ if } \ q \neq n-1, \quad \beta'^i\big(\log(1+|\beta'^i|)\big)^{1-\frac{1}{n-1}} \in L^{n-1}(\partial\Omega) \ \text{ if } \ q=n-1;$$
$$\gamma \in L^{\max\{q,\frac{n-1}{2}\}}(\partial\Omega) \ \text{ if } \ q \neq \frac{n-1}{2}, \quad \gamma\big(\log(1+|\gamma|)\big)^{1-\frac{1}{n-1}} \in L^{\frac{n-1}{2}}(\partial\Omega) \ \text{ if } \ q=\frac{n-1}{2}.$$

Theorem 1. *Let $1<p\leq \frac{nq}{n-1}<p^*$, $q\geq 2$, where p^* stands for the Sobolev conjugate of p. Under the above hypotheses on the coefficients of the operators $\mathcal{L}$ and $\mathcal{B}$, we have:*

1) *If $u \in V_{p,q}(\Omega)$ solves the Venttsel problem* (1) *with $f\in L^p(\Omega)$ and $g\in L^q(\partial\Omega)$, then*
$$\|u\|_{V_{p,q}(\Omega)} \leq C\Big(\|f\|_{p,\Omega}+\|g\|_{q,\partial\Omega}+\|u\|_{p,\Omega}+\|u\|_{q,\partial\Omega}\Big) \tag{5}$$
with a constant C depending on n, ν, p, q, $\operatorname{diam}\Omega$, the regularity of $\partial\Omega$, on the VMO-moduli of a^{ij} and α^{ij}, and on the moduli of continuity of b^i, c, β_0, β'^i, and γ in the corresponding functional spaces;
2) *Assume, in addition, that $p\geq n$, $c\geq 0$, $\beta_0\geq 0$ and $\gamma\geq\gamma_0=\mathrm{const}>0$. Then the problem* (1) *admits a unique solution $u\in V_{p,q}(\Omega)$ for each $f\in L^p(\Omega)$ and $g\in L^q(\partial\Omega)$.*

The proof of the first claim relies on a thorough analysis based on the coercive estimates from [6], appropriate interpolation procedures and L^p-bounds for suitable extension operators [2]. Further, the uniquess in the second claim is proved using the Aleksandrov–Bakel'man interior maximum principle and its local variant for the Venttsel problem [1], while the solvability of (1) is based on the Fredholm alternative and the estimate (5).

Moving to the *quasilinear* Venttsel problem (2), we suppose that the differential operators involved are uniformly elliptic and the principal coefficients $a^{ij}(x,z)$ and $\alpha^{ij}(x,z)$, being Carathéodory functions, belong to VMO w.r.t. x locally uniformly in z.

As for the lower order terms, we suppose these support quadratic gradient growths,

$$|a(x,z,p)| \leq \eta(|z|)\Big(|p|^2+b(x)|p|+\Phi(x)\Big) \quad \text{a.e. } x\in\Omega,$$
$$|\alpha(x,z,p')| \leq \eta(|z|)\Big(|p'|^2+\beta(x)|p'|+\Theta(x)\Big) \quad \text{a.e. } x\in\partial\Omega,$$

for all $(z,p,p')\in\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, $p'\perp\mathbf{n}$. Here $\eta\in\mathcal{C}^0(\mathbb{R}_+)$ is a non-decreasing function, and

$$b\,(\log(1+|b|))^{1-\frac{1}{n}} \in L^n(\Omega), \qquad \Phi\in L^n(\Omega);$$
$$\beta\,(\log(1+|\beta|))^{1-\frac{1}{n-1}} \in L^{n-1}(\partial\Omega), \qquad \Theta\in L^{n-1}(\partial\Omega).$$

We assume moreover that the lower order term in the Venttsel condition in (2) is "exterior" to $\partial\Omega$, that is, $\alpha(x,z,p)$ is weakly differentiable with respect to p and

$$0 \le \alpha_{p_i}(x,z,p)\mathbf{n}_i \le \eta(|z|)\beta_0(x),$$

with $\beta_0 \left(\log\left(1+|\beta_0|\right)\right)^{1-\frac{1}{n}} \in L^{n-1}(\partial\Omega)$ and η as above.

Finally, suppose that there is a $z_0 > 0$ such that for $|z| \ge z_0$ the functions $a(x,z,p)$ and $\alpha(x,z,p')$ with $p' \perp \mathbf{n}$ are weakly differentiable with respect to z, and

$$a_z(x,z,p) \ge \theta_0 \Phi(x); \qquad \alpha_z(x,z,p') \ge \theta_0 \max\left\{1, \Theta(x)\right\}, \quad \theta_0 = \text{const} > 0. \tag{6}$$

Theorem 2. *Under the above assumptions, the quasilinear Venttsel problem* (2) *admits at least one solution in the class* $V_{n,n-1}(\Omega)$.

The proof relies on the Leray–Schauder fixed point theorem that reduces the solvability of (2) to the establishment of suitable a priori estimates for all solutions to a family of Venttsel problems (see [4]). We only point out that (6) ensures an L^∞-estimate for these solutions; their uniform Hölder boundedness follows from [1]; while the estimates from Theorem 1, together with [11, Lemma 2.1] and the hypotheses guarantee the gradient a priori bound.

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