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LINEAR TWO-DIMENSIONAL MODELS OF ANISOTROPIC PLATES IN HIGH APPROXIMATIONS

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Abstract. *Thin elastic anisotropic plates heterogeneous in the thickness direction are studied. Delivering of 2D approximate models of plates and shells is one of the classical problems of mechanics. In the present study, 2D models are delivered by using asymptotic expansions of solutions of 3D equations of the theory of elasticity in power series of a small thickness parameter. The zero asymptotic approximation gives equations that coincide with the Kirchhof–Love model if we use in-plane elasticity relations. If the elastic moduli are of identical order, then this model gives results acceptable for applications. In the opposite case, it is desirable to construct higher approximations. For particular types of anisotropy, 2D models of the second-order of accuracy were constructed earlier, and here we present a second-order model for a plate made of a heterogeneous material with general anisotropy described by 21 elastic moduli. The structure of the model and the differential orders of PDE is the same as for the Timoshenko–Reissner model, but equations are significantly more cumbersome. For a plate infinite in the tangential directions, there exists a simple harmonic solution. This solution allows us to estimate an error of 2D models compared with the exact solution of 3D equations of the theory of elasticity. The model is used to solve static and vibration problems. Some numerical examples are considered.*

Keywords: anisotropic heterogeneous plate, two-dimensional model, second-order accuracy, bending, vibrations, waves propagation.

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1 INTRODUCTION

Derivation of two-dimensional approximate models of thin plates and shells is one of classical problems of mechanics [1–3]. Numerous investigations were devoted to delivering of 2D approximate models of thin plates and shells made of anisotropic materials (we mention the books [4–6], containing an extensive bibliography). Here (as in the previous studies [5, 7–9], etc.) we use asymptotic expansions of solutions of 3D equations of the theory of elasticity in power series of a small thickness parameter. The zero approximation coincides with the classical Kirchhof–Love (KL) model based on the kinematic hypotheses [1, 2, 10]. For thin plates, this approximation gives sufficiently exact results for isotropic homogeneous plates and as well for anisotropic heterogeneous plate if the elastic moduli of the latter have identical orders in the thickness direction. The opposite case calls for consideration of higher approximations. For plates which are transversely isotropic and strongly heterogeneous in the thickness direction, the 2D equations of the second order accuracy (SOA) were derived and investigated in [11–16]. This model is useful for multi-layered plates with the alternating hard and soft layers, because the classical KL and Timoshenko–Reissner (TR) models lead to the large errors. For these multi-layered plates the generalized TR model with an equivalent transversal shear modulus is constructed [13, 14]. An analysis of multi-layered orthotropic plates with the arbitrary orientation of the main directions of orthotropy is reduced to investigation of plates heterogeneous in the thickness direction monoclinic. An asymptotic analysis of monoclinic plates was performed in [17], and in [18], 2D equations of SOA were derived.

In the case of general anisotropy (with 21 elastic moduli), the classical KL and TR models are also unacceptable. In [19], for anisotropic plates, and in [20] for shells, the generalized TR model was proposed. In [21], based on the asymptotic expansions (as elaborated in [12–14]), a 2D model for a multi-layered plate with the general anisotropy was proposed. This model leads to equations of 8th differential order and it is similar to the KL model with equivalent elastic moduli. This model is of zero order of accuracy and it is not acceptable for a multi-layered plate with alternating hard and soft layers. In the present stud, we consider an anisotropic heterogeneous plate of general anisotropy (with 21 elastic moduli) and construct higher asymptotic approximations. For general anisotropy, the asymptotic solution is essentially more difficult and bulky as compared with that for monoclinic material. To construct a model of SOA for a material with general anisotropy, it is necessary to built asymptotic solutions of the zero, the first, and the second approximations, while for a monoclinic material the first approximation is absent. As a result, a PDE system with constant coefficients is obtained. The differential order of this system is the same as the order of the TR model, but this system is essentially more cumbersome. For a plate heterogeneous in the thickness direction, to find the coefficients it is necessary to calculate the repeated integrals of elastic moduli. Closed solutions of boundary value problems for a finite plate (for example, for a rectangular plate) do not exist, because a separation of variables is impossible. A plate infinite in the tangential directions was considered. In this case, the 3D problem is reduced to a 1D boundary value problem (in the thickness direction) that admits a simple numerical solution, and the 2D problem leads to a linear algebraic system. A comparison of these 3D and 2D problems allows one to estimate an error of the approximate 2D model. Here, this comparison is performed for some problems of bending and vibrations of anisotropic multi-layered plates. It is shown that the SOA model is acceptable for a very large level of heterogeneity of a plate cross-section.

2 THE EQUILIBRIUM EQUATIONS AND ELASTICITY RELATIONS

Consider a thin elastic plate of constant thickness h. In the Cartesian co-ordinate system $x_1, x_2, x3$, the equilibrium equations read as

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0, \quad i = 1,2,3, \qquad -\frac{h}{2} \le x_3 = z \le \frac{h}{2}, \tag{2.1}$$

where σ_{ij} are the stresses and f_i are the intensities of external forces.

Following [3], we present the stresses σ_{ij} and the strains ε_{ij} as 6D vectors, and write the elasticity relations in the matrix form:

$$\begin{gathered} \sigma = \mathbf{E}\cdot\varepsilon, \quad \mathbf{E} = (E_{ij})_{i,j=1,\dots,6}\,, \\ \sigma = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12})^T, \quad \varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{13}, \varepsilon_{12})^T, \\ \varepsilon_{ii} = \frac{\partial u_i}{\partial x_i}, \quad \varepsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \quad i \ne j, \quad i,j = 1,2,3. \end{gathered} \tag{2.2}$$

Here and in what follows, ^{T} denotes transposition, while bold letters are used for vectors, for matrices and for operators, the product of vectors and matrices is denoted by the dot sign $(\cdot)$.

The matrix $\mathbf{E}$ of elastic moduli is symmetric and positively definite. In the considered case of a general anisotropy, it contains 21 independent elastic moduli. It is assumed that the elastic moduli E_{ij} are independent of the tangential co-ordinates x_1, x_2, but they may depend on $x_3 = z$. A dependence on z takes place for functionally graded plates, as well as for multilayered plates for which the moduli E_{ij} are piecewise constant functions of z.

Following [14], for an asymptotic analysis we divide the stresses σ_{ij} and the strains ε_{ij} in groups of tangential $\sigma_t,\ \varepsilon_t$ and transversal $\sigma_n,\ \varepsilon_n$ stresses and strains with various asymptotic behavior and put

$$\begin{aligned} \sigma_t &= (\sigma_{11},\, \sigma_{22},\, \sigma_{12})^T, & \sigma_n &= (\sigma_{13},\, \sigma_{23},\, \sigma_{33})^T, \\ \varepsilon_t &= (\varepsilon_{11},\, \varepsilon_{22},\, \varepsilon_{12})^T, & \varepsilon_n &= (\varepsilon_{13},\, \varepsilon_{23},\, \varepsilon_{33})^T, \end{aligned} \tag{2.3}$$

where

$$\begin{gathered} \mathbf{A} = \{A_{ij}\} = \begin{pmatrix} E_{11} & E_{12} & E_{16} \\ E_{12} & E_{22} & E_{26} \\ E_{16} & E_{26} & E_{66} \end{pmatrix}, \quad \mathbf{B} = \{B_{ij}\} = \begin{pmatrix} E_{15} & E_{25} & E_{56} \\ E_{14} & E_{24} & E_{46} \\ E_{13} & E_{23} & E_{36} \end{pmatrix}, \\ \mathbf{C} = \{C_{ij}\} = \begin{pmatrix} E_{55} & E_{45} & E_{35} \\ E_{45} & E_{44} & E_{34} \\ E_{35} & E_{34} & E_{33} \end{pmatrix}. \end{gathered} \tag{2.4}$$

Then the elasticity relations (2.2) assume the form

$$\sigma_t = \mathbf{A}\cdot\varepsilon_t + \mathbf{B}\cdot\varepsilon_n, \quad \sigma_n = \mathbf{B}^T\cdot\varepsilon_t + \mathbf{C}\cdot\varepsilon_n, \tag{2.5}$$

Excluding the small transversal strains ε_n, we obtain

$$\sigma_t = \mathbf{A}^*\cdot\varepsilon_t + \mathbf{B}\cdot\mathbf{C}^{-1}\cdot\sigma_n, \qquad \varepsilon_n = \mathbf{C}^{-1}\cdot\sigma_n - \mathbf{C}^{-1}\cdot\mathbf{B}^T\cdot\varepsilon_t \tag{2.6}$$

where

$$\mathbf{A}^* = \mathbf{A} - \mathbf{B}\cdot\mathbf{C}^{-1}\cdot\mathbf{B}^T. \tag{2.7}$$

Taking into account that $|\sigma_n| \ll |\sigma_t|$, we obtain the approximate elasticity relations

$$\sigma_t = \mathbf{A}^* \cdot \varepsilon_t, \tag{2.8}$$

that relate the tangential strains and stresses. The matrix $\mathbf{A}^*$ of the in-plane elastic moduli plays an important role in the following approximations.

We next introduce dimensionless variables (with the sign^) as follows:

$$\begin{aligned}&\{x_1, x_2, u_1, u_2, u_3\} = l\{\hat{x}_1, \hat{x}_2, \hat{u}_1, \hat{u}_2, w\}, \quad z = l\hat{z}, \quad \mu = h/l,\\ &\{A_{ij}, B_{ij}, C_{ij}, \sigma_{ij}\} = E\{\hat{A}_{ij}, \hat{B}_{ij}, \hat{C}_{ij}, \hat{\sigma}_{ij}\}, \quad \{f_i\} = (E/l)\{\hat{f}_i\}, \quad i,j = 1,2,3,\end{aligned} \tag{2.9}$$

here l is the characteristic length of waves in tangential directions, E is the characteristic value of elastic moduli, μ is the small parameter. In what follows, the hat-sign^ will be omitted. As a result, we get a system of 6th order equations with small parameter μ,

$$\begin{aligned}&\frac{\partial w}{\partial z} = \mu\varepsilon_{33},\\ &\frac{\partial u_i}{\partial z} = -\mu(p_i w - \varepsilon_{i3}), \quad i = 1,2,\\ &\frac{\partial \sigma_{i3}}{\partial z} = -\mu(p_1\sigma_{1i} + p_2\sigma_{2i} + f_i), \quad i = 1,2,\\ &\frac{\partial \sigma_{33}}{\partial z} = -\mu(p_1\sigma_{13} + p_2\sigma_{23} + f_3),\end{aligned} \tag{2.10}$$

where $p_1 = \partial(\,)/\partial x_1,\ p_2 = \partial(\,)/\partial x_2$.

3 TRANSFORMATION OF SYSTEM (2.10)

To represent Eqs. (2.10) in the vector form, we introduce the vectors

$$\mathbf{u} = (u_1, u_2)^T, \quad \sigma_s = (\sigma_{13}, \sigma_{23})^T, \quad \varepsilon_s = (\varepsilon_{13}, \varepsilon_{23})^T, \quad \mathbf{f}_t = (f_1, f_2)^T, \tag{3.1}$$

the differential operators

$$\mathbf{p} = (p_1, p_2)^T, \qquad \mathbf{P} = \begin{pmatrix} p_1 & 0 & p_2 \\ 0 & p_2 & p_1 \end{pmatrix}^T, \tag{3.2}$$

and the integral operators

$$\mathbf{I}_a(Z) \equiv \int_{-1/2}^{1/2} Z dz, \quad \mathbf{I}(Z) \equiv \int_{-1/2}^{z} Z(z)\, dz, \quad \mathbf{I}_0(Z) \equiv \int_0^z Z(z)\, dz. \tag{3.3}$$

Now Eqs. (2.10) in the vector form read as

$$\begin{aligned}&\frac{\partial w}{\partial z} = \mu\varepsilon_{33}, &&\frac{\partial \sigma_s}{\partial z} = -\mu(\mathbf{P}^T \cdot \sigma_t + \mathbf{f}_t)\\ &\frac{\partial \mathbf{u}}{\partial z} = -\mu(\mathbf{p}\, w - \varepsilon_s), &&\frac{\partial \sigma_{33}}{\partial z} = -\mu(\mathbf{p}^T \cdot \sigma_s + f_3),\end{aligned} \tag{3.4}$$

Here the main unknowns are $w, \mathbf{u}, bs\sigma_s, \sigma_{33}$. The remaining unknowns can be expressed as

$$\begin{aligned}&\varepsilon_n = (\varepsilon_s^T, \varepsilon_{33})^T = \mathbf{C}^{-1} \cdot (\sigma_n - \mathbf{B}^T \varepsilon_t), &&\varepsilon_t = \mathbf{P} \cdot \mathbf{u},\\ &\sigma_t = \mathbf{A}^* \cdot \mathbf{P} \cdot \mathbf{u} + \mathbf{B} \cdot \mathbf{C}^{-1} \cdot \sigma_n, &&\sigma_n = (\sigma_s^T, \sigma_{33})^T.\end{aligned} \tag{3.5}$$

The orders of the functions σ_s and σ_{33} are different, and so we rewrite Eqs. (3.5) with the help of block-structure of the matrices $\mathbf{C}^{-1} = \{G_{ij}\}$ and $\mathbf{C}^{-1}\cdot\mathbf{B}^T = \{S_{ij}\}$ as follows:

$$\begin{aligned}
&\mathbf{C}^{-1} = \begin{pmatrix} \mathbf{G} & \mathbf{g} \\ \mathbf{g}^T & c_3 \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} G_{11} & G_{12} \\ G_{12} & G_{22} \end{pmatrix}, \quad \mathbf{g} = (G_{13}, G_{23})^T, \quad c_3 = G_{33}, \\
&\mathbf{C}^{-1}\cdot\mathbf{B}^T = \begin{pmatrix} \mathbf{S} \\ \mathbf{s} \end{pmatrix}, \qquad \mathbf{S} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \end{pmatrix}, \quad \mathbf{s} = (S_{31}, S_{32}, S_{33}).
\end{aligned} \tag{3.6}$$

Now Eqs. (3.5) read as

$$\begin{aligned}
\sigma_t &= \mathbf{A}^*\cdot\mathbf{P}\cdot\mathbf{u} + \mathbf{S}^T\cdot\sigma_s + \mathbf{s}^T\sigma_{33}, \\
\varepsilon_s &= \mathbf{G}\cdot\sigma_s + \mathbf{g}\,\sigma_{33} - \mathbf{S}\cdot\mathbf{P}\cdot\mathbf{u}, \\
\varepsilon_{33} &= \mathbf{g}^T\cdot\sigma_s + c_3\,\sigma_{33} - \mathbf{s}\cdot\mathbf{P}\cdot\mathbf{u}.
\end{aligned} \tag{3.7}$$

As a result, the right-hand sides of Eqs. (3.4) can be expressed in terms of the main unknowns.

The right-hand sides of Eqs. (3.4) are small, and so to solve this system we use the method of iterations. For majority of bending and tangential deformations the iterations are different, and we consider here the bending deformations. For convenience, we change the scale of unknown functions according with their orders and put

$$\mathbf{u} = \mu\hat{\mathbf{u}}, \quad \sigma_s = \mu^2\hat{\sigma}_s, \quad \sigma_{33} = \mu^3\hat{\sigma}_{33}, \quad \mathbf{f}_t = \mu\hat{\mathbf{f}}_t, \quad f_3 = \mu^2\hat{f}_3. \tag{3.8}$$

Again the sign$\,\hat{}\,$ is omitted.

Supposing that the planes $z = \pm 1/2$ are free, we get the boundary conditions

$$\sigma_s = \sigma_{33} = 0, \qquad z = \pm 1/2. \tag{3.9}$$

The external surface forces can be included in the body forces by using the Dirac delta-function.

Now in the notation of (3.1)–(3.3), (3.6), and (3.8), Eqs. (3.4) can be written as a system of integral equations

$$\begin{aligned}
w &= w_0 - \mu^2\mathbf{I}_0(\mathbf{s}_P\cdot\mathbf{u})) + \mu^3\mathbf{I}_0(\mathbf{g}^T\cdot\sigma_s) + \mu^4\mathbf{I}_0(c_3\,\sigma_{33}), \\
\mathbf{u} &= \mathbf{u}_0 - \mathbf{I}_0(\mathbf{p}\,w) - \mu\mathbf{I}_0(\mathbf{S}_P\cdot\mathbf{u}) + \mu^2\mathbf{I}_0(\mathbf{G}\cdot\sigma_s) + \mu^3\mathbf{I}_0(\mathbf{g}\,\sigma_{33}), \\
\sigma_s &= -\mathbf{I}(\mathbf{L}_0\cdot\mathbf{u}) - \mu\mathbf{I}(\mathbf{S}_P^T\cdot\sigma_s) - \mu^2\mathbf{I}(\mathbf{s}_P^T\sigma_{33}) - \mathbf{I}(\mathbf{f}_t), \\
\sigma_{33} &= -\mathbf{I}(\mathbf{p}^T\cdot\sigma_s) - \mathbf{I}(f_3),
\end{aligned} \tag{3.10}$$

where we use the following notation

$$\mathbf{L}_0(z) = \mathbf{P}^T\cdot\mathbf{A}^*(z)\cdot\mathbf{P}, \quad \mathbf{S}_P(z) = \mathbf{S}(z)\cdot\mathbf{P}, \quad \mathbf{s}_P(z) = \mathbf{s}(z)\cdot\mathbf{P}. \tag{3.11}$$

In Eqs. (3.10), the functions $w_0(x_1, x_2)$, $\mathbf{u}_0(x_1, x_2)$ are arbitrary functions. The functions σ_s, σ_{33} due Eqs. (3.3) satisfy the boundary conditions (3.9) at $z = -1/2$, and the functions w_0, $\mathbf{u}_0$ are to be found from conditions (3.9) at $z = 1/2$.

All the unknowns in Eqs. (3.10) are of the order of unity. System (3.10) is exact and it is convenient to construct a solution by using the method of iterations. In the next sections, the solution of the SOA with respect to the small thickness parameter μ will be constructed. Therefore, for simplicity the small summands of the orders of μ^3 and of μ^4 can be omitted in the first two Eqs. (3.10).

4 THE ZERO APPROXIMATION

All the unknowns in Eqs. (3.10) will be sought in the form

$$Z(x_1,x_2,z,\mu)=Z^{(0)}(x_1,x_2,z)+\mu Z^{(1)}(x_1,x_2,z)+\mu Z^{(2)}(x_1,x_2,z),\quad Z=\{w,\mathbf{u},\sigma_s,\sigma_{33}\}. \tag{4.1}$$

The zero approximation was obtained in [21], and here we give shortly the results in our notation

$$\begin{aligned}
&w^{(0)}(x_1,x_2,z)=w_0(x_1,x_2),\\
&\mathbf{u}^{(0)}(x_1,x_2,z)=\mathbf{u}_0(x_1,x_2)-z\,\mathbf{p}\,w_0(x_1,x_2)),\\
&\sigma_s^{(0)}(x_1,x_2,z)=-\mathbf{I}(\mathbf{L}_0(z)\cdot\mathbf{u}^{(0)})-\mathbf{I}(\mathbf{f}_t),\\
&\sigma_{33}^{(0)}(x_1,x_2,z)=\mathbf{II}(\mathbf{p}^T\cdot\mathbf{L}_0(z)\cdot\mathbf{u}^{(0)})+\mathbf{I}(\mathbf{p}^T\cdot\mathbf{f}_t)-\mathbf{I}(f_3).
\end{aligned} \tag{4.2}$$

The boundary conditions (3.3) at $z=1/2$ lead to a system of equations for w_0 and $\mathbf{u}_0$ in the zero approximation:

$$\begin{aligned}
&\mathbf{L}_0\cdot\mathbf{u}_0-\mathbf{N}_1 w_0+\mathbf{F}_t=\mathbf{0},\\
&\mathbf{N}_1^T\cdot\mathbf{u}_0-Q_2 w_0+m+F_3=0,
\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
&\mathbf{L}_0=\mathbf{P}^T\cdot\mathbf{I}_a(\mathbf{A}^*(z))\cdot\mathbf{P},\quad \mathbf{N}_0=\mathbf{P}^T\cdot\mathbf{I}_a(z\,\mathbf{A}^*(z))\cdot\mathbf{P}\cdot\mathbf{p},\quad \mathbf{F}_t=\mathbf{I}_a(\mathbf{f}_t),\\
&Q_0=\mathbf{p}^T\cdot\mathbf{P}^T\cdot\mathbf{I}_a(z^2\,\mathbf{A}^*(z))\cdot\mathbf{P}\cdot\mathbf{p},\quad F_3=\mathbf{I}_a(f_3),\quad m=\mathbf{I}_a(\mathbf{p}^T\cdot\mathbf{f}_t).
\end{aligned} \tag{4.4}$$

The detailed expressions of operators in Eqs. (4.3) read as [21]

$$\begin{aligned}
&\mathbf{L}_0=\begin{pmatrix} L_{11} & L_{12}\\ L_{12} & L_{22}\end{pmatrix},\qquad
\begin{aligned}
L_{11}&=a_{11}^{(0)}p_1^2+2a_{13}^{(0)}p_1p_2+a_{33}^{(0)}p_2^2,\\
L_{12}&=a_{13}^{(0)}p_1^2+(a_{12}^{(0)}+a_{33}^{(0)})p_1p_2+a_{23}^{(0)}p_2^2,\\
L_{22}&=a_{33}^{(0)}p_1^2+2a_{23}^{(0)}p_1p_2+a_{22}^{(0)}p_2^2,
\end{aligned}\\
&\mathbf{N}_0=\begin{pmatrix} N_1\\ N_2\end{pmatrix},\qquad
\begin{aligned}
N_1&=a_{11}^{(1)}p_1^3+3a_{13}^{(1)}p_1^2p_2+(a_{12}^{(1)}+2a_{33}^{(1)})p_1p_2^2+a_{23}^{(1)}p_2^3,\\
N_2&=a_{13}^{(1)}p_1^3+(a_{12}^{(1)}+2a_{33}^{(1)})p_1^2p_2+3a_{23}^{(1)}p_1p_2^2+a_{22}^{(1)}p_2^3,
\end{aligned}\\
&Q_0=a_{11}^{(2)}p_1^4+a_{13}^{(2)}p_1^3p_2+2(a_{12}^{(2)}+2a_{33}^{(2)})p_1^2p_2^2+4a_{23}^{(2)}p_1p_2^3+a_{22}^{(2)}p_2^4,\\
&a_{ij}^{(k)}=\mathbf{I}(z^kA_{ij}^*(z))=\textstyle\int_{-1/2}^{1/2}z^kA_{ij}^*(z)\,dz,\quad i,j=1,2,3,\ \ k=0,1,2.
\end{aligned} \tag{4.5}$$

Equations (4.3) describes approximately the bending deformations of an anisotropic plate in the frames of the KL hypotheses. In [17] it was shown that in the case when all the elastic moduli are of the same order, the zero approximation gives an acceptable exactness for approximate calculations. If some elements of the matrix $\mathbf{C}$ are small, then according to Eqs. (3.6) the corresponding summands that were not included in the zero approximation are large, and the exactness of the zero approximation is not enough. The main effect that is not described by the zero approximation is the transversal shear, which is essential for multi-layered plates with hard and soft alternating layers.

5 THE FIRST APPROXIMATION

The first approximation takes into account the summands of the order of μ in Eqs. (3.10). If $\mathbf{S}=\mathbf{0}$, then the summands of the order of μ in Eqs. (3.10) are absent. For monoclinic materials (in particular, for isotropic, for tranversely isotropic, and for orthotropic materials),

$\mathbf{S} = \mathbf{0}$. Materials with $\mathbf{S} \neq \mathbf{0}$ are used very rarely. We shall call such an anisotropy an inclined anisotropy, because it can be obtained with a composite plate consisting of an orthotropic matrix reinforced by a system of fibres inclined to a plane of plate [22].

For an inclined anisotropy, the first approximation reads as

$$\begin{aligned}
&w^{(1)} = w_0,\\
&\mathbf{u}^{(1)} = \mathbf{u}_0 - \mathbf{I}_0(\mathbf{p}\, w^{(1)}) - \mu \mathbf{I}_0(\mathbf{S}_P\cdot\mathbf{u}^{(0)}) = \mathbf{u}^{(0)} - \mu\mathbf{I}_0(\mathbf{S}_P\cdot\mathbf{u}^{(0)}),\\
&\sigma_s^{(1)} = -\mathbf{I}(\mathbf{L}_0\cdot\mathbf{u}^{(1)}) - \mu\mathbf{I}(\mathbf{S}_P^T\cdot\sigma_s^{(0)}) - \mathbf{I}(\mathbf{f}_t) =\\
&\quad = -\mathbf{I}(\mathbf{L}_0\cdot\mathbf{u}^{(0)}) + \mu\mathbf{I}(\mathbf{L}_0\cdot\mathbf{I}_0(\mathbf{S}_P\cdot\mathbf{u}^{(0)})) + \mu\mathbf{I}(\mathbf{S}_P^T\cdot\mathbf{I}(\mathbf{L}_0\cdot\mathbf{u}^{(0)} + \mathbf{f}_t)) - \mathbf{I}(\mathbf{f}_t),\\
&\sigma_{33}^{(1)} = -\mathbf{I}(\mathbf{p}^T\cdot\sigma_s^{(1)}) - \mathbf{I}(f_3) =\\
&\quad = \mathbf{II}(\mathbf{p}^T\cdot\mathbf{L}_0\cdot\mathbf{u}^{(0)}) - \mu\mathbf{II}\left(\mathbf{p}\cdot\mathbf{L}_0\cdot\mathbf{I}_0(\mathbf{S}_P\cdot\mathbf{u}^{(0)}) + \mathbf{p}\cdot\mathbf{S}_P^T\cdot\mathbf{I}(\mathbf{L}_0\cdot\mathbf{u}^{(0)})\right) +\\
&\quad + \mathbf{II}(\mathbf{p}^T\cdot\mathbf{f}_t) - \mathbf{I}(f_3).
\end{aligned} \tag{5.1}$$

Here the normal deflection $w^{(1)}$ is the same as in the zero approximation, and the tangential deflections $\mathbf{u}^{(1)}$ and the stresses $\sigma_s^{(1)}, \sigma_{33}^{(1)}$ are changed.

The equations for w_0 and $\mathbf{u}_0$ are not given here, because they follow from the equations (6.*) of second approximation if we omit the summands with μ^2.

6 THE SECOND APPROXIMATION

The second approximation reads as

$$\begin{aligned}
&w^{(2)} = w_0 + \mu^2\mathbf{I}_0(\sigma_{33}^{(0)}) = w_0 - \mu^2\mathbf{I}_0(\mathbf{s}_P\cdot\mathbf{u}^{(0)})),\\
&\mathbf{u}^{(2)} = \mathbf{u}_0 - \mathbf{I}_0(\mathbf{p}\, w^{(2)}) - \mu\mathbf{I}_0(\mathbf{S}_P\cdot\mathbf{u}^{(1)}) + \mu^2\mathbf{I}_0(\mathbf{G}\cdot\sigma_s^{(0)}) =\\
&\quad = \mathbf{u}^{(0)} - \mu\mathbf{I}_0(\mathbf{S}_P\cdot\mathbf{u}^{(0)}) + \mu^2\mathbf{I}_0\mathbf{I}_0(\mathbf{p}\cdot\mathbf{s}_P\cdot\mathbf{u}^{(0)}) +\\
&\quad + \mu^2\mathbf{I}_0(\mathbf{S}_P\cdot\mathbf{I}_0(\mathbf{S}_P\cdot\mathbf{u}^{(0)})) - \mu^2\mathbf{I}_0(\mathbf{G}\cdot\mathbf{I}(\mathbf{L}_0\cdot\mathbf{u}^{(0)})) - \mu^2\mathbf{I}_0(\mathbf{G}\cdot\mathbf{I}(\mathbf{f}_t)),\\
&\sigma_s^{(2)} = -\mathbf{I}(\mathbf{L}_0\cdot\mathbf{u}^{(2)}) - \mu\mathbf{I}(\mathbf{S}_P^T\cdot\sigma_s^{(1)}) - \mu^2\mathbf{I}(\mathbf{s}_P^T\sigma_{33}^{(0)}) - \mathbf{I}(\mathbf{f}_t),\\
&\sigma_{33}^{(2)} = -\mathbf{I}(\mathbf{p}^T\cdot\sigma_s^{(2)}) - \mathbf{I}(f_3).
\end{aligned} \tag{6.1}$$

Here it is necessary to substitute the values $\mathbf{u}^{(1)}$ and $\sigma_s^{(1)}$ from Eq. (5.1) and the values $\mathbf{u}^{(2)}$ and $\sigma_s^{(2)}$ from Eqs. (6.1). As a result, we can write $\sigma_s^{(2)}$ in the form

$$\sigma_s^{(2)} = -\mathbf{I}(\mathbf{L}^*\cdot\mathbf{u}^{(0)} + \mathbf{f}^*), \quad \mathbf{L}^* = \mathbf{L}_0 + \mu\mathbf{L}_1 + \mu^2\mathbf{L}_2, \tag{6.2}$$

with

$$\begin{aligned}
&\mathbf{L}_0 = \mathbf{P}^T\cdot\mathbf{A}^*(z)\cdot\mathbf{P}, \qquad \mathbf{L}_1 = -\mathbf{L}_0\cdot\mathbf{I}_0(\mathbf{S}_P) - \mathbf{S}_P^T\cdot\mathbf{I}(\mathbf{L}_0),\\
&\mathbf{L}_2 = \mathbf{L}_0\cdot\mathbf{I}_0\mathbf{I}_0(\mathbf{p}\cdot\mathbf{s}_P) + \mathbf{L}_0\cdot\mathbf{I}_0(\mathbf{S}_P\cdot\mathbf{I}_0(\mathbf{S}_P)) - \mathbf{L}_0\cdot\mathbf{I}_0(\mathbf{G}\cdot\mathbf{I}(\mathbf{L}_0)) +\\
&\quad + \mathbf{S}_P^T\cdot\mathbf{I}(\mathbf{L}_0\cdot\mathbf{I}_0(\mathbf{S}_P)) + \mathbf{S}_P^T\cdot\mathbf{I}(\mathbf{S}_P^T\cdot\mathbf{I}(\mathbf{L}_0)) + \mathbf{s}_P^T\cdot\mathbf{II}(\mathbf{p}^T\cdot\mathbf{L}_0)],\\
&\mathbf{f}^* = \mathbf{f}_t - \mu\mathbf{S}_P^T\cdot\mathbf{I}(\mathbf{f}_t) - \mu^2[\mathbf{L}_0\cdot\mathbf{I}_0(\mathbf{G}\cdot\mathbf{I}(\mathbf{f}_t)) + \mathbf{s}_P^T\cdot\mathbf{p}^T\cdot\mathbf{I}(\mathbf{f}_t) - \mathbf{s}_P^T\,\mathbf{I}(f_3)].
\end{aligned} \tag{6.3}$$

Using the forth equation in (6.1), we find that

$$\sigma_{33}^{(2)} = \mathbf{II}(\mathbf{L}^*\cdot\mathbf{u}^{(0)} + \mathbf{f}^*) - \mathbf{I}(f_3). \tag{6.4}$$

The boundary conditions $\sigma_s^{(2)} = \sigma_{33}^{(2)} = 0$ at $z = 1/2$ lead to equations for $\mathbf{u}_0$ and w_0,

$$\begin{aligned} &\mathbf{I}_a(\mathbf{L}^*\cdot\mathbf{u}^{(0)} + \mathbf{f}^*) = 0, \qquad \mathbf{u}^{(0)} = \mathbf{u}_0 - \mathbf{p}\, z\, w_0 \\ &\mathbf{I}_a\mathbf{I}(\mathbf{p}^T\cdot(\mathbf{L}^*\cdot\mathbf{u}^{(0)} + \mathbf{f}^*)) - \mathbf{I}_a(f_3) = 0. \end{aligned} \tag{6.5}$$

Employing the equality $\mathbf{I}_a\mathbf{I}(Z(z)) = (1/2)\mathbf{I}_a(Z(z)) - \mathbf{I}_a(z\,Z(z))$ at $Z = \sigma^*\cdot\mathbf{u}^{(0)} + \mathbf{f}^*$ with $\mathbf{I}_a(Z) = 0$, we can write Eqs. (6.5) in the final form

$$\begin{aligned} &\mathbf{I}_a(\mathbf{L}^*)\cdot\mathbf{u}_0 - \mathbf{I}_a(\mathbf{L}^*\cdot\mathbf{p}\, z)\, w_0 + \mathbf{I}_a(\mathbf{f}^*) = 0, \qquad \mathbf{I}_a(Z(z)) = \textstyle\int_{-1/2}^{1/2} Z(z)dz, \\ &\mathbf{I}_a(z\,\mathbf{p}^T\cdot\mathbf{L}^*)\cdot\mathbf{u}_0 - \mathbf{I}_a(z\,\mathbf{p}^T\cdot\mathbf{L}^*\cdot\mathbf{p}\, z)\, w_0 + \mathbf{I}_a(z\,\mathbf{p}^T\cdot\mathbf{f}^*) + F_3 = 0. \end{aligned} \tag{6.6}$$

Therefore, the model of the SOA is built. As one can see, the coefficients of Eqs. (6.6) are expressed through a very bulky operator $\mathbf{L}^*$ depending on the integrals of elastic moduli. Equations(6.6) form a PDE system with constant coefficients with respect to the unknown functions $\mathbf{u}_0 = (u_{10}, u_{20}),\ w_0$. The differential order of system (6.6) is the same as the order of the TR model, but the system (6.6) is essentially more complex.

Let us discuss possible simplifications of Eqs. (6.6).

For a monoclinic material, $\mathbf{S} = \mathbf{0}$, and all summands of the order of μ are equal to zero.

For a plate symmetric in the thickness direction (namely, for a plate with even or constant in z elastic moduli), in the zero approximations the tangential and the transversal deflections can be investigated separately. In higher approximations, these deflections can be tied by small terms. For an inclined anisotropy, this relation is of the order of μ and for a monoclinic material, it is of the order of μ^2.

The second-order operator $\mathbf{L}_2$ in Eqs. (6.2) and (6.3) can be simplified if we approximately put $\mathbf{L}_2 = -\mathbf{L}_0\cdot\mathbf{I}_0(\mathbf{G}\cdot\mathbf{I}(\mathbf{L}_0)$, because this term is large for a plate with large transversal shear compliance. The rest 5 summands in $\mathbf{L}_2$ are not large as a rule.

7 HARMONIC SOLUTION

The main difficulty in obtaining a closed-form solution of Eqs. (6.6) for a finite plate (for example, for a rectangular plate) consists in satisfying the boundary conditions, which for the general anisotropy do not admit the separation of variables. Further we consider an infinite plate and investigate the harmonic solutions admitting closed-form solutions.

We consider a static harmonic problem. Let the external forces be

$$\{\mathbf{f}_t,\ f_3\}(x_1, x_2, z) = \{\mathbf{f}_t(z),\ f_3(z)\}e^{iY}, \quad Y = x_1q_1 + x_2q_2, \quad i = \sqrt{-1}, \tag{7.1}$$

where $q_1,\ q_2$ are the real-valued wave numbers. We seek the solution of the three-dimensional equations (3.4) in the same harmonic form,

$$\{w, \mathbf{u}, \sigma_t, \sigma_{33}\}(x_1, x_2, z) = \{w, \mathbf{u}, \sigma_t, \sigma_{33}\}(z)e^{iY)} \tag{7.2}$$

and the solution of two-dimensional equations Eqs. (6.6) has the form:

$$w(x_1, x_2) = We^{iY)}, \quad \mathbf{u}(x_1, x_2) = \mathbf{U}e^{iY}, \quad \mathbf{U} = (U_1, U_2)^T, \tag{7.3}$$

where $\mathbf{U},\ W$ are the unknown complex amplitudes of deflection of plane $z = 0$. We consider a bending deformation under the action of a periodic compression applied to a lower plane,

$$\mathbf{f}_t = 0, \quad f_3(x_1, x_2, z) = F_3\delta(z + 1/2)e^{i(x_1q_1+x_2q_2)}. \tag{7.4}$$

Inserting Eqs. (7.1) and (7,3) into Eqs. (6.6) yields the following linear algebraic system the for unknown variables $\mathbf{U}$, W,

$$\begin{aligned}&(\tilde{\mathbf{L}}_0+\mu\tilde{\mathbf{L}}_1+\mu^2\tilde{\mathbf{L}}_2)\cdot\mathbf{U}-(\tilde{\mathbf{N}}_0+\mu\tilde{\mathbf{N}}_1+\mu^2\tilde{\mathbf{N}}_2)W=0,\\&(\tilde{\mathbf{N}}'_0+\mu\tilde{\mathbf{N}}'_1+\mu^2\tilde{\mathbf{N}}'_2)\cdot\mathbf{U}-(\tilde{Q}_0+\mu\tilde{Q}_1+\mu^2\tilde{Q}_2)W+F_3=0,\end{aligned}\tag{7.5}$$

with

$$\tilde{\mathbf{L}}_k=\mathbf{I}_a(\mathbf{L}_k),\quad \tilde{\mathbf{N}}_k=\mathbf{I}_a(\mathbf{L}_k\cdot\mathbf{p}\,z),\quad \tilde{\mathbf{N}}'_k=\mathbf{I}_a(z\mathbf{p}^T\cdot\mathbf{L}_k),\quad \tilde{Q}_k=\mathbf{I}_a(z\mathbf{p}^T\cdot\mathbf{L}_k\cdot\mathbf{p}\,z),\ k=1,2,3,$$

and the differential operators $\mathbf{p}$ and $\mathbf{P}$ are to be formally replaced by

$$\tilde{\mathbf{p}}=i\begin{pmatrix}q_1\\q_2\end{pmatrix},\qquad \tilde{\mathbf{P}}=i\begin{pmatrix}q_1&0&q_2\\0&q_2&q_1\end{pmatrix}^T.\tag{7.6}$$

We represent the solutions of Eqs. (7.5) as expansions in power series in μ,

$$W_s=W_0+\mu\,W_1+\mu^2W_2,\quad \mathbf{U}=\mathbf{U}_0+\mu\,\mathbf{U}_1+\mu^2\mathbf{U}_2.\tag{7.7}$$

In the zero approximation, we have

$$\mathbf{U}_0=\tilde{\mathbf{L}}_0^{-1}\cdot\tilde{\mathbf{N}}_0\,W_0,\qquad W_0=-\frac{F_3}{\tilde{Q}_0-Q_*},\qquad Q_*=\tilde{\mathbf{N}}'_0\cdot\tilde{\mathbf{L}}_0^{-1}\cdot\tilde{\mathbf{N}}_0\tag{7.8}$$

where the value Q_* takes into account the asymmetry of the cross-section.

We consider deformation of a plate under unit normal external compression applied to plane $z=-1/2$, that is, $\sigma_{33}(-1/2)=-1$. Next, we introduce the vector $\mathbf{Y}$ of unknown variables of Eq. (3.4),

$$\mathbf{Y}(z)=\{y_1,y_2,\ldots,y_6\}=(w,u_1,u_2,\sigma_{13},\sigma_{23},\sigma_{33})..\tag{7.9}$$

To find a solution of Eq. (3.4) satisfying the boundary conditions (3.9) we solve the four Cauchy problems with the boundary conditions

$$\begin{aligned}&\mathbf{Y}^{(1)}(-1/2)=(1,0,0,0,0,0),\quad \mathbf{Y}^{(2)}(-1/2)=(0,1,0,0,0,0),\\&\mathbf{Y}^{(3)}(-1/2)=(0,0,1,0,0,0),\quad \mathbf{Y}^{(4)}(-1/2)=(0,0,0,0,0,-1).\end{aligned}\tag{7.10}$$

Now the sought-for solution reads as

$$\mathbf{Y}(z)=C_1Y^{(1)}(z)+C_2Y^{(2)}(z)+C_3Y^{(3)}(z)+Y^{(4)}(z),\tag{7.11}$$

where the constants C_1,C_2,C_3 should be found from the equations

$$\sigma_{i3}(1/2)=0,\qquad i=1,2,3.\tag{7.12}$$

Solving Eqs. (7.12), we have $w(-1/2)=C_1$, $u_1(-1/2)=C_2$, $u_2(-1/2)=C_3$.

Consider bending deformations of an infinite multi-layered plate consisting of 5 isotropic layers of equal thickness ($h_k=0.2$) under the action of the harmonic compression $F_3(x_1,x_2)=F_3e^{i(q_1x_1+q_2x_2)}$, as applied to the lower plane $z=-1/2$. The Young moduli and the Poisson ratios of layers are as follows:

$$E_k=1,\ k=1,3,5,\qquad E_k=\eta,\ k=2,4,\qquad \nu_k=0.3,\ k=1,\ldots,5.$$

We take $\mu=0.1,\ q_1=0.6,\ q_2=0.8$. The deflection amplitudes W, as found for various values η, are given in Table 1.

Table 1. Deflection amplitude versus η.

η	δ_s	W_e	W_0	$\varepsilon_0\%$	W_s	$\varepsilon_s\%$
1	0.0029	1094	1092	-0.2	1095	0.1
0.1	0.012	1358	1344	-1.0	1360	0.1
0.01	0.11	1527	1375	-10.0	1529	0.1
0.001	1.1	2857	1378	-52.2	2908	1.8

Here W_e is the exact value found by numerical solution of Eqs. (3.4), W_0 is the first approximation (7.8) at $Q_*=0$, W_s is the second approximation (7.7), ε_0 and e_s are the corresponding relative errors of approximate values W_0 and W_0. The parameter η^{-1} describes the level of a plate heterogeneity, and the parameter $\delta_s=\mu^2\tilde{Q}_2/\tilde{Q}_0$, the influence of the transversal shear. From Table 1 it follows that the first approximation is acceptable for a comparatively small level of heterogeneity, while the second approximation gives exact enough results for wide limits of the ratio E_1/E_2.

8 HARMONIC VIBRATIONS

We consider free vibrations of an infinite anisotropic plate, and seek solutions of the 3D and of the 2D equations in the same forms (7.2) and (7.3), respectively, after changing the factor $e^{i(x_1q_1+x_2q_2)}$ by $e^{i(x_1q_1+x_2q_2+\omega t)}$, where ω is the natural frequency. According to the notation of Eqs. (2.9) and (3.8), the external (inertia) forces in the dimensionless form read as

$$\mathbf{f}_t(x_1,x_2,z,t)=\lambda\,\hat{\rho}(z)\mathbf{u}(z)\,e^{iY},\ f_3(x_1,x_2,z,t)=\frac{\lambda}{\mu^2}\hat{\rho}(z)w(z)\,e^{iY} \tag{8.1}$$

with

$$Y=x_1q_1+x_2q_2+\omega t,\qquad \lambda=\frac{\rho_0\,l^2\omega^2}{E},\quad \hat{\rho}(z)=\frac{\rho(z)}{\rho_0},\quad \rho_0=\mathbf{I}_a(\rho(z)), \tag{8.2}$$

where λ is the unknown frequency parameter, and $\rho(z)$ is the density, ρ_0 is the average density. The sign $\hat{}$ is again omitted.

The inertia forces (8.1) are to be inserted into 3D exact Eqs. (3.10) or (3.4) and into 2D approximate Eqs. (6.6). These equations contain the unknown frequency parameter λ.

The system (6.6) is a linear algebraic one, and its determinant satisfies the equation

$$\Delta(\lambda,q_1,q_2,\mu)=0. \tag{8.3}$$

This equation is cubic in λ. It is easy to prove that the roots of Eq. (8.3) are real positive numbers, $\lambda_j>0$. The smallest root gives a natural frequency of bending vibrations, $\omega=\sqrt{\lambda_1(q_1,q_2,\mu)}$.

As in Section 7, to find the exact eigenvalue λ we solve numerically the three Cauchy problems for Eqs. (3.4) with the initial conditions

$$\mathbf{Y}^{(1)}(-1/2)=(1,0,0,0,0,0),\ \mathbf{Y}^{(2)}(-1/2)=(0,1,0,0,0,0),\mathbf{Y}^{(3)}(-1/2)=(0,0,1,0,0,0) \tag{8.4}$$

Using the first root of the equation

$$\Delta(\lambda)=\begin{vmatrix}\sigma_{13}^{(1)}(1/2) & \sigma_{23}^{(1)}(1/2) & \sigma_{33}^{(1)}(1/2)\\ \sigma_{13}^{(2)}(1/2) & \sigma_{23}^{(2)}(1/2) & \sigma_{33}^{(2)}(1/2)\\ \sigma_{13}^{(3)}(1/2) & \sigma_{23}^{(3)}(1/2) & \sigma_{33}^{(3)}(1/2)\end{vmatrix}=0, \tag{8.5}$$

we get the exact value $\lambda = \lambda_e$.

Proceeding in this way it is possible to estimate the error of approximate equations (6.6).

If we put $Y = x_1 q_1 + x_2 q_2 - vt$, then Eqs. (7.2) and (7.3) describes the wave propagation in the direction $\mathbf{q} = (q_1, q_2)$ with velocity v, and Eqs. (8.3) and (8.5) are the dispersion equations. The velocity $v = v(q_1, q_2)$ depends on the direction of wave propagation.

As a numerical example, we consider the bending vibrations of a symmetric plate consisting of 5 layers with equal thickness. The first, the third, and the fifth layers are orthotropic. The third (middle) layer is turned by an angle $\alpha = \pi/2$ with respect to the rest of the layers. The second and the forth layers are isotropic, and we will change the stiffness of these layers within the wide limits. The elastic moduli of the layers in Eqs. (3.4) are given in Table 2 in the dimensionless form. Here, N is the number of layers.

Table 2. Elastic moduli of layers.

N	E_{11}	E_{22}	E_{33}	$E_{12} = E_{13} = E_{23}$	$E_{44} = E_{55} = E_{66}$
1,5	12.0	2.0	2.0	0.59	0.69
2,4	1.1	1.1	1.1	0.33	0.38
3	1.0	11.0	1.0	0.30	0.35

We take the mass density $\rho = 1$ for all layers, the wave numbers $q_1 = 0.6,\ q_2 = 0.4$, and the small thickness parameter $\mu = 0.1$.

For elastic moduli of the second and the forth layers, we consider 4 variants. The moduli of the first variant are given in Table 1. The moduli of the remaining variants can be obtained by multiplying the moduli of the first variant by the values $\eta = 0.1$, $\eta = 0.01$, and $\eta = 0.001$, respectively. By using the algorithm and formulas of Section 6, we get the following results.

Table 3. The frequency parameter versus the ratio η of elastic moduli.

1	2	3	4	5	6	7
η	λ_e	λ_0	$\varepsilon(\%)$	λ_s	$\varepsilon_0(\%)$	δ_s
1	0.002211	0.002229	0.9	0.002212	0.0	0.0079
0.1	0.001985	0.002073	4.5	0.001986	0.1	0.044
0.01	0.001482	0.002058	38.5	0.001459	1.6	0.41
0.001	0.000511	0.002056	303.0	0.000405	20.9	4.07

In Table 3, for four values of the parameter η (i.e., for the measure of small stiffness of the second and the fourth layers), the corresponding values of the frequency parameter λ and the shear parameter δ_s are presented. Here λ_e is the exact value, as found by numerical solution of Eqs. (8.5), λ_0 is the first approximation, and λ_s is the second approximation, as obtained from Eqs. (6.6). In columns 4 and 6, the relative errors of the approximate values are given.

As for the problem of plate bending (Section 7), here the first approximation is acceptable for a comparatively small level of heterogeneity, while the second approximation gives exact enough results for wide limits of the parameter η.

9 CONCLUSIONS

The 2D linear model (7.5) of the SOA describing deformations of a heterogeneous plate in the case of general anisotropy (with 21 elastic moduli) is constructed. It is necessary to continue investigations of the problem under consideration. It is desirable to investigate the peculiarities that appear in the case of plates with inclined anisotropy. Of great value is also to estimate the errors for test examples of the 2D models and compare them with the exact solutions of 3D equations of the theory of elasticity. The model presented here is bulky, and so it is desirable to propose, in particular cases, more simple but sufficiently precise models.

The next step is to solve some static, vibration, and buckling problems for particular kinds of anisotropy and heterogeneity.

The algorithm used here is based on the Cartesian coordinate system. It is interesting to apply the obtained results for heterogeneous anisotropic shells, in partial, to shallow shells for which the metric is close to the Cartesian metric.

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REFERENCES

[1] G. Kirchhoff, Vorlesungen über Matematische Physik. Mechanik. Leipzig, 1876. [in German].

[2] A.E.H. Love, A treatise on the mathematical theory of elasticity. Cambridge Univ. Press, 1927.

[3] L.H. Donnell, Beams, Plates and Shells. McGraw-Hill, 1976.

[4] K.F. Chernykh, V.A. Rodionova, B.F. Titaev, Applied theory of anisotropic plates and shells. St. Petersburg Univ. Press, 1996. [in Russian].

[5] L.A. Agolovyan, Asymptotic theory of anisotropic plates and shells. Moscow. Nauka. 414 p. 1997. [in Russian].

[6] J.N. Reddy, Mechanics of laminated composite plates and shells. CRC Press, 831 p. 2004.

[7] A.L. Goldenweizer, Theory of Elastic Thin Shells. Nauka, Moscow, 1976. [in Russian].

[8] Y. Vetukov, A. Kuzin, M. Krommer, Int. J. of Solids and Structures, **40**, 12-23, 2011.

[9] P. Schnieder, R. Kienzler, In: Shell-like structures. Non-classical theories and applications. Berlin, Springer, 377-390, 2011.

[10] P.E. Tovstik, T.P. Tovstik, Mechanics of Solids. **52**(2), 144-154, 2017.

[11] R. Kienzler, P. Shneider, Shell Structures: Theory and Applications. Proc. 10th SSTA, 2013 Conf. **3**, 109-112, 2014.

[12] P.E. Tovstik, T.P. Tovstik, Doklady Physics, **59**(8), 389-392, 2014.

[13] P.E. Tovstik, T.P. Tovstik, ZAMM, **97**(3), 296-308, 2017.

[14] P.E. Tovstik, T.P. Tovstik, Acta Mechanica, **228**(10), 3403-3419, 2017.

[15] N.F. Morozov, P.E. Tovstik, T.P. Tovstik, Mechanics of Solids. **51**(5), 527-537, 2016.

[16] N.F. Morozov, P.E. Tovstik, T.P. Tovstik, Doklady Physics, **61**(11), 567-570, 2016.

[17] P. Schneider, R. Kienzler, A Reissner-type plate theory for monoclinic material derived by extending the uniform-approximation technique by orthogonal tensor decompositions of nth-order gradients. Meccanica 52:21432167. DOI 10.1007/s11012-016-0573-1, 2017.

[18] N.F. Morozov, A.K. Belyaev, P.E. Tovstik, T.P. Tovstik, Doklady Physics, **63**(11), 471-475, 2018.

[19] P.E. Tovstik, Doklady Physics, **425**(4), 2009.

[20] P.E. Tovstik, T.P. Tovstik, Acta Mechanica. **225**(3), 647-661, 2014.

[21] P.E. Tovstik, T.P. Tovstik, Trudy Seminara “Komp’yuternye metody v mekhanike sploshnoi sredy”, **3**, St. Petersburg Publ. 4-16, 2008.