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A Stochastic Control Model for the Average Price of Manufacturer Sales on Commodity Exchanges

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Abstract—We propose an approach for controlling the weighted average price of a manufacturer's sales on commodity exchanges. This problem is highly relevant due to the need for the manufacturer to hedge their profits in case of a sharp drop in market prices. We consider applications of the proposed control to executing trading operations on real commodity exchanges in order to demonstrate its efficiency.

Keywords: commodity exchanges, random processes, sales management

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1. INTRODUCTION

The instability of price behavior in modern commodity markets, which include the trading of oil, gas, wood and other popular consumer goods, forces us to reconsider a number of established and long-standing relationships between sellers of goods and their buyers. First of all, this concerns the refusal to conclude long-term contracts, when a buyer, being aware of the unpredictabile future behavior of the market value of the goods, prefers to purchase them within the framework of specialized exchanges with obligatory delivery of purchased goods over a relatively short period of time. This tendency manifests itself very convincingly, for example, in relationships between producers and suppliers of Russian natural gas and their European consumers. Currently, Europe has 13 gas storages (hubs) and 7 specialized gas exchanges organized together with them, selling various volumes of gas with obligatory two-day delivery from the moment of the deal. At the same time, trade operations are characterized by high liquidity and high total volume that covers a large part of the global gas trade. Naturally, such a turn of events requires a certain reaction from the seller/manufacturer of a product related to constructing its own sales management strategy that would at least hedge the resulting cash revenue from a sharp drop in the market value of the goods. One such strategy can be the development of a control associated with an increase in the weighted average price of the goods sold by the manufacturer.

Of course, such a strategy requires the seller to carry out transactions that are not only uneven in terms of the volume but also conduct so-called "reverse transactions" that correspond to purchases of the same goods made in certain volumes and at certain times. The purpose of this work is to describe one possible way to implement this strategy and demonstrate its effectiveness using the example of trading in the gas industry at the European Energy Exchange.[1]

[1] European Energy Exchange AG [Electronic resource]. Leipzig, 2017.
URL: https://www.eex.com (accessed on 05/15/2017).

2. FORMAL PROBLEM SETTING

Let us assume that over a given time interval $[0, T]$, exchange transaction prices x_t, $t \in [0, T]$ with respect to a unit of the goods in question satisfy the following stochastic differential equation:

$$dx_t = c\,(t, x_t)\,dt + \sigma_t x_t dW_t, \tag{1}$$

where $\sigma_t = \sigma(t, \omega)$ is the volatility coefficient, which is an unknown and generally speaking random function of time independent of x_t, and W_t is the standard Wiener process [1]. The structure of the drift coefficient $c(t, x_t)$ does not require a separate specification for this work; however, it is assumed that the implementation of the random process x_t does not take negative values with probability one and, moreover, the Cauchy problem for Eq. (1) has a unique strong solution [2]. The question of the adequacy of the chosen pricing model for the control problem considered below is discussed separately in Appendix A. In what follows, we denote with corresponding letters, but with a wave, observable implementations of the considered random processes, e.g., $\tilde{x}_t$ and x_t.

By the weighted average sales price of the manufacturer over a time interval $[0, T]$ we mean the value

$$\tilde{x}_T^{av} = \frac{\tilde{V}_T}{\tilde{a}_T},$$

where $\tilde{V}_T$ is the amount of money gained over the time interval $[0, T]$, $\tilde{a}_T$ is the quantity of units of goods sold over the same time interval.

By the control objective, we mean the number of items sold over the time interval $[0, t]$, where $t \in [0, T]$, product units $\tilde{a}_t$ ($\tilde{a}_0 = 0$, $\tilde{a}_t < 0$, with $t > 0$ and $\tilde{a}_t$ corresponding to the price realization $\tilde{x}_t$) that would ensure the following condition:

$$\tilde{x}_T^{av} > \max_{t \in [0,T]} \tilde{x}_t, \tag{2}$$

and here the construction of the control $\tilde{a}_t$ can use as feedback only the prices of exchange deals made during the time interval $[0, t]$. In other words, we presume that the coefficients in Eq. (1) cannot be estimated in real time, at least not with the required degree of accuracy. It is intuitively clear that condition (2) can be met only by increasing some speculative profit (a precise definition will be given below) resulting from gradual reinvestment of some part of the cash resulting from sales of the goods and subsequent resale of the goods, based on from the dynamics of stock transactions since the beginning of the control period. In the next section, we will construct one such possible control, which under certain conditions will ensure the inequality (2).

3. CONSTRUCTION OF THE CONTROL

Below, we will follow a continuous model of financial markets [3] and, in particular, an approach related to constructing a control alternative to the self-financing strategy, taking into account the process of reinvesting money into the product originally proposed in [4] and then detailed in [5] (see also [6]), but now, unlike the work [5], we will consider the case of a short position and in application to operations on a commodity exchange.

We introduce a random function based on the formula

$$f_t = a_t x_t + m_t, \tag{3}$$

where $a_t = a\,(t, \omega)$ is a measurable random function that determines the number of units of goods sold over the period of time $[0, t]$; $m_t = m\,(t, \omega)$ is a measurable random function that will be explicitly defined below. We will proceed from the control of the function f_t defined at every time moment by

$$df_t = a_t dx_t + l(t, x_t)\, dt, \tag{4}$$

where dx_t is the right-hand side of Eq. (1). The second term in (4) will be interpreted as the amount of cash produced by sales and withdrawn from the control process over the time interval $[t, t + dt]$. The dependency $l(t, x_t)$ will act below as the control function. Applying to the left- and right-hand sides of the relation (3) the procedure for calculating the stochastic differential, we obtain the formula

$$df_t = a_t dx_t + x_t da_t + da_t dx_t + dm_t.$$

Using the dependency (4), the latter can be rewritten as

$$dm_t = -x_{t+dt} da_t + l(t, x_t)\, dt,$$

where $x_{t+dt} = x_t + dx_t$, or in integral form

$$m_t = -\int_0^t x_{\tau+d\tau} da_t + \int_0^t l(\tau, x_\tau)\, d\tau. \tag{5}$$

We define the concept of "speculative profit" obtained as a result of trading operations performed over the time interval $[0, t]$ as follows:

$$p_t = a_t x_t - \int_0^t x_{\tau+d\tau} da_t; \tag{6}$$

here we will consider the case of a short position, when $a_t < 0$ for $t > 0$.

Taking into account (3)–(5), it is easy to see that dependency (6) is equivalent to

$$p_t = f_t - \int_0^t l(\tau, x_\tau)\ d\tau. \tag{7}$$

Let us explain the concept of "speculative profit" in this particular situation in more detail. Suppose that at fixed points in time $1, \ldots, n$, at prices $x_1, \ldots, x_n$, $b_1, \ldots, b_n$ units of goods were sold or purchased, where $b_i < 0$ corresponds to selling and $b_i > 0$ corresponds to buying the specified quantity $|b_i|$. We will assume that we are holding a short position, which means that the inequality $\sum_{i=1}^{j} b_i < 0$ holds for every $j = 1, \ldots, n$. Then it is clear that the amount of money received as a result of these trading operations will be determined by the formula $-\sum_{i=1}^{n} x_i b_i$. On the other hand, the integral $-\int_0^t x_{\tau+d\tau} da_\tau$ is a continuous counterpart of the relation above, and the symbol $x_{\tau+d\tau}$ means that when this integral is represented as a limit, the integrand is calculated on the right end of the corresponding splitting intervals. At the same time, the following identity holds:

$$-\int_0^t x_{\tau+d\tau} da_\tau = a_t x_t - \int_0^t x_{\tau+d\tau} da_\tau + |a_t|\, x_t,$$

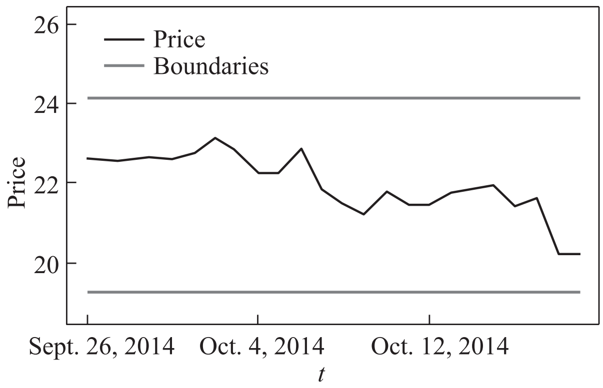

Fig. 1. Choice of the price corridor.

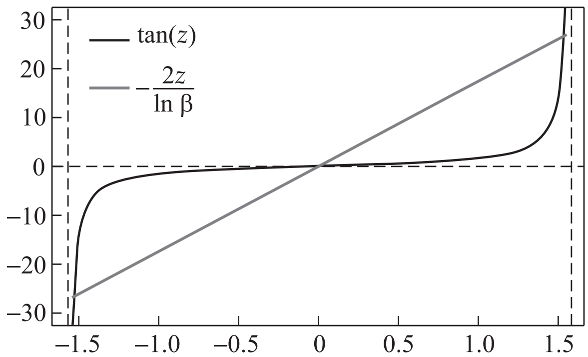

Fig. 2. Choosing a root of the transcendental equation.

i.e., in other words, the amount of money received over the time interval $[0, t]$ consists of "speculative profit" and the amount of money received from the sale of goods at time moment t.

Note that the difference between the model (3), (4) and control models with consumption [7, 8] is that the value m_t, as seen from (5), corresponds to the process of reinvestment, when the funds raised in the process of making speculative operations are invested in the purchase of a certain number of commodity units. Although m_t is not explicitly included in (7), its absence would lead to the fact that in (7) the value f_t would be equal to $a_t x_t$. This would mean the absence of the above effect of reinvestment and, as a consequence, the absence of speculative profits.

We introduce into consideration the price corridor $[a, b]$, initially assuming that the observed values of the unit cost of a commodity, i.e., the implementation of the random process $\tilde{x}_t$ over the entire time span $t \in [0, T]$ will be in the specified price corridor and, accordingly, $\tilde{x}_t \in [a, b]$. Below, we present a method of expanding the price corridor in the process of making trade operations when such a need arises, i.e., in a situation when the observed value of the price $\tilde{x}_t$ crosses one of its boundaries. In addition, for convenience of calculations, we normalize the price per unit of goods by the upper boundary of the price corridor and, accordingly, in the process of further exposition, we assume that $\tilde{x}_t \in (\beta, 1)$, where $\beta = a/b$, as shown in Fig. 1.

Next, we introduce the dependence $\varphi(x)$ that corresponds to the first eigenfunction of the Sturm–Liouville problem

$$\frac{d^2\varphi}{dx^2} + \frac{\lambda_1^2}{x^2}\varphi = 0, \tag{8}$$

$$\varphi(1) = \varphi'(\beta) = 0.$$

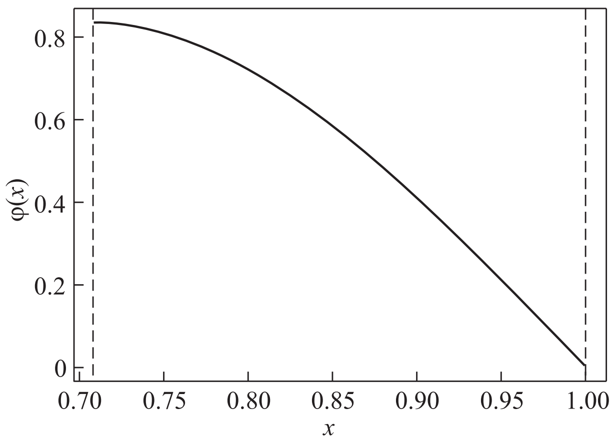

Fig. 3. $\varphi(x)$, the solution of problem (8).

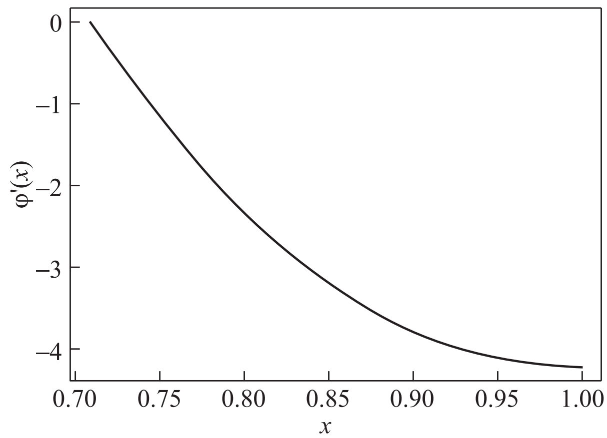

Fig. 4. $\varphi'(x)$, the derivative of the solution of problem (8).

It is easy to see [9] that as the first eigenfunction one can choose $\varphi\left(x\right)=\sqrt{x}\sin\left(b\ln x\right)$ with $b=\frac{z^{+}}{\ln\beta}<0$, where $z^{+}>0$ is minimal in absolute value strictly positive root of the transcendental equation $\tan z=-\frac{2z}{\ln\beta}$, as shown in Fig. 2; here, obviously, inequality $-\frac{2}{\ln\beta}>1$ must be satisfied, which makes the price corridor fairly narrow, namely, the value β must satisfy the inequality $\frac{1}{e^{2}}<\beta<1$.

In addition, Figs. 3 and 4 show the plots of functions $\varphi\left(x\right)$ and $\varphi'\left(x\right)$ respectively.

The first eigenvalue λ_1, which corresponds to the first eigenfunction of problem (8), satisfies the relation $\lambda_1^2=b^2+\frac{1}{4}$, while we note that $\varphi'\left(1\right)=b<0$. Let us now define the dynamics of changes in the number of units of goods a_t sold over the time interval $[0,t]$ as follows:

$$\tilde{a}_t=\int\limits_0^t\frac{u_0\left(\tau\right)}{\varphi\left(\tilde{x}_\tau\right)}d\tau\times\varphi'\left(x_t\right)\big|_{x_t=\tilde{x}_t}, \qquad (9)$$

where $u_0\left(\tau\right)>0$ is an arbitrary piecewise continuous function.

Theorem. *Suppose that the price x_t follows the stochastic differential Eq.* (1), *and the volatility coefficient $\sigma_t=\sigma\left(t,\omega\right)$ as a random function does not depend on the process x_t. Then, if the number of units of goods sold over time is given by formula* (9), *then "speculative profit" can be* *computed as*

$$\tilde{p}_t = \int\limits_0^t \frac{u_0(\tau)}{\varphi(\tilde{x}_\tau)} d\tau \times \varphi(\tilde{x}_t) - \int\limits_0^t u_0(\tau)\ e^{-\frac{1}{2}\lambda_1^2 \int\limits_\tau^t \tilde{\sigma}_s^2 ds} d\tau, \tag{10}$$

and, moreover, the formula

$$\tilde{V}_T = \tilde{p}_T + |\tilde{a}_T|\,\tilde{x}_T \tag{11}$$

determines the total amount of cash obtained over the time interval $[0,T]$.

The proof of the theorem is given in Appendix B.

Remark. Note that the choice of the first eigenfunction of the Sturm–Liouville problem (8) ensures that it is nonzero within the price corridor $(\beta,1)$ and, as a result, there are no singularities in the denominators of formulas (9) and (10). Also note that, as previously announced, the control a_t defined by the formula (9) does not explicitly contain the coefficients of the stochastic differential Eq. (1).

We also note the fact that the weighted average price of goods sold over the time interval $[0,T]$ in the amount of $|a_T|$ can be calculated as

$$\tilde{x}_T^{av} = \tilde{x}_T + \frac{\tilde{p}_T}{|\tilde{a}_T|}. \tag{12}$$

Equation (12) shows that with the same volume of goods sold $|\tilde{a}_T|$ larger weighted average price corresponds to larger speculative profit. At the same time, analysis of relation (10) shows that two factors contribute to the increase in the weighted average price. The first factor is temporary and is associated with an increase in the weighted average price as time increases, provided that the observed price value $\tilde{x}_t$ remains in the same price corridor. The second factor of the increase is due to the influence of the volatility coefficient for which the second integral in (10) decreases exponentially with it. In this case, the inequality

$$\tilde{x}_T^{av} = \tilde{x}_T + \frac{\tilde{p}_T}{|\tilde{a}_T|} > \max_{t\in[0,T]} |\tilde{x}_t|$$

ensures that we achieve the control objective (2). Finally, we note that the choice of the corresponding function $\varphi(x)$ in the statement of the theorem provides the desired increase in the weighted average price when prices fall, which follows from (9), (10). Indeed, as can be seen from the plots of functions $\varphi(x)$ and $\varphi'(x)$, shown respectively in Figs. 3 and 4, a drop in the prices $\tilde{x}_t$ leads to an increase in $\tilde{p}_t$ and at the same time to a decrease in $|\tilde{a}_t|$. In addition, we note that growing prices lead to an increase in sales.

We now consider the question of expanding the price corridor if needed. Suppose that at time point t^* the observed value of the price $\tilde{x}_t$ crosses the lower boundary of the price corridor, as shown in Fig. 5. The case of crossing the upper boundary of the corridor is considered similarly.

In either of these two cases, a new price corridor is selected corresponding to the interval $(\beta_1,1)$. We denote the first eigenfunction of the Sturm–Liouville problem (8) for the new price corridor by $\varphi_1(x)$. Instead of $u_0(\tau)$, we choose a new control function $u_1(\tau)$ over the time interval $[0,t^*]$ so that

$$\int\limits_0^t \frac{u_1(\tau)}{\varphi_1(\tilde{x}_\tau)} d\tau \times \varphi_1'(x_t)\,\big|_{x_t=\tilde{x}_t} = \tilde{a}_t^*$$

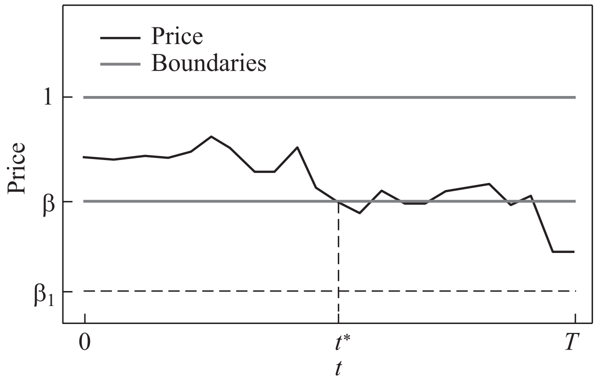

Fig. 5. Expanding the price corridor.

for all $t \in [0, t^*]$, where the dependence $\tilde{a}_t^*$ corresponds to the trade deals already completed over the time interval $[0, t^*]$. This problem reduces to considering a Volterra equation of the first kind

$$\int_0^t K(t, \tau) \, x(\tau) \, d\tau = f(t)$$

with respect to the unknown function $x(\tau) = u_1(\tau)$. Here

$$K(t, \tau) = \frac{\varphi_1'(x_t) \,|_{x_t = \tilde{x}_t}}{\varphi(\tilde{x}_\tau)}, \quad f(t) = \tilde{a}_t^*.$$

The last equation is solved with standard methods of the theory of ill-posed problems in the space L_2 on a compact set of functions with bounded variation [10]. Accordingly, for $t > t^*$ the control function $u_1(t) > 0$ can, as before, be chosen arbitrarily in the class of piecewise-continuous functions.

4. SAMPLE IMPLEMENTATION OF THE CONSTRUCTED CONTROL ON REAL STOCK EXCHANGES

Figure 6 shows a graph of prices for exchange transactions that correspond to their average values for a single day trading session in European currency over the time interval from October 1, 2014 until September 30, 2015 for 1 MW × h of tradable energy. The same plot shows the dynamics of the weighted average price for one unit of the same product, obtained as a result of the control

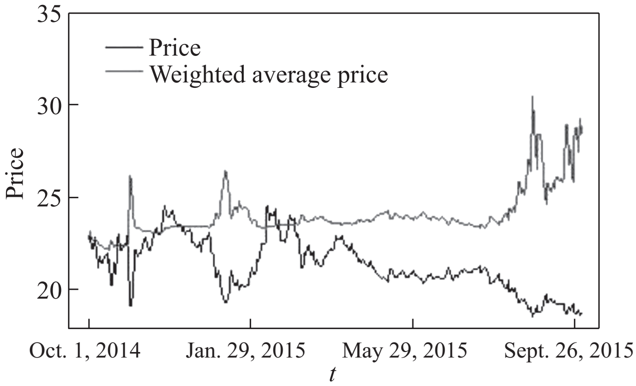

Fig. 6. Graph of the weighted average price.

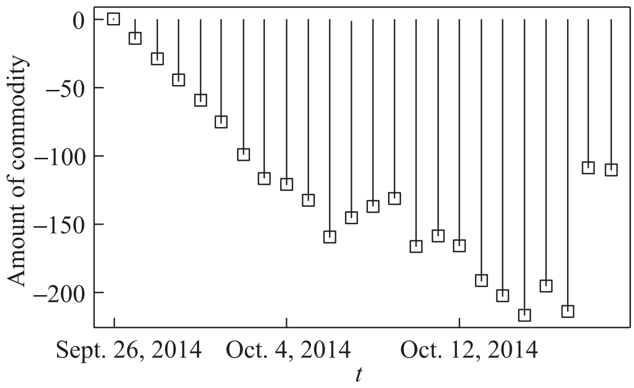

Fig. 7. Number of items sold.

constructed above (9). Here, the control function $u_0(\tau)$ is initially assumed to be always constant, and the price corridor is chosen to be symmetrical with respect to the first transaction made by the control system and with the width of two euros. In the future, the boundaries of the price corridor expand by one euro whenever the price "break through" its upper or lower boundary. In this case, $u_0(\tau)$ is replaced with the control function $u_1(\tau)$ for the values of the argument τ defined above that belong to the interval $[0, t^*]$. In addition, Fig. 7 contains a graph of changes in the quantity of goods sold, which clearly shows the process of making the corresponding "reverse transactions".

5. CONCLUSION

Our analysis of the use of the control (9) on modern gas trading exchange markets demonstrates that it is an effective method for increasing the weighted average price of goods sold in the event of a sharp decrease in its market price. Even when condition (2) is violated on some sufficiently small subset of the time interval $[0, T]$, the constructed control does not become less effective since it is impossible to guess a priori the exact values of maximal possible prices with respect to the weighted average price at time point T. However, a problem arises here with the planned volume of gas sales over a fixed period of time. The proposed construction indicates that the quantity of goods sold in relevant trading operations becomes a random value, and issues relating to the specification of the choice of the control function $u_0(\tau)$, timely closing of positions, as well as planning for the supply of goods, become independently important.

APPENDIX A

The question of the adequacy of the pricing model (1) used in this work essentially boils down to the question of the adequacy of calculating speculative profits using formula (10) when the quantity of goods sold, defined by relation (9), changes. Note that the calculation of speculative profits with formula (6) does not use a specific pricing model and is carried out directly based on the change in the quantity of goods sold $\tilde{a}_t$ and the registry of exchange transactions $\tilde{x}_t$. Therefore, without loss of generality, we assume in (9) that $u_0(\tau) \equiv 1$ and calculate speculative profits using formula (6) for a given realization of the price $\tilde{x}_t$. On the other hand, we find the same value using formula (10), which is obtained on the basis of the pricing model (1), while integral volatility in (10) is calculated on the basis of robust estimation algorithms defined in [11, 12]. It is important to note that the relative error in the calculation of the desired quantity using formula (10), as shown by the analysis of the plots on Fig. 8, does not exceed 7%. Here we need to clarify that calculations were carried out under the assumption that only closing prices of the trading sessions were available and hence used out of the above-mentioned external sources; this, of course, significantly coarsened the

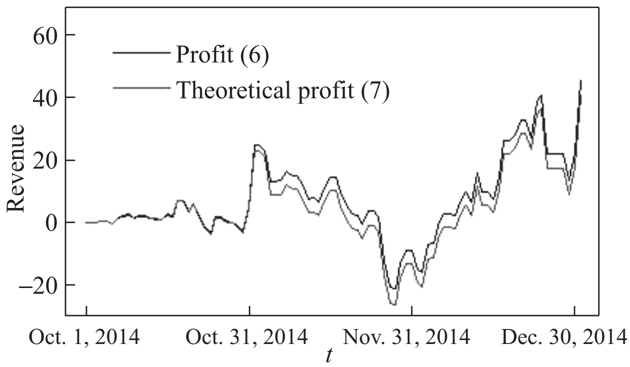

Fig. 8. Charts of speculative profits calculated by formulas (6) and (7).

accuracy of the calculations. At the same time, for highly liquid assets such as blue chip stocks traded on both Russian and US stock markets, the same value obtained from full intraday quotes data is on the order of fractions of one percent and less. In addition, we note that the theorem will remain valid in case of a diffusion process with jumps considered in [13], as well as when using the Heston model [14], when controlling Wiener processes in this model are independent or when the "volatility volatility" coefficient used in it is zero.

APPENDIX B

Proof of Theorem.

We will look for the unknown function f_t in the form $f_t = f(t, x_t)$, where x_t satisfies Eq. (1). Applying the Ito formula to the function $f(t, x_t)$ and comparing it with relation (4), we obtain the dependencies

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_t^2 x_t^2 \frac{\partial^2 f}{\partial x_t^2} = l(t, x_t); \tag{B.1}$$

$$a_t = \frac{\partial f}{\partial x_t}. \tag{B.2}$$

We will look for the control $l(t, x_t)$ in the form

$$l(t, x_t) = r(t)\varphi(x_t), \tag{B.3}$$

where $\varphi(x_t)$ is the first eigenfunction corresponding to the first eigenvalue λ_1 of the Sturm–Liouville problem (8). The structure of the function $r(t)$ will be defined below. Let us set the initial and boundary conditions:

$$f(0, x_t) = 0, \tag{B.4}$$

$$\frac{\partial f}{\partial x_t} \to 0 \quad \text{for} \quad x_t \to \beta, \tag{B.5}$$

$$f(t, x_t) \to 0 \quad \text{for} \quad x_t \to 1. \tag{B.6}$$

Taking into account relation (B.3), we will look for the solution of the mixed problem (B.4)–(B.6) for Eq. (B.1) by the method of separation of variables (since, by virtue of the conditions of Theorem σ_t, it does not depend on x_t) in the form

$$f(t, x_t) = K(t)\varphi(x_t),$$

where $K(t)$ is an unknown function. After simple transformations, we find that

$$f(t, x_t) = \int_0^t e^{\frac{1}{2}\lambda_1^2 \int_\tau^t \sigma_s^2 ds} r(\tau)\, d\tau \times \varphi(x_t). \tag{B.7}$$

Dependencies (B.2), (B.7) determine the quantity of goods sold based on the formula

$$\tilde{a}_t = \left(\frac{\partial f}{\partial x_t}\right)\Big|_{x_t=\tilde{x}_t} = \int_0^t e^{\frac{1}{2}\lambda_1^2 \int_\tau^t \tilde{\sigma}_s^2 ds} r(\tau)\, d\tau \times \varphi'(x_t)\,|_{x_t=\tilde{x}_t}. \tag{B.8}$$

One disadvantage of formula (B.8) is that, as feedback for calculating $\tilde{a}_t$, it includes not only the prices of transactions made on this commodity, but also the volatility $\tilde{\sigma}_t$. To fix this problem, we introduce the following procedure. We divide the segment $[0; t]$ into n parts as follows: $0 = t_0 < t_1 < \ldots < t_n = t$. We define the function $r(\tau)$ as the limit of the sequence of functions $r_n(\tau)$ which is pointwise converging to it and is defined by

$$r_n(\tau) = \frac{u_n(\tau)}{\varphi(\tilde{x}_\tau)} e^{-\frac{1}{2}\lambda_1^2 \int_\tau^{t_i} \tilde{\sigma}_s^2 ds}, \tag{B.9}$$

where $\tau \in (t_{i-1}, t_i]$, $u_n(\tau)$ are given functions, while the sequence $u_n(\tau)$ with $n \to \infty$ when the segment $[0; t]$ is divided uniformly is assumed to converge pointwise to some function $u(\tau) \geq 0$. Substituting in (B.8) instead of $r(\tau)$ the sequence (B.9), we get

$$\tilde{a}_{t_j} = \sum_{i=1}^{j} \int_{t_{i-1}}^{t_i} \frac{u_n(\tau)}{\varphi(\tilde{x}_\tau)} d\tau \times \varphi'(x_t)\,|_{x_t=\tilde{x}_{t_j}},$$

or

$$\tilde{a}_{t_j} = \int_0^{t_j} \frac{u_n(\tau)}{\varphi(\tilde{x}_\tau)} d\tau \times \varphi'(x_t)\,|_{x_t=\tilde{x}_{t_j}}.$$

Carrying out the limit transition for $n \to \infty$ and using the assumption that the segment $[0; t]$ is divided uniformly, we finally get the formula that defines the continuous distribution of the quantity of goods sold over time with the observed value of the price $\tilde{x}_t$:

$$\tilde{a}_t = \int_0^t \frac{u(\tau)}{\varphi(\tilde{x}_\tau)} d\tau \times \varphi'(x_t)\,|_{x_t=\tilde{x}_t}. \tag{B.10}$$

Proceeding, in turn, from relations (7), (B.3), (B.7), (B.9) and arguing in a similar way, we arrive at the formula for the value of "speculative profit" corresponding to the observed price values $\tilde{x}_t$ at time point t:

$$\tilde{p}_t = \int_0^t \frac{u(\tau)}{\varphi(\tilde{x}_\tau)} d\tau \times \varphi(\tilde{x}_t) - \int_0^t e^{-\frac{1}{2}\lambda_1^2 \int_\tau^t \tilde{\sigma}_s^2 ds} u(\tau)\, d\tau. \tag{B.11}$$

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