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MATHEMATICS

Invariant Surfaces of Periodic Systems with Conservative Cubic First Approximation

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Abstract—Two classes of time-periodic systems of ordinary differential equations with a small parameter $\varepsilon \geq 0$, those with "fast" and "slow" time, are studied. The corresponding conservative unperturbed systems $\dot{x}_i = -\gamma_i y_i \varepsilon^\nu$, $\dot{y}_i = \gamma_i(x_i^3 - \eta_i x_i)\varepsilon^\nu$ $(i = \overline{1,n}$, $\nu = 0, 1)$ have 1 to 3^n singular points. The following results are obtained in explicit form: (1) conditions on perturbations independent of the parameter under which the initial systems have a certain number of invariant surfaces of dimension $n + 1$ homeomorphic to the torus for all sufficiently small parameter values; (2) formulas for these surfaces and their asymptotic expansions; (3) a description of families of systems with six invariant surfaces.

Keywords: invariant surface, bifurcation, averaging, separatrix.

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1. INTRODUCTION

1.1. Objects of Study

Consider the following periodic ODE system of dimension $2n$ $(n \geq 2)$ with a small parameter:

$$\begin{cases} \dot{x}_i = -\gamma_i y_i \varepsilon^\nu + X_i(t, x, y, \varepsilon)\varepsilon^{1+\nu}, \\ \dot{y}_i = \gamma_i(x_i^3 - \eta_i x_i)\varepsilon^\nu + Y_i(t, x, y, \varepsilon)\varepsilon^{1+\nu} \qquad (i = \overline{1,n}), \end{cases} \tag{1.1}$$

where $\nu = 0, 1$; $\gamma_i \in (0, +\infty)$; $\eta_i = -1, 0, 1$; $\varepsilon \in [0, \varepsilon_0)$ is a small parameter; $x = (x_1, \ldots, x_n)$, and the vectors y, X, Y, γ, and η are written in a similar form; and X and Y are continuous $C^3_{x,y,\varepsilon}$ functions T-periodic in t for $t \in \mathbb{R}$, $|x_i| < x^0$, and $|y_i| < y^0$.

In essence, formula (1.1) determines two different systems, with $\nu = 0$ and with $\nu = 1$; comparing these systems, we can say that the system with $\nu = 1$ has "fast" time, because, reducing it to the system with $\nu = 0$, we obtain the period $T\varepsilon$.

Remark 1. The linear change $\tilde{x}_i = \lambda_i x_i$, $\tilde{y}_i = \mu_i y_i$ $(i = \overline{1,n})$ reduces the following system of more general form to system (1.1):

$$\dot{\tilde{x}}_i = -\tilde{a}_i \tilde{y}_i \varepsilon^\nu + \tilde{X}_i(t, \tilde{x}, \tilde{y}, \varepsilon)\varepsilon^{1+\nu}, \quad \dot{\tilde{y}}_i = (\tilde{b}_i \tilde{x}_i^3 + \tilde{c}_i \tilde{x}_i)\varepsilon^\nu + \tilde{Y}_i(t, \tilde{x}, \tilde{y}, \varepsilon)\varepsilon^{1+\nu} \quad (\tilde{a}_i, \tilde{b}_i > 0).$$

Indeed, (a) if $\tilde{c}_i = 0$, then $\lambda_i = (\tilde{a}_i \tilde{b}_i)^{-1/2}$ and $\mu_i = (\tilde{a}_i^3 \tilde{b}_i)^{-1/2}$, and in (1.1), $\gamma_i = 1$ and $\eta_i = 0$; (b) if $\tilde{c}_i \neq 0$, then $\lambda_i = \tilde{b}_i^{-1/2} |\tilde{c}_i|^{1/2}$, $\mu_i = (\tilde{c}_i \tilde{b}_i)^{-1/2} |\tilde{c}_i|$, $\gamma_i = (\tilde{a}_i |\tilde{c}_i|)^{1/2}$, and $\eta_i = -\operatorname{sign} \tilde{c}_i$.

It is natural to refer to the 2-dimensional autonomous system

$$\dot{x}_i = -\gamma_i y_i \varepsilon^\nu, \quad \dot{y}_i = \gamma_i(x_i^3 - \eta_i x_i)\varepsilon^\nu \quad (i = \overline{1,n}) \tag{1.2}$$

as the system of the first approximation, or the unperturbed system with respect to (1.1); system (1.2) decomposes into n independent conservative two-dimensional systems with variables (x_i, y_i), each of which has the only singular point (0,0) (for $\eta_i = 0, -1$) or the three singular points $(-1, 0)$, $(0, 0)$, and $(1, 0)$ (for $\eta_i = 1$).

The phase planes of such two-dimensional systems are filled, in addition to the singular points, with the closed orbits and separatrices determined by the integrals

$$x_i^4 - 2\eta_i x_i^2 + 2y_i^2 = a_i^2 - \eta_i^2, \quad \text{or} \quad (x_i^2 - \eta_i)^2 + 2y_i^2 = a_i^2 \quad (a_1, \ldots, a_n \geq 0). \tag{1.3}$$

Thus, the autonomous system (1.2) has 3^ι ($\iota \in \{\overline{0, n}\}$) singular points, depending on the values of η_1, ..., η_n, and its phase space of dimension $2n$ is filled with the invariant n-tori determined by integrals (1.3) with a_1, ..., $a_n \neq 0$, 1, as well as with the singular points, separatrix surfaces, and tori of lower dimensions determined by integrals (1.3) with other constants a_1, ..., a_n.

1.2. Formulation of the Problem and Results

The goal of this paper is to find, for system (1.1) with any sufficiently small $\varepsilon > 0$, a certain number of invariant surfaces homeomorphic to the $(n + 1)$-torus in the cylindrical phase space of (1.1), which is obtained by factoring time with respect to the period. The projections of such surfaces on the space of the variables x and y are contained in a small neighborhood of certain invariant n-tori of the unperturbed system (1.2); thereby, they are preserved under small periodic perturbations.

Systems (1.1) with $\nu = 0$ and $\nu = 1$ can be analyzed simultaneously, because the invariant tori can be found by the same method, which was developed in [1, 2] and substantially modified in [3–5].

As a result, we can explicitly write out conditions on the functions $X_i(t, x, y, 0)$ and $Y_i(t, x, y, 0)$ under which the perturbed system (1.1) has invariant surfaces specified above (these conditions depend on γ_i and η_i), derive formulas for these surfaces, and obtain the asymptotic expansion of each of them in powers of ε.

The constructiveness of the developed method is confirmed by the example of the four-dimensional system (1.1) with perturbation analytic at $\varepsilon = 0$. The choice of average value makes it possible to find six points such that, through their small neighborhoods, different invariant three-dimensional surfaces of system (1.1) homeomorphic to the torus pass, provided that Siegel's condition on the periods and the dissipativity condition hold.

2. PARAMETERIZATION OF THE ORBITS OF THE UNPERTURBED SYSTEM

2.1. Construction of the Phase Portrait

For each $i = \overline{1, n}$, consider the two-dimensional autonomous system

$$C_i'(\varphi_i) = -\gamma_i S_i(\varphi_i), \quad S_i'(\varphi_i) = \gamma_i(C_i^3(\varphi_i) - \eta_i C_i(\varphi_i)) \quad (\gamma_i > 0, \eta_i \in \{0, \pm 1\}), \tag{2.1}$$

whose integrals are the functions

$$C_i^4 - 2\eta_i C_i^2 + 2S_i^2 = a_i^2 - \eta_i^2 \quad \text{or} \quad (C_i^2 - \eta_i)^2 + 2S_i^2 = a_i^2 \quad (a_i \geq 0). \tag{2.2}$$

First, consider the case where $\eta_i = 1$.

For $a_i = 0$, integral (2.2) degenerates into the two singular points (1, 0) and (–1, 0) (in the coordinates C_i, S_i) of system (2.1).

For $a_i = 1$, integral (2.2) determines the singular point (0,0) of system (2.1) and the separatrices $\Gamma_{i,-1}$ and $\Gamma_{i,1}$ adjacent to this point. Each separatrix Γ_{ij} ($j = \pm 1$) passes through the extremal points $(2^{1/2}j, 0)$ and $(j, \pm 2^{-1/2})$; hence, on it, $jC_i \in (0, 2^{1/2}]$ and $S_i \in [-2^{-1/2}, 2^{-1/2}]$.

For $a_i \in (0, 1)$, integral (2.2) determines two closed orbits $l_{i,-1}$ and l_{i1} of system (2.1). Each orbit l_{ij} ($j = \pm 1$) lies inside Γ_{ij} and encloses the point $(j, 0)$; moreover, for it, $jC_i \in [(1 - a_i)^{1/2}, (1 + a_i)^{1/2}]$, where $(1 \mp a_i)^{1/2}$ are the intersection points of l_{ij} with the abscissa axis.

For $a_i > 1$, integral (2.2) determines the closed orbit l_{i0} of system (2.1) enclosing $\Gamma_{i,-1} \cup \Gamma_{i1}$, and for this orbit, $C_i \in [-(a_i + 1)^{1/2}, (a_i + 1)^{1/2}]$, where the boundary values are the intersection points of l_{i0} with the OC_i axis.

Now consider the case where $\eta_i = 0, -1$. Then the only rest point of system (2.1) is (0, 0), and the integral (2.2) degenerates into this point at $a_i = |\eta_i|$.

All other orbits l_{i0} are closed, enclose the origin, and are given by integrals (2.2) with $a_i > |\eta_i|$, and for these orbits, $C_i \in [-(a_i - \eta_i)^{1/2}, (a_i - \eta_i)^{1/2}]$.

2.2. Parameterization of Closed Orbits

Based on the obtained range of variation of the coordinate C_i of any closed orbit l_{ik_i} ($k_i = 0, \pm 1$) of system (2.1), we introduce the following constants c_{ik_i} ($i = \overline{1, n}$):

$$c_{i0} \in (\sqrt{\eta_i(\eta_i + 1)}, +\infty) \quad \text{if} \quad \eta_i = 0, \pm 1, \quad jc_{i,j} \in (1, 2^{1/2}) \ (j = \pm 1) \quad \text{if} \quad \eta_i = 1. \tag{2.3}$$

Then the orbits l_{i0} for $\eta_i = 0, \pm 1$ and l_{i1}, $l_{i,-1}$, for $\eta_i = 1$ are parameterized by the solutions of the initial value problem $C_i(\varphi_i) = C_i(\varphi_i, c_{ik_i})$, $S_i(\varphi_i) = S_i(\varphi_i, c_{ik_i})$ for system (2.1) with initial conditions

$$C_i(0) = c_{ik_i}, \quad S_i(0) = 0 \quad (k_i = 0, \pm 1). \tag{2.4}$$

Thereby, geometrically, each c_{ik_i} determines the maximum absolute value of C_i on the parameterized closed orbit l_{ik_i}.

The solution of system (2.1) with initial condition (2.4) is a real-analytic $\omega(c_{ik_i})$-periodic function φ_i. Moreover, the function $C_i(\varphi_i)$ is even and $S_i(\varphi_i)$ is odd, because the solution $(C_i(-\varphi_i), -S_i(-\varphi_i))$ of system (2.1) also satisfies the initial condition (2.4).

Moreover, $C_i'(0) = 0$ and $S_i'(0) = \gamma_i c_{ik_i}(c_{ik_i}^2 - \eta_i) > 0$ for all c_{ik_i} in (2.3). Therefore, for $\varphi_i = 0$, any orbit l_{ik_i} from the point $(c_{ik_i}, 0)$ goes counterclockwise with increasing φ_i.

Convention. All functions and constants introduced in what follows depend in some way on the initial data c_{ik_i} ($k_i = 0, \pm 1$); we sometimes omit the second subscript k_i, when its particular value does not matter. For example, $c_i = c_{ik_i}$.

Let $b_i = b_{ik_i}$ be the second intersection point of the orbit l_{ik_i} with the OC_i axis.

The choice of c_{ik_i} in (2.3) fixes $b_i = b_{ik_i}$ and the constant $a_i = a_{ik_i}$ in integral (2.2), namely,

$$a_i = c_i^2 - \eta_i; \quad b_{i0} = -c_{i0}, \quad b_{ij} = j(2 - c_{ij}^2)^{1/2} \quad (j = \pm 1); \tag{2.5}$$

moreover, $0 < |b_{ij}| < 1 < |c_{ij}| < 2^{1/2}$ and $|c_{ij}| - 1 < 1 - |b_{ij}|$.

2.3. Calculation of Periods

Let us calculate the period $\omega(c_{ik_i})$ of the trajectory of a given closed orbit l_{ik_i} of system (2.1). There exists a φ_i^* such that $C_i(\varphi_i^*) = b_i$ and $S_i(\varphi_i^*) = 0$ ($0 < \varphi_i^* < \omega(c_i)$).

Let $\varphi_i \in [0, \varphi_i^*]$. In this case, if $c_i > 0$, then $S_i(\varphi_i) > 0$, and if $c_i < 0$, then $S_i(\varphi_i) < 0$. Therefore, in integral (2.2), $S_i = \operatorname{sign}(c_i) \times 2^{-1/2}(a_i^2 - (C_i^2 - \eta_i)^2)^{1/2}$, and in system (2.1), $d\varphi_i = -\operatorname{sign}(c_i) \times 2^{1/2}\gamma_i^{-1}(a_i^2 - (C_i^2(\varphi_i) - \eta_i)^2)^{-1/2} dC_i(\varphi_i)$. Integrating this equality with respect to φ_i from 0 to φ_i^*, we obtain $\varphi_i^* = I_*$, where $I_* = \operatorname{sign}(c_i)\dfrac{2^{1/2}}{\gamma_i}\displaystyle\int_{b_i}^{c_i} \frac{dC_i}{((c_i^2 - \eta_i)^2 - (C_i^2 - \eta_i)^2)^{1/2}}$.

Let us continue to move on the orbit, considering $\varphi_i \in [\varphi_i^*, \omega(c_{ik_i})]$. We have $c_i S_i(\varphi_i) < 0$. Therefore, the sign on the right-hand sides of the expressions for S_i and $d\varphi_i$ changes, and integrating the latter expression with respect to φ_i from φ_i^* to $\omega(c_i)$ yields $\omega(c_j) - \varphi_i^* = I_*$.

As a result, we obtain $\varphi_i^* = \omega(c_i)/2$, $C_i(\omega(c_i)/2) = b_i$, $S_i(\omega(c_i)/2) = 0$, and

$$\omega(c_i) = \operatorname{sign}(c_i)\frac{2^{3/2}}{\gamma_i}\int_{b_i}^{c_i} \frac{dC_i}{((c_i^2 - \eta_i)^2 - (C_i^2 - \eta_i)^2)^{1/2}} \quad (i = \overline{1, n}). \tag{2.6}$$

2.4. The Choice of Generating Invariant Surfaces

We set $\varphi = (\varphi_1, \ldots, \varphi_n)$, $C(\varphi) = (C_1(\varphi_1), \ldots, C_n(\varphi_n))$, $S(\varphi) = (S_1(\varphi_1), \ldots, S_n(\varphi_n))$, $k = (k_1, \ldots, k_n)$, $c^k = (c_{1k_1}, \ldots, c_{nk_n})$, $\omega(c^k) = (\omega(c_{1k_1}), \ldots, \omega(c_{nk_n}))$, and

$$\int_0^{\omega(c^k)} v(\varphi)d\varphi = \int_0^{\omega(c_1)} \ldots \int_0^{\omega(c_n)} v(\varphi_1, \ldots, \varphi_n)d\varphi_1 \ldots d\varphi_m.$$

For system (1.1), we introduce the defining system of equations

$$\int_0^T \int_0^{\omega(c^k)} (S_i' X_i(t, C, S, 0) - C_i' Y_i(t, C, S, 0))d\varphi dt = 0 \quad (i = \overline{1, n}), \tag{2.7}$$

in which $C_i = C_i(\varphi_i, c_{ik_i})$, $S_i = S_i(\varphi_i, c_{ik_i})$ is an $\omega(c_{ik_i})$-periodic (in φ_i) solution of the Cauchy problem for system (2.1) with initial condition (2.4) satisfying conditions (2.3), C_i' and S_i' are the derivatives of this solution with respect to φ_i, and X_i and Y_i are perturbations of (1.1) T-periodic in t.

We refer to a vector c^k which is a solution of system (2.7) as an admissible vector.

Remark 2. In what follows, we shall be interested in systems (1.1) with nonempty sets of admissible vectors. For such systems, is will be proved that any admissible vector c^k satisfying two additional conditions, which will be specified for perturbations of system (1.1) with $\varepsilon = 0$ in Sections 4.3 and 5.1, determines the point $(c^k, 0)$; through a small neighborhood of this point, for all sufficiently small ε, an invariant $(n + 1)$-surface retained in the T-periodically perturbed system (1.1) passes.

3. PASSAGE TO A NEIGHBORHOOD OF THE CHOSEN INVARIANT SURFACE OF THE UNPERTURBED SYSTEM

3.1. Shifts of the Unperturbed System to Singular Points

Take any admissible vector c^k. For each $i = \overline{1, n}$, it fixes a solution of system (2.1) parameterizing a closed orbit l_{ik_i} enclosing a singular point $(k_i, 0)$. This distinguishes the singular point $(k_1, \ldots, k_n, 0, \ldots, 0)$ of the unperturbed system (1.2) and the invariant surface $l_{1k_1} \times \ldots \times l_{nk_n}$ enclosing this point; in a small neighborhood of this surface, we shall seek an invariant surface of system (1.1).

It will be seen in what follows that, for $\eta_i = 1$, we have to move the origin in system (2.1) to the singular point $(k_i, 0)$. Therefore, in addition to system (2.1), we shall consider two more systems obtained from (2.1) by shifting the variable C_i by 1 to the right or left.

In other words, in system (2.1), we must make the change

$$C_i = \breve{C}_i + k_i, \quad S_i = \breve{S}_i \quad (k_i = 0, \pm 1); \tag{3.1}$$

as a result, we obtain the system

$$\breve{C}_i'(\varphi_i) = -\gamma_i \breve{S}_i(\varphi_i), \quad \breve{S}_i'(\varphi_i) = \gamma_i(\breve{C}_i^3(\varphi_i) + 3k_i\breve{C}_i^2(\varphi_i) + (3k_i^2 - \eta_i)\breve{C}_i(\varphi_i)) \quad (i = \overline{1, n}), \tag{3.2}$$

which has integrals

$$((\breve{C}_i + k_i)^2 - \eta_i)^2 + 2\breve{S}_i^2 = \breve{a}_i^2 \quad (\breve{a}_i \geq 0). \tag{3.3}$$

Now, any closed orbit l_{ik_i} is determined both by integral (2.2) and by the corresponding integral (3.3), and it is parameterized by a solution of the initial value problem $\breve{C}_i(\varphi_i) = \breve{C}_i(\varphi_i, \breve{c}_{ik_i})$, $\breve{S}_i(\varphi_i) = \breve{S}_i(\varphi_i, \breve{c}_{ik_i})$ for system (3.2) with initial condition

$$\breve{C}_i(0) = \breve{c}_i, \quad \breve{S}_i(0) = 0 \quad (k_i = 0, \pm 1), \tag{3.4}$$

where $\breve{c}_i = \breve{c}_{ik_i} = c_{ik_i} - k_i$; hence, it follows from (2.3) that $\breve{c}_{i0} \in (\sqrt{\eta_i(\eta_i + 1)}, -\infty)$ for $\eta_i = 0, \pm 1$ and $\breve{c}_{i,-1} \in (1 - 2^{1/2}, 0)$ and $\breve{c}_{i1} \in (0, 2^{1/2} - 1)$ for $\eta_i = 1$.

The choice of $\breve{c}_i$ fixes $\breve{a}_i = \breve{a}_{ik_i}$ in (3.3) and $\breve{b}_i = \breve{b}_{ik_i} = b_i - k_i$, which is the second intersection point of the orbit l_{ik_i} ($k_i = 0, \pm 1$) with the $O\breve{C}_i$ axis, namely,

$$\begin{aligned} &\breve{a}_{i0} = a_{i0}(= c_{i0}^2 - \eta_i), \quad \breve{b}_{i0} = b_{i0}(= -c_{i0}), \\ &\breve{a}_{ij} = (\breve{c}_{ij} + 2j)\breve{c}_{ij}, \quad \breve{b}_{ij} = j((2 - (\breve{c}_{ij} + j)^2)^{1/2} - 1)) \quad (j = \pm 1); \end{aligned} \tag{3.5}$$

moreover, $\breve{a}_{ij} \in (0, 1)$, $|\breve{b}_{ij}| > |\breve{c}_{ij}|$, and $j\breve{C}_i(\varphi_i, \breve{c}_{ij}) \in [\breve{b}_{i1}, \breve{c}_{i1}]$ for $\breve{c}_{ij}$ from (3.4).

Obviously, the period of the solution of system (2.1) parameterizing an arbitrary closed orbit l_{ik_i} does not change under the passage to system (3.2); i.e., for any initial value $\breve{c}_i$ in (3.4), we have

$$\omega(\breve{c}_{ik_i}) = \omega(c_{ik_i}) \quad (i = \overline{1, n}). \tag{3.6}$$

3.2. Study of the Monotonicity of the Angular Variable

For each $\breve{c}_{ik_i}$ in (3.4), we introduce the function $\alpha_i(\breve{C}_i) = \alpha_{ik_i}(\breve{C}_i(\varphi_i, \breve{c}_{ik_i}))$ defined by

$$\alpha_i(\breve{C}_i) = \{\breve{a}_{i0}^2 - \eta_i^2 + \eta_i\breve{C}_i^2 \ (\eta_i = 0, \pm 1), \breve{a}_{ij}^2 - j\breve{C}_i^3 - 2\breve{C}_i^2 \ (\eta_i = 1, j = \pm 1)\}. \tag{3.7}$$

Then, for the solutions of system (3.2) with initial values 0, $(\breve{c}_{ik_i}, 0)$, we have

$$\alpha_i = \gamma_i^{-1}(\breve{C}_i\breve{S}_i' - 2\breve{C}_i'\breve{S}_i) \quad (k_i = 0, \pm 1). \tag{3.8}$$

Indeed, substituting the right-hand sides of system (3.2) into (3.8) instead of $\breve{C}_i'(\varphi_i)$ and $\breve{S}_i'(\varphi_i)$ and using formula (3.3), we obtain (3.7).

Now let us show that, for any c_{ik_i}, the function α_{ik_i} is positive.

A. Let $k_i = 0$. If $\eta_i = 0$, then $\alpha_{i0} = c_{i0}^4 > 0$; if $\eta_i = 1$, then $\alpha_{i0} = (c_{i0}^2 - 1)^2 - 1 + \breve{C}_i^2 > 0$, because $c_{i0}^2 > 2$; if $\eta_i = -1$, then $\alpha_{i0} = (c_{i0}^2 + 1)^2 - 1 - \breve{C}_i^2 \geq 2c_{i0} > 0$, because $|\breve{C}_i| = |C_i| \leq |c_{i0}|$.

B. Let $k_i = j$ ($\eta_i = 1$). Since $\alpha_{ij}'(\breve{C}_i) = -(3j\breve{C}_i + 4)\breve{C}_i$ and $|\breve{C}_i(\varphi_i)| < 1$, then $\alpha_{ij}(\breve{C}_i)$ increases for $\breve{C}_i < 0$ and decreases for $\breve{C}_i > 0$. Therefore, in view of (3.5), it suffices to check its positivity at the endpoints:

$$\begin{aligned} \alpha_{ij}(\breve{c}_{ij}) &= (\breve{c}_{ij} + 2j)^2\breve{c}_{ij}^2 - (j\breve{c}_{ij} + 2)\breve{c}_{ij}^2 = (j\breve{c}_{ij} + 2)(j\breve{c}_{ij} + 1)\breve{c}_{ij}^2 > 0, \\ \alpha_{ij}(\breve{b}_{ij}) &= (\breve{c}_{ij} + 2j)^2\breve{c}_{ij}^2 - (j\breve{b}_{ij} + 2)\breve{b}_{ij}^2 = (\breve{c}_{ij} + 2j)\breve{c}_{ij}((2 - (\breve{c}_{ij} + j)^2)^{1/2} - (2 - (\breve{c}_{ij} + j)^2)) > 0, \end{aligned}$$

given that

$$2 - (\breve{c}_{ij} + j)^2 \in (0,1).$$

Remark 3. If $\eta_i = 1$ and the closed orbits l_{ij} lying inside Γ_{ij} are parameterized by solutions of system (2.1), i.e., the c_{ij} are chosen from (2.4) without passing to system (3.2), then the function $\alpha_{ij}(C_i) = \gamma_i^{-1}(C_iS_i' - 2C_i'S_i)$ introduced on the solutions of system (2.1) is alternating. Indeed, $\alpha_{ij}(c_{ij}) = (c_{ij}^2 - 1)c_{ij}^2 > 0$ and $\alpha_{ij}(b_{ij}) = (c_{ij}^2 - 2)(c_{ij}^2 - 1) < 0$, because $jc_{ij} \in (1, 2^{1/2})$.

Geometrically, this fact means the absence of monotonicity with respect to the angular variable in moving on orbits lying inside the separatrices Γ_{ij}; thus, it is necessary to perform shift (3.1), because the monotonicity of the functions α_{ik_i} makes it possible to pass to neighborhoods of such orbits.

3.3. Shifts in the Perturbed System

Before passing to a small neighborhood of the chosen invariant surface $l_{1k_1} \times \ldots \times l_{nk_n}$ of the unperturbed system (1.2), in system (1.1), we perform the shift to the point $(k_1, \ldots, k_n, 0, \ldots, 0)$ with the given k_i similar to the shift of (3.1) for each system (2.1) ($i = \overline{1, n}$), i.e., make the variable change

$$x_i = \breve{x}_i + k_i, \quad y_i = \breve{y}_i \quad (i = \overline{1, n}), \tag{3.9}$$

where $k_i \in \{0, \pm 1\}$ if $\eta_i = 1$ and $k_i = 0$ if $\eta_i = -1, 0$.

As a result, we obtain the system

$$\begin{cases}\varepsilon^{-\nu}\dot{\breve{x}}_i = -\gamma_i\breve{y}_i + \breve{X}_i(t,\breve{x},\breve{y},\varepsilon)\varepsilon,\\ \varepsilon^{-\nu}\dot{\breve{y}}_i = \gamma_i(\breve{x}_i^3 + 3k_i\breve{x}_i^2 - (\eta_i - 3k_i^2)\breve{x}_i) + \breve{Y}_i(t,\breve{x},\breve{y},\varepsilon)\varepsilon \quad (\nu\in\{0,1\}),\end{cases} \tag{3.10}$$

in which $\breve{X}_i(t,\breve{x},\breve{y},\varepsilon) = X_i(t,\breve{x}+k,\breve{y},\varepsilon)$ and $\breve{Y}_i$ is defined in a similar way ($i=\overline{1,n}$).

Obviously, the closed orbits l_{ik_i} of system (3.10) without the perturbations $\breve{X}_i$ and $\breve{Y}_i$ are parameterized by solutions of the n initial value problems $\breve{C}_i(\varphi_i) = \breve{C}_i(\varphi_i,\breve{c}_{ik_i})$, $\breve{S}_i(\varphi_i) = \breve{S}_i(\varphi_i,\breve{c}_{ik_i})$ of system (3.2) with the $\breve{c}_{ik_i}$ determined by (3.4).

3.4. A Nonnormalized Generalized Polar Change

Now, in system (3.10), we make the nonnormalized generalized polar change

$$\breve{x}_i = \breve{C}_i(\varphi_i)(1+r_i), \quad \breve{y}_i = \breve{S}_i(\varphi_i)(1+r_i)^2 \quad (|r_i| < r_0 \le 1, i=\overline{1,n}), \tag{3.11}$$

where each $\breve{C}_i(\varphi_i)$, $\breve{S}_i(\varphi_i)$ is the $\omega(\breve{c}_i)$-periodic real analytic solution of the initial value problem for system (3.2) with initial values 0, ($\breve{c}_{ik_i}$, 0); here the $\breve{c}_{ik_i}$ are determined by (3.4) and (3.6) holds.

For brevity, we introduce the notation

$$\begin{gathered} s_i = 1 + r_i;\quad v_i^{\xi} = \partial v_i(\xi,\varepsilon)/\partial\xi, \quad v_i^{\varepsilon} = \partial v_i(\xi,\varepsilon)/\partial\varepsilon;\\ (\kappa) = (t,\breve{C}_1s_1,\dots,\breve{C}_ns_n,\breve{S}_1s_1^2,\dots,\breve{S}_ns_n^2,\varepsilon), \quad (\kappa_0) = (t,\breve{C},\breve{S},0);\\ R_i(t,\varphi,r,\varepsilon) = \gamma_i^{-1}(\breve{S}_i'\breve{X}_i(\kappa) - \breve{C}_i'(1+r_i)^{-1}\breve{Y}_i(\kappa)),\\ \Phi_i(t,\varphi,r,\varepsilon) = \gamma_i^{-1}\alpha_i^{-1}(1+r_i)^{-1}(\breve{C}_i(1+r_i)^{-1}\breve{Y}_i(\kappa) - 2\breve{S}_i\breve{X}_i(\kappa)).\end{gathered} \tag{3.12}$$

Let us differentiate change (3.11) with respect to system (3.10):

$$\varepsilon^{-\nu}(\breve{C}_i\dot{r}_i + \breve{C}_i's_i\dot{\varphi}_i) = -\gamma_i\breve{S}_is_i^2 + \breve{X}_i(\kappa)\varepsilon,$$

$$\varepsilon^{-\nu}(2\breve{S}_is_i\dot{r}_i + \breve{S}_i's_i^2\dot{\varphi}_i) = \gamma_i(\breve{C}_i^3s_i^3 + 3k_i\breve{C}_i^2s_i^2 - (\eta_i - 3k_i^2)\breve{C}_is_i) + \breve{Y}_i(\kappa)\varepsilon.$$

Then, we solve the system thus obtained with respect to $\dot{r}_i$ and $\dot{\varphi}_i$:

$$\varepsilon^{-\nu}(\breve{C}_i\breve{S}_i' - 2\breve{C}_i'\breve{S}_i)s_i\dot{r}_i = -\gamma_is_i(\breve{S}_i\breve{S}_i's_i^2 + (\breve{C}_i^2s_i^2 + 3k_i\breve{C}_i^2s_i - (\eta_i - 3k_i^2)\breve{C}_i)\breve{C}_i') + (\breve{S}_i's_i\breve{X}_i - \breve{C}_i'\breve{Y}_i)\varepsilon,$$

$$\varepsilon^{-\nu}(\breve{C}_i\breve{S}_i' - 2\breve{C}_i'\breve{S}_i)s_i^2\dot{\varphi}_i = \gamma_is_i(\breve{C}_i^4s_i^2 + 2\breve{S}_i^2s_i^2 + 3k_i\breve{C}_i^3s_i - (\eta_i - 3k_i^2)\breve{C}_i^2) + (\breve{C}_i\breve{Y}_i - 2\breve{S}_is_i\breve{X}_i)\varepsilon.$$

Taking into account relations (3.8) and (3.12) and reducing by γ_is_i, we obtain the system

$$\begin{cases}\varepsilon^{-\nu}\alpha_i\dot{r}_i = -\breve{C}_i'\Theta_i(\breve{C}_i,r_i) + R_i(t,\varphi,r,\varepsilon)\varepsilon,\\ \varepsilon^{-\nu}\alpha_i\dot{\varphi}_i = \alpha_is_i + \breve{C}_is_i^{-1}\Theta_i(\breve{C}_i,r_i) + \alpha_i\Phi_i(t,\varphi,r,\varepsilon)\varepsilon,\end{cases} \tag{3.13}$$

where $\Theta_i = (1-s_i)(3k_i\breve{C}_is_i - (\eta_i - 3k_i^2)(s_i+1))\breve{C}_i = \alpha_i'r_i + (\alpha_i'/2 - 3k_i\breve{C}_i^2/2)r_i^2$, because, by virtue of (3.7), we have $\alpha_{i0}'(\breve{C}_i) = 2\eta_i\breve{C}_i$ and $\alpha_{ij}'(\breve{C}_i) = -(3j\breve{C}_i + 4)\breve{C}_i$.

To complete change (3.11), we divide the equations of system (3.13) by the function $\alpha_i(\breve{C}_i(\varphi_i))$, which is of fixed sign for the chosen initial values $\breve{c}_{1k_1}$, ..., $\breve{c}_{nk_n}$, and single out terms of lower order in r_1, ..., r_n and ε in the perturbations R_i and Φ_i of system (3.13) described in (3.12).

As a result, making change (3.11) and taking into account (3.12), we reduce system (3.10) to the form

$$\begin{cases}\varepsilon^{-\nu}\dot{r}_i = -\dfrac{\breve{C}_i'}{\alpha_i}(\alpha_i'r_i + \alpha_i^*r_i^2) + \alpha_i^{-1}\left(R_{io} + \displaystyle\sum_{l=1}^{n}R_{io}^{r_l}r_l + R_{io}^{\varepsilon}\varepsilon + O((|r|+\varepsilon)^2)\right)\varepsilon,\\ \varepsilon^{-\nu}\dot{\varphi}_i = 1 + \alpha_i\rho_ir_i + O(|r|^2) + (\Phi_{io} + O(|r|+\varepsilon))\varepsilon \quad (i=\overline{1,n}),\end{cases} \tag{3.14}$$

where $R_{io}(t,\varphi) = R_i(t,\varphi,0,0) = \gamma_i^{-1}(\breve{S}_i'\breve{X}_i(\kappa_0) - \breve{C}_i'\breve{Y}_i(\kappa_0))$, $\Phi_{io}(t,\varphi) = \Phi_i(t,\varphi,0,0) = \gamma_i^{-1}\alpha_i^{-1}(\breve{C}_i\breve{Y}_i(\kappa_0) - 2\breve{S}_i\breve{X}_i(\kappa_0))$, $R_{io}^{r_l}(t,\varphi) = \gamma_i^{-1}\breve{S}_i'(\breve{C}_l\breve{X}_i^{x_l}(\kappa_0) + 2\breve{S}_l\breve{X}_i^{y_l}(\kappa_0)) - \gamma_i^{-1}\breve{C}_i'(\breve{C}_l\breve{Y}_i^{x_l}(\kappa_0) + 2\breve{S}_l\breve{Y}_i^{y_l}(\kappa_0) - \sigma_{il}\breve{Y}_i(\kappa_0))$,

$R^\varepsilon_{io}(t,\varphi) = \gamma_i^{-1}(\breve{S}_i'\breve{X}_i^\varepsilon(\kappa_0) - \breve{C}_i'\breve{Y}_i^\varepsilon(\kappa_0))$ (σ_{ij} is the Kronecker delta), $\alpha_i^*(\breve{C}_i) = \alpha_i'/2 - 3k_i\breve{C}_i^2/2$ is from Θ_i, and $\rho_i(\breve{C}_i) = \alpha_i^{-1}(1 + \alpha_i^{-1}\alpha_i'\breve{C}_i)$ ($\alpha_i(\breve{C}_i)$ is from (3.7)).

4. PRIMARY AVERAGING IN THE RADIAL EQUATIONS OF THE SYSTEM

4.1. Decompositions of Two-Periodic Functions

For continuous functions $\upsilon^\nu(t,\varphi)$ T-periodic in t and $\omega(\breve{c}_i) = \omega(c_i)$-periodic in φ_i, we use the following decomposition depending on the parameter ν:

$$\upsilon^\nu(t,\varphi) = \bar{\upsilon}^\nu + \nu\hat{\upsilon}^\nu(\varphi) + \tilde{\upsilon}^\nu(t,\varphi) \quad (\upsilon = 0,1),$$

in which $\bar{\upsilon}^\nu = \left(T\prod_{m=1}^n \omega(c_m)\right)^{-1}\int_0^T\int_0^{\omega(c^k)} \upsilon^\nu(t,\varphi)d\varphi dt$ is the average value of υ^ν and $\hat{\upsilon}^1 = T^{-1}\int_0^T \upsilon^1(t,\varphi)dt - \bar{\upsilon}^1$. Therefore, $\tilde{\upsilon}^1 = \upsilon^1(t,\varphi) - T^{-1}\int_0^T \upsilon^1(t,\varphi)dt$ has zero average value with respect to t, which implies the periodicity of the function $\int_{t_0}^t \tilde{\upsilon}^1(\tau,\varphi)d\tau$, which also has zero average value by virtue of the choice of the constant $t_0 \in [0, T]$.

For the subsequent changes and systems to look uniformly, we introduce the functions

$$\check{\upsilon}^0 = \tilde{\upsilon}^0(t,\varphi), \quad \check{\upsilon}^1 = \upsilon^1(\varphi).$$

4.2. The Mean Value of R_{io}.

First, let us show that the functions $R_{io}(t, \varphi)$ in system (3.14) have zero average value, i.e.,

$$\bar{R}_{io} = 0 \quad (i = \overline{1,n}). \tag{4.1}$$

Relation (4.1) holds because we have chosen an admissible vector c^k of initial values, which satisfies system (2.7) by definition.

Indeed, we have $\gamma_i R_{io} \overset{(3.14),(3.12)}{=} \breve{S}_i'\breve{X}_i(t,\breve{C},\breve{S},0) - \breve{C}_i'\breve{Y}_i(t,\breve{C},\breve{S},0) \overset{(3.10)}{=} S_i'X_i(t, \breve{C} + k, \breve{S}, 0) - C_i'Y_i(t, \breve{C} + k, \breve{S}, 0) \overset{(3.1)}{=} S_i'X_i(t, C, S, 0) - C_i'Y_i(t, C, S, 0)$, whence $\bar{R}_{io} = \left(T\prod_{m=1}^n \omega(c_m)\right)^{-1}\int_0^T\int_0^{\omega(c^k)} R_{io}(t,\varphi)d\varphi dt = 0$ by virtue of (2.7).

Thus, $R_{io} = \tilde{R}_{io}$ for $\nu = 0$ and $R_{io} = \hat{R}_{io} + \tilde{R}_{io}$ for $\nu = 1$.

4.3. Preparation of Functions for the Primary Averaging Change

A. For each $i = \overline{1,n}$, consider the equation

$$\frac{\partial\tilde{g}_i^\nu}{\partial t} + \sum_{m=1}^n \frac{\partial \check{g}_i^\nu}{\partial\varphi_m} = R_{io}(t,\varphi) \quad (\nu = 0,1). \tag{4.2}$$

In view of (4.1), it decomposes into the three equations

$$\frac{\partial\tilde{g}_i^0}{\partial t} + \sum_{m=1}^n \frac{\partial\tilde{g}_i^0}{\partial\varphi_m} = \tilde{R}_{io}(t,\varphi), \quad \sum_{m=1}^n \frac{\partial\hat{g}_i^1}{\partial\varphi_m} = \hat{R}_{io}(\varphi), \quad \frac{\partial\tilde{g}_i^1}{\partial t} = \tilde{R}_{io}(t,\varphi). \tag{4.3}$$

For the solvability of Eqs. (4.3), it is sufficient that the periods $\omega_i = \omega(c_{ik_i})$ (calculated by (2.6)) of the solutions C_i, S_i specified in (2.7) and the period T, if $\nu = 0$, satisfy the Siegel condition

$$|l_0T(1-\nu) + l_1\omega_1 + \ldots + l_n\omega_n| > K(|l_0|(1-\nu) + |l_1| + \ldots + |l_n|)^{-\tau}, \tag{4.4}$$

in which $K > 0$, $\tau \geq 1$, l_0 and l_i are integers, and T and ν are the constants in system (1.1).

Then, by Lemma V.5 [6, p. 17], Eqs. (4.3) have solutions $\tilde{g}_i^0(t,\varphi)$ and $\hat{g}_i^1(\varphi)$ of the same differentiability class as the right-hand sides, i.e., continuous, real-analytic, and $\omega(c^k)$-periodic in φ; moreover, the solution $\tilde{g}_i^0(t,\varphi)$ is also T-periodic in t. These solutions are uniquely determined by their zero average value.

In particular, let $g_{io}^1(t,\varphi) = \int_{t_0}^t \tilde{R}_{io}(\tau,\varphi)d\tau$ be a function with zero average value periodic in t, φ. Then $\tilde{g}_i^1 = g_{io}^1(t,\varphi) - \hat{g}_{io}^1(\varphi)$ is the unique periodic solution of Eq. (4.3_3) with zero mean.

The solutions of Eqs. (4.3) determine the unique solution of Eq. (4.2).

Remark 4. The Siegel condition holds for almost all vectors with respect to the Lebesgue measure.

B. For each $k_i = 0, \pm 1$, we introduce the following auxiliary constants $c_i^* = c_{ik_i}^*$:

$$c_{i0}^* = \breve{c}_{i0}^2(\breve{c}_{i0}^2 - 2\eta_i), \quad c_{ij}^* = \breve{c}_{ij}^2(\breve{c}_{ij} + 2j)^2 \quad (i = \overline{1,n};\ j = \pm 1).$$

It is convenient to define a function $\beta_i = \beta_{ik_i}(\breve{C}_i)$ as a solution of the equation

$$\beta_i' = \alpha_i^{-2}(\alpha_i\alpha_i'\rho_i - \alpha_i^*). \tag{4.5}$$

To facilitate integration, we rewrite Eq. (4.5) in the form

$$\beta_i' = \alpha_i^{-3}\alpha_i'(5\alpha_i/2 - 2c_i^*) + k_i\alpha_i^{-3}\breve{C}_i^2(3\alpha_i/2 - \alpha_i'\breve{C}_i),$$

where $\alpha_i(\breve{C}_i)$ is the function from (3.7), $\alpha_i'(\breve{C}_i)$, $\alpha_i^*(\breve{C}_i)$, and $\rho_i(\breve{C}_i)$ are from (3.14), and the constants are from (3.5). Then we have

$$\beta_i(\breve{C}_i) = c_i^*\alpha_i^{-2}(\breve{C}_i) - 5/(2\alpha_i(\breve{C}_i)) + k_i\beta_i^*(\breve{C}_i) \quad (k_i = 0, \pm 1), \tag{4.6}$$

where $\beta_{ij}^* = \int_{\breve{c}_i}^{\breve{C}_i} \frac{\tau^2(3\alpha_{ij}(\tau) - 2\tau\alpha_{ij}'(\tau))}{2\alpha_{ij}^3(\tau)}d\tau = \int_{\breve{c}_i}^{\breve{C}_i} \frac{\tau^2(3c_{ij}^* + 3j\tau^3 + 2\tau^2)}{2(c_{ij}^* - j\tau^3 - 2\tau^2)^3}d\tau$ for $j = \pm 1$.

C. Using the obtained solution of Eq. (4.2) and the β_i' from Eq. (4.5), for any i, $l = \overline{1,n}$ and $\nu = 0, 1$, we introduce the periodic functions

$$V_{il}^\nu(t,\varphi) = \alpha_l^{-1}R_{io}^{r_l} - \rho_l\frac{\partial \breve{g}_i^\nu}{\partial\varphi_l} + \sigma_{il}\breve{C}_i'(\alpha_i^{-1}\alpha_i'\Phi_{io} + 2\breve{g}_i^\nu\beta_i'),$$

$$V_{i\varepsilon}^\nu(t,\varphi) = R_{io}^\varepsilon + \sum_{l=1}^n \alpha_l^{-1}R_{io}^{r_l}\breve{g}_l^\nu + \breve{C}_i'\breve{g}_i^\nu(\alpha_i^{-1}\alpha_i'\Phi_{io} + \breve{g}_i^\nu\beta_i') - \sum_{m=1}^n\left(\frac{\partial\breve{g}_i^\nu}{\partial\varphi_m}\Phi_{m\varepsilon}^\nu + \nu\frac{\partial\tilde{g}_i^\nu}{\partial\varphi_m}\right), \tag{4.7}$$

$$\Phi_{i\varepsilon}^\nu(t,\varphi) = \Phi_{io} + \rho_i\breve{g}_i^\nu.$$

D. Finally, for the same i, l, and ν, we consider the equation

$$\frac{\partial\tilde{h}_{il}^\nu}{\partial t} + \sum_{m=1}^n\frac{\partial\breve{h}_{il}^\nu}{\partial\varphi_m} = V_{il}^\nu - \bar{V}_{il}^\nu; \tag{4.8}$$

obviously, its right-hand side has zero average value. Therefore, Eq. (4.8) is similar to Eq. (4.2), and the functions $\tilde{h}_{il}^\nu(t,\varphi)$ and $\hat{h}_{il}^\nu(\varphi)$ in its solution are continuous, real-analytic, $\omega(c^k)$-periodic in φ, and T-periodic in t, and they are uniquely determined by their zero average value.

4.4. The Primary Averaging Change of Radial Variables

Let us show that, under condition (4.4), the doubly periodic (in φ and t) averaging change

$$r_i = \alpha_i^{-1}\left((1 + \beta_i v_i)v_i + (\breve{g}_i^\nu + \nu\tilde{g}_i^\nu\varepsilon)\varepsilon + \sum_{l=1}^n(\breve{h}_{il}^\nu + \nu\tilde{h}_{il}^\nu\varepsilon)v_l\varepsilon\right), \tag{4.9}$$

in which α_j is defined by (3.7), β_i is defined by (4.6), $\hat{g}_i^1$ and $\tilde{g}_i^\nu$ are the solutions of Eq. (4.2), and $\hat{h}_{il}^1$ and $\tilde{h}_{il}^\nu$ are the solutions of Eq. (4.8), transforms system (3.14) into the system

$$\varepsilon^{-\nu}\dot{v}_i = \sum_{l=1}^{n}\bar{V}_{il}^\nu v_l\varepsilon + V_{i\varepsilon}^\nu\varepsilon^2 + O((|v|+\varepsilon)^3), \quad \varepsilon^{-\nu}\dot{\varphi}_i = 1 + \rho_i v_i + \Phi_{i\varepsilon}^\nu\varepsilon + O((|v|+\varepsilon)^2), \tag{4.10}$$

where the function ρ_i, V_{il}^ν, $\Phi_{i\varepsilon}^\nu$, and $V_{i\varepsilon}^\nu$ are defined by (3.14) and (4.7) ($\nu = 0, 1$; $i = \overline{1, n}$).

Substituting (4.9) into the equations for $\dot{\varphi}_i$ in (3.14) and taking into account (4.7_3), we obtain equations for the angular variables of system (4.10).

Then, differentiating change (4.9) with respect to systems (3.14) and (4.10) and multiplying by $\alpha_i\varepsilon^\nu$, we obtain the following identity for each $i = \overline{1, n}$:

$$-\breve{C}_i'\alpha_i'\alpha_i^{-1}\left(v_i + \beta_i v_i^2 + \breve{g}_i^\nu\varepsilon + \nu\tilde{g}_i^\nu\varepsilon^2 + \sum_{l=1}^{n}\breve{h}_{il}^\nu v_l\varepsilon\right) - \breve{C}_i'\alpha_i^*\alpha_i^{-2}(v_i + \breve{g}_i^\nu\varepsilon)^2$$

$$+\left(R_{io} + \sum_{l=1}^{n}R_{io}^n\alpha_l^{-1}(v_l + \breve{g}_l^\nu\varepsilon) + R_{io}^\varepsilon\varepsilon\right)\varepsilon + O((|v|+\varepsilon)^3)$$

$$\equiv -\alpha_i^{-1}\alpha_i'\breve{C}_i'(1 + \rho_i v_i + \Phi_{i\varepsilon}^\nu\varepsilon)\left(v_i + \beta_i v_i^2 + \breve{g}_i^\nu\varepsilon + \nu\tilde{g}_i^\nu\varepsilon^2 + \sum_{l=1}^{n}\breve{h}_{il}^\nu v_l\varepsilon\right)$$

$$+\sum_{l=1}^{n}\bar{V}_{il}^\nu v_l\varepsilon + V_{i\varepsilon}^\nu\varepsilon^2 + \beta_i'\breve{C}_i'v_i^2 + \frac{\partial\tilde{g}_i^\nu}{\partial t}\varepsilon + \frac{\partial\tilde{h}_{il}^\nu}{\partial t}v_l\varepsilon$$

$$+\sum_{m=1}^{n}\left(\frac{\partial\breve{g}_i^\nu}{\partial\varphi_m}\varepsilon + \nu\frac{\partial\tilde{g}_i^\nu}{\partial\varphi_m}\varepsilon^2 + \sum_{l=1}^{n}\frac{\partial\breve{h}_{il}^\nu}{\partial\varphi_m}v_l\varepsilon\right)(1 + \rho_m v_m + \Phi_{m\varepsilon}^\nu\varepsilon).$$

Let us equate the coefficients of respective powers of v_i and ε.

For v_i, the identity degenerates, and for ε, we obtain Eq. (4.2).

For v_i^2, we obtain Eq. (4.5); its solution is given in (4.6).

For ε^2, we obtain the expression (4.7_2) for $V_{i\varepsilon}^\nu$, because, in this formula, according to (4.7_3) and (4.5), $\breve{C}_i'\breve{g}_i^\nu(\alpha_i^{-1}\alpha_i'\Phi_{io} + \breve{g}_i^\nu\beta_i') = \alpha_i^{-1}\alpha_i'\breve{C}_i'\breve{g}_i^\nu\Phi_{i\varepsilon}^\nu - \alpha_i^{-2}\alpha_i^*\breve{C}_i'(\breve{g}_i^\nu)^2$.

Finally, for $v_l\varepsilon$, we obtain Eq. (4.8), because in V_{il}^ν, we have $\sigma_{il}\breve{C}_i'(\alpha_i^{-1}\alpha_i'\Phi_{io} + 2\breve{g}_i^\nu\beta_i') = \sigma_{il}\alpha_i^{-2}\breve{C}_i'(\alpha_i\alpha_i'(\rho_i\breve{g}_i^\nu + \Phi_{i\varepsilon}^\nu) - 2\alpha_i^*\breve{g}_i^\nu)$ from (4.7_1).

Thus, we have obtained the partially averaged system (4.10).

5. FINAL AVERAGINGS UNDER THE DISSIPATIVITY CONDITION

5.1. The Dissipativity Condition

To performing further averagings of system (3.14), in addition to the already used conditions (2.7) and (4.4), we need one more assumption, which can be called the dissipativity condition. This condition consists in that the matrix $\bar{V}^\nu$ composed of the average values of the functions $V_{il}^\nu(t, \varphi)$ defined in (4.7) has no eigenvalues with zero real parts.

Thus, we shall assume that, in system (4.10),

$$\bar{V}^\nu = \{\bar{V}_{il}^\nu\}_{i,l=1}^{n} \text{ is a noncritical matrix.} \tag{5.1}$$

Remark 5. It is easy to check that, in the matrix V^ν, only the functions $X(t, x, y, 0)$ and $Y(t, x, y, 0)$ from the perturbed part of system (1.1) are used.

Certainly, the violation of condition (5.1) is an exceptional case, in which it is possible to continue seeking an invariant surface but of higher dimension. Thus, in [7], a constructive process was proposed for obtaining classes of real autonomous systems of order 2^d ($d \geq 1$), including polynomial ones, in which the bifurcation of the birth of an invariant torus of codimension 1 occurs for all sufficiently small positive values of the parameter.

5.2. The Secondary Averaging Change of Radial Variables

Assumption (5.1) makes it possible to annihilate the functions $V^\nu_{i\varepsilon}(t,\varphi)$ in the radial equations of system (4.10) by means of the two-periodic change

$$v_i = u_i + \bar{f}^\nu_i\varepsilon + \check{f}^\nu_i\varepsilon^2 + \nu^2\tilde{f}^\nu_i\varepsilon^3 \quad (\check{f}^0_i = \tilde{f}^0_i(t,\varphi), \check{f}^1_i = \hat{f}^1_i(\varphi)), \tag{5.2}$$

which transforms system (4.10) into the system

$$\varepsilon^{-\nu}\dot{u}_i = \sum_{l=1}^{n}\bar{V}^\nu_{il}u_l\varepsilon + O((|u| + \varepsilon)^3), \quad \varepsilon^{-\nu}\dot{\varphi}_i = 1 + \rho_i u_i + \Psi^\nu_{i\varepsilon}\varepsilon + O((|u| + \varepsilon)^2) \tag{5.3}$$

with $\Psi^\nu_{i\varepsilon}(t,\varphi) = \Phi^\nu_{i\varepsilon} + \bar{f}^\nu_i\rho_i$ ($\nu = 0, 1$; $i = \overline{1,n}$).

Differentiating change (5.2) with respect to (4.10) and (5.3), we obtain the equation

$$\frac{\partial\tilde{f}^\nu_i}{\partial t} + \sum_{m=1}^{n}\frac{\partial\check{f}^\nu_i}{\partial\varphi_m} = \sum_{l=1}^{n}\bar{V}^\nu_{il}\bar{f}^\nu_l + V^\nu_{i\varepsilon}; \tag{5.4}$$

to make it solvable, it is necessary to zero the average value of the right-hand side, which can easily be done, thanks to condition (5.1), by setting

$$\bar{f}^\nu = -(\bar{V}^\nu)^{-1}\bar{V}^\nu_\varepsilon \quad (f^\nu = (f^\nu_1,\dots,f^\nu_n)), V^\nu_\varepsilon = (V^\nu_{1\varepsilon},\dots,V^\nu_{n\varepsilon})). \tag{5.5}$$

After this, Eq. (5.4) takes the form $\frac{\partial\tilde{f}^\nu_i}{\partial t} + \sum_{m=1}^{n}\frac{\partial\check{f}^\nu_i}{\partial\varphi_m} = \hat{V}^\nu_{i\varepsilon} + \tilde{V}^\nu_{i\varepsilon}$, that matches Eq. (4.2). This allows us to explicitly find the required solutions $\check{f}^\nu_i$ and $\hat{f}^1_i$, along with the constant vector $\bar{f}^\nu$, from (5.5).

5.3. The Averaging Change of Angular Variables

Now we simplify the equations for the angular variables of system (5.3). We average the functions $\Psi^\nu_{i\varepsilon}$ in these equations by means of the following periodic change of angular variables:

$$\varphi_i = \psi_i + \check{\xi}^\nu_i\varepsilon + \nu\tilde{\xi}^\nu_i\varepsilon^2 \quad (\check{\xi}^0_i = \tilde{\xi}^0_i(t,\psi), \check{\xi}^1_i = \hat{\xi}^1_i(\psi)). \tag{5.6}$$

As a result, we obtain the system

$$\varepsilon^{-\nu}\dot{u}_i = \sum_{l=1}^{n}\bar{V}^\nu_{il}u_l\varepsilon + O((|u| + \varepsilon)^3), \quad \varepsilon^{-\nu}\dot{\psi}_i = 1 + \bar{\Psi}^\nu_{i\varepsilon}\varepsilon + \rho_i u_i + O((|u| + \varepsilon)^2). \tag{5.7}$$

Indeed, differentiating change (5.6) with respect to (5.3) and (5.7) and taking into account the relation $\Psi^\nu_{i\varepsilon}(t, \psi_i + \check{\xi}^\nu_i\varepsilon + \nu\tilde{\xi}^\nu_i\varepsilon^2) = \Psi^\nu_{i\varepsilon}(t,\psi_i) + O(\varepsilon)$, we obtain the following equation similar to (4.2):

$$\frac{\partial\tilde{\xi}^\nu_i}{\partial t} + \sum_{m=1}^{n}\frac{\partial\check{\xi}^\nu_i}{\partial\psi_m} = \Psi^\nu_{i\varepsilon} - \bar{\Psi}^\nu_{i\varepsilon}. \tag{5.8}$$

Note that the change inverse to (5.6) can be written in the form

$$\psi_i = \varphi_i + \chi^\nu_i(t,\varphi,\varepsilon)\varepsilon, \tag{5.9}$$

where $\chi^\nu_i = -\check{\xi}^\nu_i + \left(\sum_{m=1}^{n}\check{\xi}^\nu_i\frac{\partial\check{\xi}^\nu_i}{\partial\varphi_m} - \nu\tilde{\xi}^\nu_i\right)\varepsilon + O(\varepsilon^2)$ is a continuous real-analytic function $\omega(c^k)$-periodic in φ and T-periodic in t; here, the $\xi^\nu_i = \xi^\nu_i(t,\varphi)$ are determined from (5.8) ($\nu = 0, 1$; $i = \overline{1,n}$).

6. INVARIANT SURFACES OF SYSTEM (1.1) AND THEIR ASYMPTOTICS

6.1. The Final Scaling Change

For convenience, we introduce the vectors $u = (u_1, \ldots, u_n)$, $w = (w_1, \ldots, w_n)$, and $\psi = (\psi_1, \ldots, \psi_n)$.

In system (5.7), we make the additional scaling change

$$u = w\varepsilon^{3/2}, \tag{6.1}$$

which transforms this system into the following system written in the vector form:

$$\dot{w} = (\bar{V}^{\nu} w\varepsilon + W^{\nu}(t, \psi, w, \varepsilon)\varepsilon^{3/2})\varepsilon^{\nu}, \quad \dot{\psi} = (1 + \bar{\Psi}^{\nu}_{\varepsilon}\varepsilon + \Xi^{\nu}(t, \psi, w, \varepsilon)\varepsilon^{3/2})\varepsilon^{\nu}, \tag{6.2}$$

where W^{ν} and Ξ^{ν} are real continuous vector functions of their arguments in a small neighborhood of the point $w = 0$, $\varepsilon = 0$, which are continuously differentiable with respect to w and ψ, $\omega(c^k)$-periodic in ψ, and T-periodic in t.

Indeed, we have $W^{\nu}(t, \psi, w, \varepsilon) = O((|w|\varepsilon^{3/2} + \varepsilon)^3)\varepsilon^{-3}$ and $\Xi^{\nu}(t, \psi, w, \varepsilon) = \mathrm{diag}\{\rho_1, \ldots, \rho_n\}w + O((|w|\varepsilon^{3/2} + \varepsilon)^2)\varepsilon^{-3/2}$, and the functions $O((|w|\varepsilon^{3/2} + \varepsilon)^2)$ are real-analytic with respect to ψ and three times continuously differentiable in a small neighborhood of the point $w = 0$, $\varepsilon = 0$. Therefore, in particular, $(W^{\nu})'_w$ and $(\Xi^{\nu})'_w$ are continuous at this point.

6.2. The Application of Hale's Lemma

System (6.2) satisfies the conditions of Hale's lemmas 2.1 and 2.2 in [8]; hence, for all sufficiently small $\varepsilon > 0$, it has an invariant surface of the form

$$w = H^{\nu}(t, \psi, \varepsilon)\varepsilon^{1/2}, \tag{6.3}$$

where H^{ν} is a continuous continuously differentiable function T-periodic in t and $\omega(c^k)$-periodic in ψ.

Thus, we have proved the following assertions.

Lemma 1. *Under condition* (4.10), *for any sufficiently small* $\varepsilon > 0$, *system* (4.10) *has the continuous continuously differentiable invariant surface*

$$v = \Theta^{\nu}(t, \varphi, \varepsilon), \tag{6.4}$$

where $\Theta^{\nu} = \bar{f}^{\nu}\varepsilon + (\check{f}^{\nu}(t, \varphi) + H^{\nu}(t, \varphi + \chi^{\nu}(t, \varphi, \varepsilon)\varepsilon, \varepsilon))\varepsilon^2 + \nu\tilde{f}^{\nu}(t, \varphi)\varepsilon^3$, *which is T-periodic in t and* $\omega(c^k)$-*periodic in* φ; *this surface is obtained by substituting the invariant surface* (6.3) *into the composition of changes* (5.2), (5.9), *and* (6.1).

Moreover, the function f is uniquely determined from Eqs. (5.4) *and* (5.5) *and* χ^{ν} *is uniquely determined from* (5.9).

Lemma 2. *Under conditions* (4.4) *and* (5.1), *for any sufficiently small* $\varepsilon > 0$, *system* (3.14) *has the continuous continuously differentiable invariant surface*

$$r_i = \Upsilon^{\nu}_i(t, \varphi, \varepsilon) \quad (i = \overline{1, n}), \tag{6.5}$$

where $\Upsilon^{\nu}_i = \alpha_i^{-1}(\varphi_i)(\Theta^{\nu}_i(t, \varphi, \varepsilon) + \check{g}^{\nu}_i(t, \varphi)\varepsilon + \beta_i(\varphi_i)(\Theta^{\nu}_i(t, \varphi, \varepsilon))^2 + \nu\tilde{g}^{\nu}_i(t, \varphi)\varepsilon^2) + \alpha_i^{-1}(\varphi_i)\left(\sum_{l=1}^{n}(\check{h}^{\nu}_{il}(t, \varphi) + \nu\tilde{h}^{\nu}_{il}(t, \varphi)\varepsilon)\Theta^{\nu}_i(t, \varphi, \varepsilon)\varepsilon\right)$, *which is T-periodic in t and* $\omega(c^k)$-*periodic in* φ. *This surface is obtained by substituting the invariant surface* (6.4) *into change* (4.9). *The functions* α_i *are given by* (3.7), *the* β_i *are given by* (4.6), *the* g_i *are uniquely determined by* (4.3), *and the* h_i *are uniquely determined by* (4.8).

Corollary 1. *The invariant surface* (6.5) *of system* (3.14) *has asymptotic expansion* $\Upsilon^{\nu}_i = \alpha_i^{-1}(\varphi_i)((\bar{f}^{\nu}_i + \check{g}^{\nu}_i(t, \varphi))\varepsilon + (\check{f}^{\nu}_i(t, \varphi) + H^{\nu}_i(t, \varphi, 0) + (\bar{f}^{\nu}_i)^2\beta_i(\varphi_i) + \nu\tilde{g}^{\nu}_i(t, \varphi) + \sum_{l=1}^{n}\bar{f}^{\nu}_l\check{h}^{\nu}_{il}(t, \varphi))\varepsilon^2) + o(\varepsilon^2)$.

6.3. Results of the Study

Theorem 1. *For any admissible vector c^k such that conditions* (4.4) *and* (5.1) *hold and any sufficiently small positive* ε, *system* (1.1) *has the* $(n+1)$*-dimensional continuous continuously differentiable invariant surface*

$$x = \Gamma_x^{\nu}, \quad y = \Gamma_y^{\nu}, \tag{6.6}$$

where $\Gamma_{x,i}^{\nu} = k_i + \breve{C}_i(\varphi_i, \breve{c}_{ik_i})(1 + \Upsilon_i^{\nu}(t,\varphi,\varepsilon))$ *and* $\Gamma_{y,i}^{\nu} = \breve{S}_i(\varphi_i, \breve{c}_{ik_i})(1 + \Upsilon_i^{\nu}(t,\varphi,\varepsilon))^2$ $(i = \overline{1,n})$, *which is T-periodic in t and* $\omega(c^k)$*-periodic in* φ. *This surface is obtained by substituting the invariant surface* (6.5) *into the composition of changes* (3.9) *and* (3.11). *Moreover, the invariant surface* (6.6) *passes through a small neighborhood of the point* $(c^k, 0)$ *and is homeomorphic to the* $(n+1)$*-torus, provided that time t is factored by the period.*

Corollary 2. *The invariant surface* (6.6) *in Corollary* 1 *has asymptotic expansion*

$$\Gamma_{x,i}^{\nu} = k_i + \breve{C}_i(\varphi_i)\left(1 + \alpha_i^{-1}(\varphi_i)\left((\bar{f}_i^{\nu} + \check{g}_i^{\nu}(t,\varphi))\varepsilon + \left(\check{f}_i^{\nu}(t,\varphi) + \Gamma_i^{\nu}(t,\varphi,0)\right.\right.\right.$$
$$\left.\left.\left. + (\bar{f}_i^{\nu})^2\beta_i(\varphi_i) + \nu\tilde{g}_i^{\nu}(t,\varphi) + \sum_{l=1}^{n}\bar{f}_l^{\nu}\,\check{h}_{il}^{\nu}(t,\varphi)\right)\varepsilon^2\right)\right) + o(\varepsilon^2),$$

$$\Gamma_{y,i}^{\nu} = \breve{S}_i(\varphi_i)\left(1 + 2\alpha_i^{-1}(\varphi_i)\left((\bar{f}_i^{\nu} + \check{g}_i^{\nu}(t,\varphi))\varepsilon + \left((2\alpha_i(\varphi_i))^{-1}(\bar{f}_i^{\nu} + \check{g}_l^{\nu}(t,\varphi))^2\right.\right.\right.$$
$$\left.\left.\left. + \check{f}_i^{\nu}(t,\varphi) + \Gamma_i^{\nu}(t,\varphi,0) + (\bar{f}_i^{\nu})^2\beta_i(\varphi_i) + \nu\tilde{g}_i^{\nu}(t,\varphi) + \sum_{l=1}^{n}\bar{f}_l^{\nu}\,\check{h}_{il}^{\nu}(t,\varphi)\right)\varepsilon^2\right)\right) + o(\varepsilon^2).$$

Thus, if a system of the form (1.1) has m different admissible vectors for which conditions (4.4) and (5.1) hold, then, for any sufficiently small $\varepsilon > 0$, system (1.1) has m invariant surfaces homeomorphic to the $(n+1)$-torus and passing through small neighborhoods of the corresponding generating points.

7. THE DEFINING SYSTEM IN THE ANALYTIC CASE

7.1. The Structure of the Defining System

Let us study system (2.7) in the important special case where the functions $X_i(t, x, y, 0)$ and $Y_i(t, x, y, 0)$ in system (1.1) are continuous, T-periodic in t, and analytic in x_i and y_i in the open connected set $G = \{(t, x, y) : t \in \mathbb{R}, |x_1|, \ldots, |x_n| < x_0, |y_1|, \ldots, |y_n| < y_0\}$ and, moreover, $x_0 > 2^{1/2}$ and $y_0 > 2^{-1/2}$.

In other words, we assume that, in system (1.1),

$$X_i(t,x,y,0) = \sum_{p,q=0}^{\infty} X_i^{(p,q)}(t)x^p y^q, \quad Y_i(t,x,y,0) = \sum_{p,q=0}^{\infty} Y_i^{(p,q)}(t)x^p y^q \tag{7.1}$$

are power series absolutely convergent in the open connected set G uniformly in t with real continuous coefficients T-periodic in t, the vectors p and q have nonnegative integer components, and $x^p = x_1^{p_1}\ldots x_n^{p_n}$; the numbers y^q are defined in a similar way.

As a result, in the equation with index i of system (2.7), the integrand contains the series

$$\sum_{p,q=0}^{\infty}(X_i^{(p,q)}(t)S_i'(\varphi_i) - Y_i^{(p,q)}(t)C_i'(\varphi_i))C^p(\varphi)S^q(\varphi)$$

absolutely convergent for any real $\varphi_1, \ldots, \varphi_n$ uniformly in t.

Since the integral over the period of the product of the even function $C_i(\varphi_i)$ and the odd function $S_i(\varphi_i)$ vanishes, it follows that system (2.7) takes the form

$$\sum_{p=e^i}^{\infty}\sum_{q=0}^{\infty}\overline{X_i^{(p,2q)}}\int_0^{\omega(c^k)} S_i'C^pS^{2q}d\varphi + \sum_{p=0}^{\infty}\sum_{q=e^i}^{\infty}\gamma_i\overline{Y_i^{(p,2q-e^i)}}\int_0^{\omega(c^k)} C^pS^{2q}d\varphi = 0,$$

where the bar denotes mean value with respect to t and $e^i = (0, \ldots, 1_{(i)}, \ldots, 0)$.

Using the identity $(2q_i+1)C_i^{p_i}S_i^{2q_i}S_i' = (C_i^{p_i}S_i^{2q_i+1})' + \gamma_i p_i C_i^{p_i-1}S_i^{2q_i+2}$, we reduce system (2.7) to the form

$$\sum_{p=0}^{\infty}\sum_{q=e^i}^{\infty}\left(\frac{p_i+1}{2q_i-1}\overline{X_i^{(p+e^i,2(q-e^i))}}+\overline{Y_i^{(p,2q-e^i)}}\right)\prod_{\iota=1}^{n}\int_0^{\omega(c_\iota)}C_\iota^{p_\iota}S_\iota^{2q_\iota}d\varphi_\iota = 0.$$

Expressing $d\varphi_\iota$ in the first equation of system (2.1) and $S(\varphi_\iota)$ on the half-periods from integral (2.2) and taking into account (2.5), as in Section 2.3, we obtain

$$\int_0^{\omega(c_\iota)}C_\iota^{p_\iota}S_\iota^{2q_\iota}d\varphi_\iota = \operatorname{sign}(c_\iota)\frac{2^{3/2-q_\iota}}{\gamma_\iota}\int_{b_\iota}^{c_\iota}C_\iota^{p_\iota}((c_\iota^2-\eta_\iota)^2-(C_\iota^2-\eta_\iota)^2)^{q_\iota-1/2}dC_\iota.$$

After reduction by $\prod_{\iota=1}^{n}\operatorname{sign}(c_\iota)\frac{2^{3/2}}{\gamma_\iota}$, system (2.7) takes the form

$$\sum_{p=0}^{\infty}\sum_{q=e^i}^{\infty}d_i^{(p,q)}\prod_{\iota=1}^{n}\theta_{p_\iota,q_\iota}(c_{\iota k_\iota}) = 0 \quad (p_\iota, q_\iota \ge 0;\ k_\iota = 0,\pm 1;\ i=\overline{1,n}), \tag{7.2}$$

where $\theta_{p_\iota,q_\iota}(c_\iota) = \int_{b_\iota}^{c_\iota}\tau^{p_\iota}(c_\iota^2-\eta_\iota)^2 - (\tau_\iota^2-\eta_\iota)^2)^{q_\iota-1/2}d\tau$, the $b_\iota = b_\iota(c_\iota)$ are from (2.5), $d_i^{(p,q)} = 2^{-(q_1+\ldots+q_n)}\left(\frac{p_i+1}{2q_i-1}\overline{X_i^{(p+e^i,2(q-e^i))}}+\overline{Y_i^{(p,2q-e^i)}}\right)$, and the $\overline{X_i^{(p,q)}}$ and $\overline{Y_i^{(p,q)}}$ are the average values of the coefficients in expansions (7.1).

The functions $\theta_{p_\iota q_\iota}(c_{\iota k_\iota})$ can be written in a form more convenient for calculations as

$$\begin{aligned}\theta_{p_\iota,q_\iota}(c_{\iota 0}) &= c_{\iota 0}^{p_\iota+2q_\iota}\int_{-1}^{1}\varsigma^{p_\iota}((c_{\iota 0}^2\varsigma^2+c_{\iota 0}^2-2\eta_\iota)(1-\varsigma^2))^{q_\iota-1/2}d\varsigma,\\ \theta_{p_\iota,q_\iota}(c_{\iota j}) &= \frac{j^{p_\iota+1}(c_{\iota j}^2-1)^{2q_\iota}}{2}\int_{-1}^{1}((c_{\iota j}^2-1)\varsigma+1)^{(p_\iota-1)/2}(1-\varsigma^2)^{q_\iota-1/2}d\varsigma,\end{aligned} \tag{7.3}$$

where, according to (2.3), if $k_\iota = 0$, then $c_{\iota 0} \in (\sqrt{\eta_\iota(\eta_\iota+1)}, +\infty)$ $(\eta_\iota = 0, \pm 1)$ and if $k_\iota = j$ $(j = \pm 1)$, then $\eta_\iota = 1$ and $jc_{\iota j} \in (1, 2^{1/2})$.

Indeed, if $k_\iota = 0$ in (7.2), then $b_{\iota 0} = -c_{\iota 0}$ and the change $\tau = c_{\iota 0}\varsigma$ should be made, and if $k_\iota = j$, then $b_{\iota j} = j(2-c_{\iota j}^2)^{1/2}$ and the change $\tau = j((c_{\iota j}^2-1)\varsigma+1)^{1/2}$ should be made.

Note that, in (7.3), we have $\theta_{p_\iota q_\iota}(c_{\iota 0}) = 0$ for odd p_ι.

In terms of the $\theta_{0,0}(c_{\iota k_\iota})$ defined by (7.3), it is also convenient to rewrite the expression (2.6) for the period $\omega(c_{\iota k_\iota})$ of trajectory of the closed orbit $l_{\iota k_\iota}$ of system (2.1):

$$\omega(c_{\iota k_\iota}) = \frac{2^{3/2}}{\gamma_\iota}\int_{-1}^{1}\frac{d\varsigma}{(\zeta(\varsigma,c_{\iota k_\iota})(1-\varsigma^2))^{1/2}} \quad (\iota = \overline{1,n}), \tag{7.4}$$

where $\zeta = \{c_{\iota 0}^2(1+\varsigma^2)-2\eta_\iota$ for $k_\iota = 0$, $4(c_{\iota j}^2\varsigma+1-\varsigma)$ for $k_\iota = j$ $(j = \pm 1)\}$.

7.2. The Application of the Obtained Results in Practice

Suppose that system (1.1) satisfies the following conditions:

$$n = 2;\quad \eta_1 = -1,\quad \eta_2 = 1;\quad \gamma_1 = 1,\quad \gamma_2 = 2;\quad (7.1)\text{ holds};$$

all coefficients $d_i^{(p,q)}$ $(i = 1, 2)$ of system (7.2) are zero, except

$$\begin{aligned}&d_1^{(\breve p^1,\breve q^1)} = 1,\quad (\breve p^1,\breve q^1) = (0,0;3,1);\quad d_1^{(\hat p^1,\hat q^1)} = -10,\quad (\hat p^1,\hat q^1) = (0,0;2,1);\\ &d_1^{(\tilde p^1,\tilde q^1)} = 18,\quad (\tilde p^1,\tilde q^1) = (0,0;1,1);\quad d_2^{(\tilde p^2,\tilde q^2)} = 20,\quad (\tilde p^2,\tilde q^2) = (0,0;1,1);\\ &d_2^{(\hat p^2,\hat q^2)} = -22,\quad (\hat p^2,\hat q^2) = (0,2;1,1);\quad d_2^{(\breve p^2,\breve q^2)} = 1,\quad (\breve p^2,\breve q^2) = (0,0;1,1).\end{aligned} \tag{7.5}$$

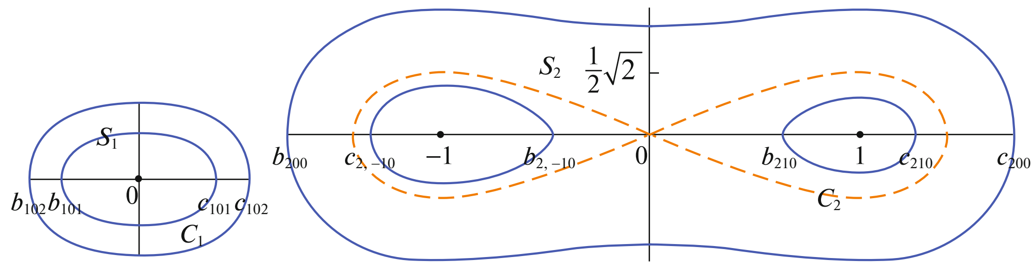

Fig. 1. Phase portraits.

Then system (1.2) unperturbed with respect to (1.1) has the three singular points $(x_1, x_2, y_1, y_2) = (0, -1, 0, 0)$, $(0, 0, 0, 0)$, $(0, 1, 0, 0)$.

In turn, the defining system (7.2) takes the form

$$\begin{cases}(18\theta_{0,1}(c_1) - 10\theta_{0,2}(c_1) + \theta_{0,3}(c_1))\theta_{0,1}(c_2) = 0,\\(20\theta_{0,1}(c_2) - 22\theta_{2,1}(c_2) + \theta_{1,1}(c_2))\theta_{0,1}(c_1) = 0,\end{cases} \tag{7.6}$$

where, according to (7.3) and by virtue of the choice of η_1 and η_2,

$$c_1 = c_{10} \in (0, +\infty) \quad (k_1 = 0),$$

$$c_2 = \{c_{2,-1} \in (-2^{1/2}, -1), c_{21} \in (1, 2^{1/2}), c_{20} \in (2^{1/2}, +\infty)\} \quad (k_2 = 0, \pm 1);$$

$$\theta_{0,m}(c_{10}) = c_{10}^{2m}\int_{-1}^{1}((c_{10}^2\varsigma^2 + c_{10}^2 + 2)(1-\varsigma^2))^{m-1/2}d\varsigma \quad (m = 1, 2, 3),$$

$$\theta_{0,1}(c_{20}) = c_{20}^2\int_{-1}^{1}((c_{20}^2\varsigma^2 + c_{20}^2 - 2)(1-\varsigma^2))^{-1/2}d\varsigma,$$

$$\theta_{1,1}(c_{20}) = 0, \quad \theta_{1,1}(c_{2j}) = \frac{(c_{2j}^2 - 1)^2}{2}\int_{-1}^{1}(1-\varsigma^2)^{1/2}d\varsigma = \frac{\pi}{4}(c_{2j}^2 - 1)^2,$$

$$\theta_{2,1}(c_{20}) = c_{20}^4\int_{-1}^{1}\varsigma^2((c_{10}^2\varsigma^2 + c_{10}^2 + 2)(1-\varsigma^2))^{1/2}d\varsigma,$$

$$\theta_{2,1}(c_{2j}) = \frac{j(c_{2j}^2 - 1)^2}{2}\int_{-1}^{1}((c_{2j}^2 - 1)\varsigma + 1)^{1/2}(1-\varsigma^2)^{1/2}d\varsigma \quad (j = \pm 1).$$

Let $L_1(c_{10}) = 18\theta_{0,1}(c_{10}) - 10\theta_{0,2}(c_{10}) + \theta_{0,3}(c_{10})$. Then $L_1(1.01) \approx 0.614$, $L_1(1.02) \approx -0.29$, $L_1(1.44) \approx -1.14$, and $L_1(1.45) \approx 3.756$. Hence, there exist $c_{101} \in (1.01, 1.02)$ and $c_{102} \in (1.44, 1.45)$ such that $L_1(c_{101}) = L_1(c_{102}) = 0$.

Now suppose that $L_{2k_2}(c_{2k_2}) = 20\theta_{0,1}(c_{2k_2}) - 22\theta_{2,1}(c_{2k_2}) + \theta_{1,1}(c_{2k_2})$. We have $L_{20}(1.73) \approx 0.43$, $L_{20}(1.74) \approx -0.14$, $L_{21}(1.26) \approx -0.001$, $L_{21}(1.27) \approx 0.015$, $L_{2,-1}(-1.34) \approx -0.03$, and $L_{2,-1}(-1.33) \approx 0.02$. Hence, there exist $c_{200} \in (1.73, 1.74)$, $c_{210} \in (1.26, 1.27)$, and $c_{2,-10} \in (-1.34, -1.33)$ such that $L_{20}(c_{200}) = L_{21}(c_{210}) = L_{2,-1}(c_{2,-10}) = 0$.

Thus, system (7.6) has the six solutions $(c_{10\iota}, c_{2k_20})$ $(\iota = 1, 2;\ k_2 = 0, \pm 1)$.

According to formula (7.4), the solutions of the initial value problems for system (2.1) with $i = 1$ have periods $\omega(c_{101}) \approx 4.7$ and $\omega(c_{102}) \approx 4.0$, and those for system (2.1) with $i = 2$ have periods $\omega(c_{200}) \approx 3.0$, $\omega(c_{210}) \approx 2.4$, and $\omega(c_{2,-10}) \approx 2.6$.

By Theorem 1, through a small neighborhood of each of the six points $(c_{10\iota}, c_{2k_20}, 0, 0)$, for any small ε, a three-dimensional invariant torus passes, provided that Siegel's condition (4.4) on the periods and the dissipativity condition (5.1) hold; if needed, the latter can be verified by straightforward calculations.

In the phase portraits (see the figure 1), the orbits of systems (2.1) whose pairwise products determine the six invariant surfaces of the unperturbed system (1.2) are constructed. In their small neighborhood, the invariant surfaces of any system (1.1) are preserved under conditions (7.5). As mentioned, the violation of Siegel's conditions and the dissipativity condition is an exceptional case. Therefore, slightly changing the coefficients $d_{ik_i}^{(p,q)}$, which entails a small continuous change of the admissible solutions of the defining system, we can satisfy conditions (4.4) and (5.1).

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