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MATHEMATICS

Solving a Tropical Optimization Problem with Application to Optimal Scheduling

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Abstract—A multidimensional optimization problem is formulated and solved in terms of tropical mathematics that is concerned with the theory and applications of semi-rings with idempotent addition. The problem, whose objective function is defined by a matrix, is proposed to be solved via idempotent algebra and tropical optimization tools. A strict lower bound is first derived for the objective function, used for solving the problem, to allow the evaluation of its minimum value. The objective function and its minimum value are then combined into an equation whose complete solution is obtained in the form of all eigenvectors of the matrix. A practical application of the problem is considered using the example of an explicit solution for the optimal scheduling of a project that consists of a set of activities defined by constraints on their start and end times. The optimality criterion for scheduling is defined to minimize the maximum, over all activities, of the working cycle time, which is described as the time interval between the start and the end of the activity. The analytical result extends and supplements the existing algorithmic numerical solutions to optimal scheduling problems. As an illustrative example, the solution of a problem to schedule a project consisting of three activities is presented to illustrate the result.

Keywords: idempotent semi-field, (max, +)-algebra, eigenvalues and eigenvectors of matrices, tropical optimization, scheduling problem.

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1. INTRODUCTION

Tropical optimization problems present an important class of problems in tropical (idempotent) mathematics that consists in studying semi-rings with idempotent addition. The first works devoted to tropical mathematics [1–4] appeared in the 1960s. In particular, one of the first problems of tropical optimization is examined in [2]. Further development of tropical mathematics and optimization tools was reflected in numerous publications, including monographs [5–10], dedicated to specific optimization problems in various areas, such as engineering, economics, and management.

One of the features of tropical optimization problems is that the solution of the problems often reduces to solving linear vector equations, investigating spectrum of linear operators, and other computational issues in tropical algebra. In many cases, it allows the direct solution to be obtained in a compact vector form. Tropical optimization finds application in multi-criterion decision making problems, minimax problems of placing objects on a plane and in a space, as well as in project scheduling problems.

Optimal project scheduling is an important project management problem that is aimed at determining the optimum start and end times of activities in the project under various temporal constraints and optimality criteria. In 1910, American engineer G.L. Gantt developed a project scheduling technique involving horizontal diagrams [11]. The Gantt diagrams were used afterwards in other algorithms based on the graph theory, known as network planning.

Methods of network planning were first developed in the 1950s in the US. On the one hand, DuPont and Remington Rand corporation staff, performing maintenance repair of the DuPont plants, proposed the critical path method (CPM) [12]. On the other hand, the development of a Polaris ballistic rocket, commissioned by the US Navy, by the staff of Lockheed Company and Booz Allen Hamilton Consulting Corporation resulted in the Program Evaluation and Review Technique (PERT) [13].

Among the shortcomings of project scheduling problems are the non-linearity and non-smoothness of the objective function and constraints. The conventional methods for solving this type of problems based on linear programming or graph optimization allow the numerical solution to be obtained in the form of iterative computation algorithms, which does not ensure an explicit analytical solution. Unlike the above methods, solving the scheduling problems in the form of multidimensional tropical optimization problems enables the explicit result to arise in the compact vector form [14–17]. This analytical solution opens up prospects for using the formal methods in the analysis of problems and their solutions, yielding the finite-step computational algorithms for which one can determine the exact number of operations needed.

There is a set of optimal scheduling problems that require the minimization of the maximum working cycle time over all activities, which is defined as the difference between the start and end times of activities [2, 5, 14, 15, 17]. In terms of tropical mathematics, these problems lead to the minimization of the function $\mathbf{x}^{-}\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a given quadratic matrix, and $\mathbf{x}$ and $\mathbf{x}^{-}$ are the unknown vector and its multiplicatively conjugate vector, respectively. As shown in [2], the minimum of this function is equal to the tropical spectral radius of the matrix $\mathbf{A}$. Implicit solutions of the problem in the form of a vector inequality were described in [18, 19]. A direct complete solution of the problem and of some its generalizations were obtained in the monographs [14, 15].

In this paper, a problem is considered to minimize the function $\mathbf{x}^{-}\mathbf{A}\mathbf{x}(\mathbf{A}\mathbf{x})^{-}\mathbf{x}$, which appear in scheduling problems that require to minimize the maximum deviation of the cycle time of activities. A complete solution to the problem is obtained, which formulated in terms of a general semi-field. An example is given, which illustrates the obtained results with the solution of a problem to schedule a project consisting of three activities.

2. IDEMPOTENT ELEMENTS

This section briefly overviews the main definitions and designations of idempotent algebra, which are essential in formulating and solving the optimization problem in the following parts of this work. Additional information on the theory, methods and applications of tropical mathematics is available in works [5–10].

2.1. Idempotent semi-field. Consider a nonempty set $\mathbb{X}$, that is closed with respect to two operations, addition $\oplus$ and multiplication $\odot$. In terms of addition, the set $\mathbb{X}$ is the idempotent commutative monoid with a neutral element $\mathbb{0}$. The idempotency means that any element $x \in \mathbb{X}$ obeys the equality $x \oplus x = x$. With respect to multiplication, $\mathbb{X}\backslash\{\mathbb{0}\}$ forms a commutative group with a neutral element $\mathbb{1}$. Any nonzero x has an inverse element x^{-1} such that $x \odot x^{-1} = \mathbb{1}$. Furthermore, multiplication $\odot$ distributes over addition $\oplus$ and has $\mathbb{0}$ as an absorbing element. The algebraic system $\langle\mathbb{X}, \oplus, \odot, \mathbb{0}, \mathbb{1}\rangle$ with the specified characteristics is called the idempotent semi-field.

Due to the associativity of multiplication, the integer powers can be introduced in a standard way. For any nonzero $x \in \mathbb{X}$ and a natural number n, define $x^0 = \mathbb{1}$, $x^n = x^{n-1} \odot x$, $x^{-n} = (x^{-1})^n$, and $\mathbb{0}^n = \mathbb{0}$. We assume that the integer powers can be extended in the semi-field to the case of real exponents.

Below the multiplication sign will be dropped in the algebraic expression for the sake of simplicity, i.e., $x \odot y = xy$.

The idempotencity of addition generates the partial order relation $\leqslant$ on $\mathbb{X}$ so that $x \leqslant y$ if and only if $x \oplus y = y$. From this definition follows the fulfillment of the inequalities $x \leqslant x \oplus y$ and $y \leqslant x \oplus y$, as well as the equivalence of the inequality $x \oplus y \leqslant z$ and the system of inequalities $x \leqslant z$ and $y \leqslant z$ for any $x, y, z \in \mathbb{X}$. The operations $\oplus$ and $\odot$ are monotonic in terms of the above order with respect to each argument, i.e., if $x \leqslant y$, then any $z \in \mathbb{X}$ obeys the inequalities $x \oplus z \leqslant y \oplus z$ and $xz \leqslant yz$. Notice that any nonzero elements $x, y \in \mathbb{X}$ that satisfy the inequality $x \leqslant y$ give the inequality $x^{-1} \geqslant y^{-1}$. Here and hereinafter the above introduced partial order is assumed to be linear.

As an example of the idempotent semi-field, consider a real semi-field $\mathbb{R}_{\max,+} = \langle\mathbb{R} \cup \{-\infty\}, \max, +, -\infty, 0\rangle$, where the zero element is $-\infty$ and the unit element is 0. For any $x \in \mathbb{R}$, there is an inverse element x^{-1} that is equal to $-x$ in the conventional arithmetic. For any $x, y \in \mathbb{R}$, there is a degree x^y with a value matching the arithmetic product xy. The order, generated by the idempotent addition on $\mathbb{R}_{\max,+}$, corresponds to the ordinary linear order on $\mathbb{R}$. This semi-field is usually called the (max, +)-algebra.

2.2. Vectors and matrices. Denote a set of matrices over $\mathbb{X}$, composed of m lines and n columns, by $\mathbb{X}^{m\times n}$. A matrix with all zero elements is the zero matrix. A matrix without zero columns is called column-regular.

The operations of addition and multiplication of matrices of suitable size, as well as the multiplication of a matrix by a scalar, are performed by the standard rules with substituting the corresponding componentwise operations by $\oplus$ and $\odot$.

Consider the square matrices from $\mathbb{X}^{n\times n}$. A matrix $\mathbf{I} = \mathrm{diag}(\mathbb{1}, ..., \mathbb{1})$ whose nondiagonal elements are $\mathbb{0}$ is called the unit matrix. For any square matrix $\mathbf{A}$ and a natural n the matrix powers are defined as follows: $\mathbf{A}^0 = \mathbf{I}$, $\mathbf{A}^n = \mathbf{A}^{n-1}\mathbf{A}$.

A square matrix is reducible if the same permutations of rows and columns allow to obtain a block (triangular) form of the matrix, where all blocks above (below) the diagonal blocks are zero. Otherwise the matrix is irreducible.

A trace of a matrix $\mathbf{A} = (a_{ij}) \in \mathbb{X}^{n\times n}$ is the number $\mathrm{tr}\mathbf{A} = a_{11} \oplus ... \oplus a_{nn}$.

Denote a set of column vectors over $\mathbb{X}$ of the order n by $\mathbb{X}^n$. A vector is called regular if it has no zero components.

The vector operations of addition and multiplication by a scalar are performed in accordance with the standard rules with substituting the corresponding scalar operations by $\oplus$ and $\odot$. The monotonicity properties of operations $\oplus$ and $\odot$ for scalars are extended to the vector operations, where the inequalities are assumed to be componentwise.

It is evident that the vector $\mathbf{Ax}$ for an irreducible matrix $\mathbf{A}$ and a regular vector $\mathbf{x}$ is also regular.

For any nonzero column vector $\mathbf{x} = (x_i) \in \mathbb{X}^n$, there is a multiplicatively conjugate row vector $\mathbf{x}^- = (x_i^-)$ with elements $x_i^- = x_i^{-1}$, if $x_i \neq \mathbb{0}$, otherwise $x_i^- = \mathbb{0}$.

If $\mathbf{x}$, $\mathbf{y}$ are nonzero vectors, then the following equality is valid: $(\mathbf{xy}^-)^- = \mathbf{yx}^-$. For regular vectors $\mathbf{x}, \mathbf{y} \in \mathbb{X}^n$ from the inequality $\mathbf{x} \leqslant \mathbf{y}$, it follows that $\mathbf{x}^- \geqslant \mathbf{y}^-$.

Any regular vector $\mathbf{x}$ obeys the inequality $\mathbf{xx}^- \geqslant \mathbf{I}$. If a vector $\mathbf{x}$ is nonzero, then $\mathbf{x}^-\mathbf{x} = \mathbb{1}$.

A vector $\mathbf{y} \in \mathbb{X}^n$ is linearly dependent on vectors $\mathbf{x}_1, ..., \mathbf{x}_m \in \mathbb{X}^n$, if it can be represented by a linear combination $\mathbf{y} = a_1\mathbf{x}_1 \oplus \cdot ... \oplus a_m\mathbf{x}_m$ with coefficients $a_1, ..., a_m \in \mathbb{X}$. Vectors $\mathbf{x}$ and $\mathbf{y}$ are collinear, if $\mathbf{y} = a\mathbf{x}$, where $a \in \mathbb{X}$.

2.3. Spectral radius and eigenvectors of a matrix. A number λ is an eigenvalue of a matrix $\mathbf{A} \in \mathbb{X}^{n\times n}$, if there is a nonzero vector $\mathbf{x} \in \mathbb{X}^n$, which satisfies the equality

$$\mathbf{Ax} = \lambda\mathbf{x}.$$

Any nonzero vector $\mathbf{x} \in \mathbb{X}^n$, satisfying this equality, is called eigenvector of the matrix $\mathbf{A}$, corresponding to the eigenvalue λ. The eigenvectors of the irreducible matrices are regular.

A maximum eigenvalue of the matrix $\mathbf{A}$ is called its spectral radius. If $\mathbf{A}$ is an irreducible matrix, it has a single eigenvalue $\lambda > \mathbb{0}$, which coincides with its spectral radius and is calculated as follows:

$$\lambda = \mathrm{tr}\,\mathbf{A} \oplus \mathrm{tr}^{1/2}(\mathbf{A}^2) \oplus \ldots \oplus \mathrm{tr}^{1/n}(\mathbf{A}^n) = \bigoplus_{m=1}^{n} \mathrm{tr}^{1/m}(\mathbf{A}^m). \quad (1)$$

All eigenvectors of an irreducible matrix $\mathbf{A}$, corresponding to its single eigenvalue λ, can be found as follows.

Using a designation $\mathbf{A}_\lambda = \lambda^{-1}\mathbf{A}$, define two matrices (a Kleene star and a Kleene plus):

$$\mathbf{A}_\lambda^* = \mathbf{I} \oplus \mathbf{A}_\lambda \oplus \ldots \oplus \mathbf{A}_\lambda^{n-1} = \bigoplus_{m=0}^{n-1} \mathbf{A}_\lambda^m, \quad (2)$$

$$\mathbf{A}_\lambda^+ = \mathbf{A}_\lambda\mathbf{A}_\lambda^* = \mathbf{A}_\lambda \oplus \mathbf{A}_\lambda^2 \oplus \ldots \oplus \mathbf{A}_\lambda^n = \bigoplus_{m=1}^{n} \mathbf{A}_\lambda^m. \quad (3)$$

Let $\mathbf{a}_i^*$ denote a column i of the matrix $\mathbf{A}_\lambda^*$, and a_{ii}^+ be a diagonal element of the matrix $\mathbf{A}_\lambda^+$. Consider a subset of columns $\mathbf{a}_i^*$ with indices i, which satisfy the equality $a_{ii}^+ = \mathbb{1}$. From this subset, choose the columns that are linearly independent of others and compose a matrix $\mathbf{A}_\lambda^\times$. A set of all eigenvectors of the matrix $\mathbf{A}$ coincides with the linear span of columns in $\mathbf{A}_\lambda^\times$ and is defined by an equality:

$$\mathbf{x} = \mathbf{A}_\lambda^\times \mathbf{v},$$

where $\mathbf{v}$ is any nonzero vector of appropriate size.

3. PRELIMINARY RESULTS

Consider some known tropical mathematics results that will be used in solving the optimization problem in the forthcoming section. First of all, assume a matrix $\mathbf{A} \in \mathbb{X}^{m\times n}$ and a regular vector $\mathbf{d} \in \mathbb{X}^m$. The goal is to find all vectors $\mathbf{x} \in \mathbb{X}^n$, satisfying the inequality

$$\mathbf{A}\mathbf{x} \leqslant \mathbf{d}. \tag{4}$$

The solution of inequality (4) is described by the following statement whose complete proof is available, e.g., in [14].

Lemma 1. *For any column-regular matrix* $\mathbf{A}$ *and a regular vector* $\mathbf{d}$, *all solutions of inequality* (4) *take the following form*:

$$\mathbf{x} \leqslant (\mathbf{d}^- \mathbf{A})^-.$$

Let an irreducible matrix $\mathbf{A} \in \mathbb{X}^{n\times n}$ be defined with an eigenvalue λ. One needs to find regular vectors $\mathbf{x} \in \mathbb{X}^n$, which solve the problems

$$\min_{\mathbf{x}} \mathbf{x}^- \mathbf{A}\mathbf{x}, \tag{5}$$

$$\min_{\mathbf{x}} (\mathbf{A}\mathbf{x})^- \mathbf{x}. \tag{6}$$

Solving an optimization problem in the next section will require a result that has been obtained for an irreducible matrix in [20] as follows.

Lemma 2. *Let* $\mathbf{A}$ *be the irreducible matrix and* λ *be its eigenvalue. Then the following equalities are valid*:

$$\min_{\mathbf{x}} \mathbf{x}^- \mathbf{A}\mathbf{x} = \lambda, \tag{7}$$

$$\min_{\mathbf{x}} (\mathbf{A}\mathbf{x})^- \mathbf{x} = \lambda^{-1}, \tag{8}$$

where the minimums are achieved at any eigenvector of a matrix $\mathbf{A}$.

For an arbitrary matrix $\mathbf{A}$, problems (5) and (6) were solved in [14, 21].

4. SOLVING A TROPICAL OPTIMIZATION PROBLEM

This section is dedicated to considering a new tropical optimization problem that consists in minimizing a function defined by the idempotent product of the objective functions in problems (5) and (6). Let $\mathbf{A}$ be an irreducible matrix. The goal is to find regular vectors $\mathbf{x} \in \mathbb{X}^n$, which solve the problem

$$\min_{\mathbf{x}} \mathbf{x}^- \mathbf{A}\mathbf{x} (\mathbf{A}\mathbf{x})^- \mathbf{x}. \tag{9}$$

This problem is solved using the approach based on constructing a strict lower bound for an objective function, which allows one to determine the minimum value of the function. The problem is then reduced to solving the equation for the objective function and its minimum.

The set of all solutions of problem (9) is described by the following theorem.

Theorem 1. *Let* $\mathbf{A}$ *be an irreducible matrix with a spectral radius* λ *and* $\mathbf{A}_\lambda = \lambda^{-1}\mathbf{A}$. *Then the minimum value in problem* (9) *is equal to* $\mathbb{1}$ *and is achieved*, *if and only if* $\mathbf{x}$ *is an eigenvector of the matrix* $\mathbf{A}$, *taking the form*

$$\mathbf{x} = \mathbf{A}_\lambda^\times \mathbf{v},$$

where $\mathbf{v}$ *is any nonzero vector of appropriate size*.

Proof. First, find a lower bound of the objective function $\mathbf{x}^-\mathbf{A}\mathbf{x}(\mathbf{A}\mathbf{x})^-\mathbf{x}$ of problem (9). According to the condition of the problem, the matrix $\mathbf{A}$ is irreducible, and the vector $\mathbf{x}$ is regular. Then vectors $\mathbf{A}\mathbf{x}$ and $(\mathbf{A}\mathbf{x})^-$ are also regular, resulting in the inequality $\mathbf{A}\mathbf{x}(\mathbf{A}\mathbf{x})^- \geqslant \mathbf{I}$. Taking this inequality into consideration, it follows that $\mathbf{x}^-\mathbf{A}\mathbf{x}(\mathbf{A}\mathbf{x})^-\mathbf{x} \geqslant \mathbf{x}^-\mathbf{x} = \mathbb{1}$, allowing $\mathbb{1}$ to be a lower bound for the function $\mathbf{x}^-\mathbf{A}\mathbf{x}(\mathbf{A}\mathbf{x})^-\mathbf{x}$. Notice that this bound can also be obtained directly from equalities (7) and (8).

Since the matrix $\mathbf{A}$ is irreducible, it has a single eigenvalue $\lambda > \mathbb{0}$ and a regular eigenvector $\mathbf{x}_0$. Substituting $\mathbf{x}_0$ in the objective function gives $\mathbf{x}_0^-\mathbf{A}\mathbf{x}_0(\mathbf{A}\mathbf{x}_0)^-\mathbf{x}_0 = \lambda\mathbf{x}_0^-\mathbf{x}_0(\lambda\mathbf{x}_0)^-\mathbf{x}_0 = \lambda\lambda^{-1}\mathbf{x}_0^-\mathbf{x}_0\mathbf{x}_0^-\mathbf{x}_0 = \mathbb{1}$, meaning that $\mathbb{1}$ is the minimum of the objective function in problem (9).

Find all vectors, for which the objective function $\mathbf{x}^-\mathbf{A}\mathbf{x}(\mathbf{A}\mathbf{x})^-\mathbf{x}$ achieves its minimum value. For this, solve the equation

$$\mathbf{x}^-\mathbf{A}\mathbf{x}(\mathbf{A}\mathbf{x})^-\mathbf{x} = \mathbb{1} \tag{10}$$

and show that, besides the eigenvectors of the matrix $\mathbf{A}$, there are no other solutions.

Let $\mathbf{x}^-\mathbf{A}\mathbf{x} = \alpha$. The vector $\mathbf{x}$ is regular, resulting in $\alpha \neq \mathbb{0}$. Then there is α^{-1} such that $\alpha\alpha^{-1} = \mathbb{1}$, and equation (10) provides $(\mathbf{A}\mathbf{x})^-\mathbf{x} = \alpha^{-1}$.

Thus, equation (10) can be replaced by the equivalent system

$$\mathbf{x}^-\mathbf{A}\mathbf{x} = \alpha, \quad (\mathbf{A}\mathbf{x})^-\mathbf{x} = \alpha^{-1}, \quad \alpha > \mathbb{0}. \tag{11}$$

According to Lemma 2, the eigenvalue λ of a matrix $\mathbf{A}$ is the minimum value of the function $\mathbf{x}^-\mathbf{A}\mathbf{x}$ and λ^{-1} is that of the function $(\mathbf{A}\mathbf{x})^-\mathbf{x}$. As a result, $\alpha = \mathbf{x}^-\mathbf{A}\mathbf{x} \geqslant \lambda$ and $\alpha^{-1} = (\mathbf{A}\mathbf{x})^-\mathbf{x} \geqslant \lambda^{-1}$, giving the double inequality $\lambda \leqslant \alpha \leqslant \lambda$ and, consequently, $\alpha = \lambda$. Now system (11) takes the form:

$$\mathbf{x}^-\mathbf{A}\mathbf{x} = \lambda, \quad (\mathbf{A}\mathbf{x})^-\mathbf{x} = \lambda^{-1}.$$

It is evident that a set of solutions of the obtained system coincides with that of solutions of the system of inequalities

$$\mathbf{x}^-\mathbf{A}\mathbf{x} \leqslant \lambda, \quad (\mathbf{A}\mathbf{x})^-\mathbf{x} \leqslant \lambda^{-1}.$$

Applying Lemma 1 for solving the first inequality with respect to $\mathbf{A}\mathbf{x}$ and the second to $\mathbf{x}$ leads to the equivalent system of inequalities $\mathbf{A}\mathbf{x} \leqslant (\lambda^{-1}\mathbf{x}^-)^-$ and $\mathbf{x} \leqslant (\lambda(\mathbf{A}\mathbf{x})^-)^-$, which can be written as $\mathbf{A}\mathbf{x} \leqslant \lambda\mathbf{x}$ and $\lambda\mathbf{x} \leqslant \mathbf{A}\mathbf{x}$. Then the double inequality $\mathbf{A}\mathbf{x} \leqslant \lambda\mathbf{x} \leqslant \mathbf{A}\mathbf{x}$ holds which is equivalent to the equality $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.

Assuming that λ is the eigenvalue of the matrix $\mathbf{A}$, the last equality means that $\mathbf{x}$ is the eigenvector of $\mathbf{A}$, corresponding to λ. Hence a set of solutions of equation (10) coincides with the set of eigenvectors of the matrix $\mathbf{A}$, which consists of vectors $\mathbf{x} = \mathbf{A}_\lambda^\times\mathbf{v}$, where $\mathbf{v}$ is any nonzero vector of appropriate size. There are no other solutions of the equation.

Note that the computational complexity of the obtained solution is directly determined by computational costs of calculating the eigenvalue λ using Eq. (1) and the matrices $\mathbf{A}_\lambda^*$ and $\mathbf{A}_\lambda^+$ using Eqs. (2) and (3), which is based on finding a sum of matrix powers. Taking into account that the product of two matrices of order n requires not more than $O(n^3)$ arithmetical operations, the computational complexity of finding the sum of n powers and the solutions itself will be not higher than $O(n^4)$.

5. APPLICATION TO SCHEDULING PROBLEMS

Consider an application of the obtained results to the solution of optimal scheduling problems [22, 23]. Assume a project that involves n activities. For each activity $i = 1, \ldots, n$, denote the start time by x_i and the end time by y_i. Let a_{ij} be the minimum allowed time lag between the start of activity j and the end of activity i. The "start—finish" constraints, determining the relationships between the start and the end times of activities by the inequalities:

$$y_i \geqslant x_j + a_{ij}, \quad i, j = 1, \ldots, n,$$

are defined, as well.

Assume that each activity ends immediately, as soon as the "start—finish" constraints on its end time are fulfilled. Then at least one inequality must be fulfilled as equality, and all inequalities for the the end time of activity i can be combined in the equality

$$y_i = \max_{1 \leqslant j \leqslant n} (x_j + a_{ij}), \quad i = 1, \ldots, n.$$

For each activity i, the work cycle time is defined as the difference $y_i - x_i$ between the end and the start times. The maximum deviation of the cycle time over all activities in the project is determined, as follows:

$$\max_{1 \leqslant i \leqslant n} (y_i - x_i) - \min_{1 \leqslant i \leqslant n} (y_i - x_i) = \max_{1 \leqslant i \leqslant n} (y_i - x_i) + \max_{1 \leqslant i \leqslant n} (x_i - y_i).$$

The scheduling problem in accordance with the criterion of minimizing the maximum deviation of the cycle time is formulated to find the start time x_i and the end time y_i for each activity $i = 1, \ldots, n$, which solve the problem

$$\min_{x_1, \ldots, x_n} \left(\max_{1 \leqslant i \leqslant n} (y_i - x_i) + \max_{1 \leqslant i \leqslant n} (x_i - y_i) \right),$$
$$y_i = \max_{1 \leqslant j \leqslant n} (x_j + a_{ij}), \quad i = 1, \ldots, n.$$

This problem can be formulated as a linear programming problem and solved using a suitable numerical algorithm, e.g., a simplex-algorithm. This approach, however, does not allow the solutions to be obtained analytically in a closed form that is convenient for the formal analysis and immediate computations.

To construct the analytical solution, rewrite the problem in terms of the idempotent semi-field $\mathbb{R}_{\max,+}$. Introduce the following vectors and the matrix:

$$\mathbf{x} = (x_i), \quad \mathbf{y} = (y_i), \quad \mathbf{A} = (a_{ij}).$$

Then the problem is rewritten as

$$\min_{\mathbf{x}} \mathbf{y}^{-}\mathbf{x}\mathbf{x}^{-}\mathbf{y},$$
$$\mathbf{y} = \mathbf{A}\mathbf{x}.$$

Making the substitution $\mathbf{y} = \mathbf{A}\mathbf{x}$, one obtains the constraint-free problem

$$\min_{\mathbf{x}} \mathbf{x}^{-}\mathbf{A}\mathbf{x}(\mathbf{A}\mathbf{x})^{-}\mathbf{x},$$

whose complete solution is provided by Theorem 1.

Example 1. The obtained solution can be illustrated via the example below. Consider a project with $n = 3$ activities that are interrelated by "start—finish" constraints given by the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & 37 \\ 25 & 31 & 43 \\ 25 & 5 & 1 \end{pmatrix}.$$

Using arithmetic in the semi-field $\mathbb{R}_{\max,+}$, find the spectral radius λ of the matrix $\mathbf{A}$ from Eq. (1). The first step is to calculate the following matrices:

$$\mathbf{A}^2 = \begin{pmatrix} 62 & 42 & 43 \\ 68 & 62 & 74 \\ 29 & 36 & 62 \end{pmatrix}, \quad \mathbf{A}^3 = \begin{pmatrix} 68 & 73 & 99 \\ 99 & 93 & 105 \\ 87 & 67 & 79 \end{pmatrix}.$$

Next, one determines the traces of the matrix $\mathbf{A}$ and of its powers:

$$\operatorname{tr} \mathbf{A} = 31, \quad \operatorname{tr} \mathbf{A}^2 = 62, \quad \operatorname{tr} \mathbf{A}^3 = 93$$

as well as the spectral radius in the form

$$\lambda = \operatorname{tr} \mathbf{A} \oplus \operatorname{tr}^{1/2}(\mathbf{A}^2) \oplus \operatorname{tr}^{1/3}(\mathbf{A}^3) = 31.$$

Compose the matrix $\mathbf{A}_\lambda = \lambda^{-1}\mathbf{A}$ and its square:

$$\mathbf{A}_\lambda = \begin{pmatrix} -27 & -31 & 6 \\ -6 & 0 & 12 \\ -6 & -26 & -30 \end{pmatrix}, \quad \mathbf{A}_\lambda^2 = \begin{pmatrix} 0 & -20 & -19 \\ 6 & 0 & 12 \\ -32 & -26 & 0 \end{pmatrix}.$$

Using Eqs. (2) and (3), calculate the matrices

$$\mathbf{A}_\lambda^* = \mathbf{A}_\lambda^+ = \begin{pmatrix} 0 & -20 & 6 \\ 6 & 0 & 12 \\ -6 & -26 & 0 \end{pmatrix}.$$

Note that all columns in the matrix $\mathbf{A}_\lambda^+$ have $0 = \mathbb{1}$ on the diagonal. Taking into account that the third column is collinear (in terms of the semi-field $\mathbb{R}_{\max,+}$) with the first one, it can hence be discarded.

Making the matrix $\mathbf{A}_\lambda^\times$ from the first two columns of the matrix $\mathbf{A}_\lambda^+$, the solution of the problem can be written in the vector form

$$\mathbf{x} = \mathbf{A}_\lambda^\times \mathbf{v}, \quad \mathbf{A}_\lambda^\times = \begin{pmatrix} 0 & -20 \\ 6 & 0 \\ -6 & -26 \end{pmatrix},$$

where the vector $\mathbf{v} = (v_1, v_2)^T$ can be chosen arbitrarily.

In the ordinary notation, the coordinates of a vector $\mathbf{x} = (x_1, x_2, x_3)^T$, are represented as follows:

$$x_1 = \max(v_1, v_2 - 20), \quad x_2 = \max(v_1 + 6, v_2), \quad x_3 = \max(v_1 - 6, v_2 - 26).$$

6. CONCLUSIONS

A new tropical optimization problem is considered which requires the minimization of the function defined by an irreducible matrix on a set of vectors over an idempotent semi-field. It is shown that all solutions of the problem are the eigenvectors of the matrix, corresponding to its spectral radius and can be presented in a compact vector form, which is convenient for formal analysis and immediate computation with moderate computational complexity . The result is used for the direct explicit solution of a project scheduling problem. This analytical solution complements and expands the abilities of existing algorithmic methods in project scheduling, and is useful when the algorithmic numerical solution of the problem becomes for one reason or another unsuitable or impossible. The construction of solution to the problem with additional constraints and more complicated objective functions is of interest for further study.

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