Theoretical and Mathematical Physics, **200**(2): 1075–1093 (2019)

RADIAL SPECTRUM OF LIGHT MESONS IN PLANAR QCD SUM RULES AND THE SCALAR SIGMA-MESON

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In the framework of spectral sum rules in the planar limit of quantum chromodynamics, we propose two new methods for calculating the spectra of light mesons based on using linear radial Regge trajectories and the simplest quark–antiquark operators interpolating meson states. Both methods predict a resonance near 500 MeV in the scalar–isoscalar channel, which hypothetically corresponds to the lightest scalar hadron, the σ-meson. This can mean that the quark–antiquark component is strongly dominating in its structure even if the σ-meson is a tetraquark state. In one of the methods, we obtain a reasonable agreement with experimental data using only two input parameters: the phenomenological value of the gluon condensate and the weak decay constant of the pion. In this case, the predicted quark condensate value agrees well with contemporary lattice computation results.

Keywords: meson spectroscopy, quantum chromodynamics, spectral sum rules, planar limit

DOI: 10.1134/S0040577919080014

1. Introduction

It is well known that the nonperturbative physics of strong interactions is encoded in the values of hadron masses. This extremely complicated physics, which is still unclear in many aspects, is most clearly manifested in hadrons consisting of the lightest u- and d-quarks because the masses $m_{u,d}$ of these quarks are much less than the characteristic scale of nonperturbative strong interactions Λ_{QCD}. Moreover, precisely such hadrons determine the environment. In addition to the usual nucleons and pions, the scalar σ-meson plays an important role in nuclear physics and particle physics. For example, it is assumed that the exchange by this particle determines the main part of the attractive internucleon potential. The corresponding resonance in particle physics, denoted by $f_0(500)$ [1], appears in many low-energy models of strong interactions describing spontaneous breaking of the approximate chiral symmetry. Despite tremendous efforts to study the σ-meson over more than 60 years, its nature is still the subject of hot debates, described in detail in a recent survey [2]. Great progress in reducing the inaccuracy in determining its mass and total decay width has occurred in recent years [1]. More and more specialists are inclined to think that this wide resonance cannot be explained in the framework of the usual quark–antiquark meson picture [2]. The situation could be explained by direct computations of the σ-meson mass from the Lagrangian in quantum chromodynamics (QCD), but no convincing results in this direction have yet been obtained.

In the contemporary literature, the σ-meson, as a rule, is studied in the framework of approaches based on the analyticity and unitarity of the S-matrix [2]. The arising models are usually not directly related to QCD. The relation between QCD and the old approaches based on using the σ-meson, i.e., different effective field theories and bag models, also remains unclear [2]. In light-meson spectroscopy, the

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Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 200, No. 2, pp. 173–194, August, 2019. Received November 8, 2018. Revised December 26, 2018. Accepted February 19, 2019.

Shifman–Vainshtein–Zakharov (SVZ) method of spectral sum rules [3], [4], often called "ITPh" sum rules or simply QCD sum rules, is the phenomenological approach that is probably most explicitly related to QCD. The main idea underlying this method is the assumption that the quark–antiquark pair (or a more complicated quark current interpolating the hadron in question) arising in the QCD vacuum perturbs it relatively weakly. This idea allows parameterizing complex effects of the nonperturbative vacuum by several universal phenomenological characteristics, i.e., vacuum condensates that are the average vacuum values of various operators constructed from quark and gluon fields. The response to adding this current to the QCD vacuum yields correlation functions of this current in the vacuum shells, which can be calculated using the Wilson operator expansion in QCD where the corresponding operators are replaced with their vacuum averages. It is usually further assumed that the spectral density is saturated by the peak corresponding to the lightest resonance with quantum numbers of the constructed current after which the perturbative continuum follows. In this case, it is unimportant whether the constructed current can exist in nature, and we can formally calculate the response to any added current. But if at least one of the above assumptions is false, then the method does not work. As a result, it can be assumed that hadrons with distinct quantum numbers have distinct masses (decay constants, form-factors, etc.) because their currents interact differently with the vacuum environment leading to different responses of the QCD vacuum to the addition of a given current. The technical implementation of this concept turned out to be extremely fruitful for describing hadron phenomenology, which is described the classic reviews [5]–[7]. The most current review of the SVZ sum rules is the recent paper [8].

Here, we consider the computations of scalar and vector meson spectra in the framework of spectral sum rules in the so-called QCD planar limit for linear radial trajectories [9]. We give the corresponding definitions in the next section. In Sec. 2, our problem is to obtain sum rules with a minimal number of parameters that still ensure a reasonable description of the experimental spectra: an analysis showed that this number is two.

The most interesting results were obtained for the scalar σ-meson. It is usually assumed that the mass of the lightest scalar quark–antiquark state is near 1 GeV or above [2], [5] and the σ-meson is significantly lighter. At present, the dominating concept of the σ-meson nature is the tetraquark interpretation [2]. Our initial intention was to confirm the absence of a light scalar particle between quark–antiquark meson states by using the QCD sum rules in the planar limit matched with the Regge phenomenology. But the result turned out to be the opposite: a light scalar state can be naturally predicted. The apparent reason for this is that in the framework of the methods presented in [2] (which are mainly based on dispersion relations and unitarity), the description of the σ-meson is mainly "isolated in a certain sense" from the other physics. A conceptual distinction of the spectral sum rules considered here is the close relation between the arising σ-meson and the existence of resonances both in other channels (primarily in the vector channel) and on the corresponding radial scalar trajectory.

In Sec. 3, we propose a new approach for considering planar QCD sum rules. The idea is to use the Borel transform in the planar limit, which must significantly improve the convergence. Further, expressions for the inclination a in linear spectrum (5) can be derived by the method used to determine the ground state mass in the classical SVZ sum rules [3], [4]. In other words, we propose to consider Borelized planar sum rules and analyze them using a well-developed technique. The value m_0 (ground state mass) is regarded as the value known from old SVZ sum rules or from experimental data. In the first case, the usual computations of m_0 by using the SVZ sum rules are only a preliminary stage in determining the complete mass spectrum of radial excitations. An extension of the SVZ sum rules is therefore constructed that allows obtaining the mass spectrum of radial excitations using the same number of input parameters. The main result is the calculation of the trajectory inclination a, which is an important parameter in various theoretical studies and in hadron phenomenology. We apply this approach to light vector, axial vector, and scalar mesons.

In the last case, we obtain an unexpected result related to the appearance of a second scalar trajectory starting from the light state of a mass of nearly 500 MeV, which completely agrees with the approach used in Sec. 2.

2. Planar sum rules

2.1. General scheme. The physical characteristics of hadrons are contained in different correlation functions of quark currents with quantum numbers of the corresponding hadrons. The most important physical characteristic of a hadron is its mass. According to the general principles of quantum field theory, it appears as the real part of the pole of the two-point correlator $\langle JJ\rangle$, where the current J is constructed of quark and gluon fields and interpolates the hadron in question. For example, if the scalar–isoscalar state f_0 is a usual light quark–antiquark meson without strange quarks, then the current interpolating it must be quark and bilinear $J=\bar{q}q$, where q denotes a u- or d-quark.

The SVZ sum rules follow from an analysis of the two-point correlation function

$$\langle JJ\rangle = i\int d^4x\, e^{iqx}\langle 0|\mathrm{T}\{J(x),J(0)\}|0\rangle. \tag{1}$$

Correlator (1) contains much dynamical information. In particular, the asymptotic behavior of expression (1) at large distances in the Euclidean space is $\sim e^{-m|x|}$, where m is the mass of the lightest hadron with quantum numbers of the current J. This property underlies the lattice computations of hadron masses directly from the QCD Lagrangian. Unfortunately, there are no analytic methods for calculating correlator (1) because of the problem of the strong bond in hadronization. The main problem of the classical SVZ sum rules is to calculate the hadron masses from relation (1) using some semianalytic methods supplemented with some phenomenological input parameters. On the whole, the idea is to equate two representations of correlator (1). The first of these representations is the Wilson operator expansion in vacuum shells, and the second is a certain dispersion relation into which the ansatz for the spectral density is then introduced. In the classical sum rules, this ansatz is taken in the form of a single infinitely narrow resonance in the sum with the perturbative continuum [3]–[5], where the "continuum" origin is subsequently chosen from the phenomenology.

From the theoretical standpoint, the zero-width approximation and the simultaneous absence of multiparticle cuts of amplitude (1) on the positive axis of the squared momentum appear in the limit of a large number of colors N_c for the fixed product g^2N_c, where g is the QCD coupling constant. This limit is often called the 't Hooft limit or the planar limit [10], [11], and the latter term is related to the fact that only planar diagrams survive in this limit. It can be shown that in this case, the only poles of the two-point correlator of the quark current J are single-hadron states [11]. For mesons, the two-point correlator has the form

$$\langle J(q)J(-q)\rangle = \sum_n \frac{F_n^2}{q^2-M_n^2} \tag{2}$$

in the lowest order in $1/N_c$ (in the momentum space). The behavior of the main spectral characteristics is known for large N_c: $M_n=\mathcal{O}(1)$ for masses, $F_n^2=\langle 0|J|n\rangle^2=\mathcal{O}(N_c)$ for residues, and $\Gamma=\mathcal{O}(1/N_c)$ for the complete widths of strong decays [11]. The asymptotic freedom in QCD dictates a logarithmic asymptotic behavior for the left-hand side of (2) for large q^2. This behavior is possible if the sum has finitely many terms [11].

Because the right-hand side of relation (2) can be summed by the Euler–Maclaurin formula,

$$\sum_{n=0}^{N}\frac{F^2(n)}{q^2-M^2(n)} = \int_0^N \frac{F^2(x)\,dx}{q^2-M^2(x)} + \mathcal{O}\biggl(\frac{1}{q^2}\biggr), \tag{3}$$

a logarithmic asymptotic behavior arises for the residues for sufficiently large n:

$$F_n^2 \sim \frac{dM_n^2}{dn}. \tag{4}$$

Under the assumption that there is a certain ansatz for the mass spectrum of radial excitations M_n, we can sum expression (2), expand it for large spatially similar momenta $Q^2 = -q^2$, and compare the result with the corresponding operator expansion in QCD. We then obtain a set of sum rules where each sum rule is an equation corresponding to some k in the $1/Q^{2k}$ expansion, $k = 0, 1, 2, \ldots$, of both sides of (2).

In the simplest case, based on certain theoretical considerations (hadron string models, Veneziano amplitude, two-dimensional QCD in the planar limit), we can choose the radial mass spectrum in the planar limit to be linear:

$$M_n^2 = an + m_0^2, \quad n = 0, 1, 2, \ldots. \tag{5}$$

In this case, the inclination a in the first approximation is independent of the quantum numbers, which can be explained by the universality of the gluodynamics determining the inclination. This Regge behavior of the radial spectrum of light mesons has been confirmed experimentally [12], [13], and the inclinations indeed exhibit a nearly universal behavior in experiments. Within the accuracy of the planar limit (10 to 20%), the universal behavior of the inclination is an acceptable assumption [14], [15].

In this case, relation (4) implies $F_n^2 \sim$ const. The sum rules thus obtained are well known and have already been used many times [16]–[22]. It is interesting to note that these sum rules also appear in the "bottom-up" holographic approach to strong interactions (see, e.g., the discussions in [23]).

2.2. Vector mesons. Because the vector current $J_\mu^{\mathrm{V}} = \bar{q}\gamma_\mu q$ is conserved, the vector two-point correlator is transverse and depends only on one scalar function:

$$\langle J_\mu^{\mathrm{V}}(q) J_\nu^{\mathrm{V}}(-q)\rangle = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi_{\mathrm{V}}(q^2). \tag{6}$$

We consider the simplest linear Regge ansatz of type (5) following the motivation discussed above:

$$M_{\mathrm{V}}^2(n) = an + M_{\mathrm{V}}^2, \quad n = 0, 1, 2, \ldots. \tag{7}$$

Because the isosinglet and isotriplet states are degenerate in the limit of a large number of colors N_c [11], the spectra for ω- and ρ-mesons are indistinguishable in our case. We now consider the isosinglet states.

The available experimental data suggest that the ground state is significantly below the linear trajectory (see Fig. 1), and we hence distinguish the ground state from linear trajectory (7). Using spectral representation (2), definition (6), and ansatz (7) in the Euclidean space $Q^2 = -q^2$, we obtain

$$\Pi_{\mathrm{V}}(Q^2) = \frac{F_\omega^2}{Q^2 + M_\omega^2} + \sum_{n=0}^{\infty} \frac{F^2}{Q^2 + an + M_{\mathrm{V}}^2}. \tag{8}$$

It follows from the arguments presented in Sec. 1 that the residues F^2 of excited states in (8) can be taken to be constant.

In the chiral and planar limits (setting $N_c = 3$ at the end), the operator expansion of the vector correlator for large Q^2 has the form [3], [4]

$$\Pi_{\mathrm{V}}(Q^2) = -\frac{C_0}{8\pi^2}\log\frac{Q^2}{\mu^2} + \frac{1}{24Q^4}\frac{\alpha_s}{\pi}\langle(G_{\mu\nu}^a)^2\rangle - \frac{14}{9}\frac{\pi\alpha_s}{Q^6}\langle\bar{q}q\rangle^2 + \ldots, \tag{9}$$

where $\langle(G_{\mu\nu}^a)^2\rangle$ and $\langle\bar{q}q\rangle$ denote the gluon and quark condensates.

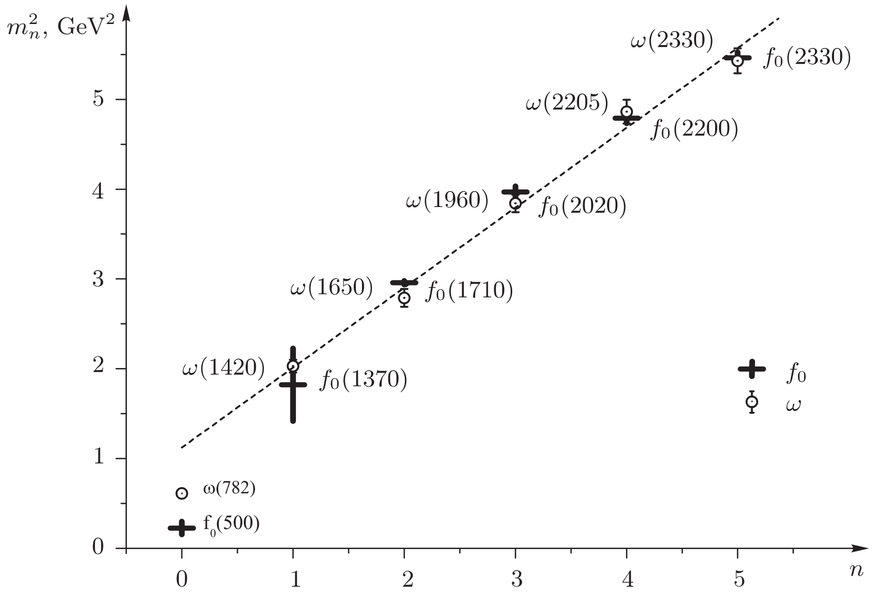

Fig. 1. Hypothetical spectrum of ω-mesons (circles) and f_0-mesons (crosses) [1]: the crosses are drawn with a large fixed horizontal size to visualize the position of scalar resonances. The resonance $f_0(1500)$ is eliminated because the available data indicate the global nature of this state (see [1]). The graph is taken from [27].

We note that the coefficient 14/9 in the last term is related to precisely the planar limit. This coefficient generally contains multipliers related to the number of colors N_c,

$$\frac{N_c^2 - 1}{N_c^2} \cdot \frac{14}{9},$$

and for $N_c = 3$, the coefficient must be 112/81, but it turns out that the correct value is 14/9 in the planar limit ($N_c \to \infty$).

It follows from the principles of the classical QCD sum rules [3], [4] that these vacuum characteristics must be universal, i.e., independent of the quantum numbers of the quark current J. The multiplier C_0 has a perturbative correction $C_0 = 1 + \alpha_s/\pi$ to the leading logarithm but can be suppressed in our case of large N_c, and we hence set $C_0 = 1$.

We can rewrite expression (8) using the function ψ, i.e., the logarithmic derivative of the function Γ:

$$\sum_{n=0}^{\infty} \frac{1}{n+a} = -\psi(a) + \text{const.} \tag{10}$$

This function has the asymptotic expansion for large argument values

$$\psi(z) = \log z - \frac{1}{2z} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k z^{2k}}, \tag{11}$$

where B_{2k} are Bernoulli numbers. These formulas can be used to expand correlator (8) for large Q^2. Introducing the dimensionless variables

$$m_v = \frac{M_{\rm V}}{\sqrt{a}}, \qquad m_\omega = \frac{M_\omega}{\sqrt{a}}, \qquad f = \frac{F}{\sqrt{a}}, \qquad f_\omega = \frac{F_\omega}{\sqrt{a}}, \tag{12}$$

we obtain

$$\Pi_{\rm V}(Q^2) = -f^2 \log\frac{Q^2}{\mu^2} + \frac{a}{Q^2}\left[f_\omega^2 - f^2\left(m_v^2 - \frac{1}{2}\right)\right] + \\ + \frac{a^2}{Q^4}\left[-f_\omega^2 m_\omega^2 + \frac{1}{2} f^2\left(m_v^4 - m_v^2 + \frac{1}{6}\right)\right] + \\ + \frac{a^3}{Q^6}\left[f_\omega^2 m_\omega^4 - \frac{1}{3} f^2 m_v^2\left(m_v^2 - \frac{1}{2}\right)(m_v^2 - 1)\right] + \ldots. \tag{13}$$

We obtain the planar sum rules for linear spectrum (7) by comparing the terms with like powers of $1/Q^2$ in (13) and (9).

2.3. Axial vector mesons. The axial vector current $J_\mu^{\rm A} = \bar{q}\gamma_\mu\gamma_5 q$ is not conserved, and the axial two-point correlator hence has two independent contributions:

$$\langle J_\mu^{\rm A}(q) J_\nu^{\rm A}(-q)\rangle = \Pi_{\rm A}(q^2) q_\mu q_\nu - \widetilde{\Pi}_{\rm A}(q^2) g_{\mu\nu}. \tag{14}$$

The sum rules for $\Pi_{\rm A}$ and $\widetilde{\Pi}_{\rm A}$ differ, and $\Pi_{\rm A}$ therefore contains an additional term from the pole of the π-meson because of the spontaneous chiral symmetry breaking:

$$J_\mu^{\rm A} \sim f_\pi\, \partial_\mu \pi.$$

In our normalization, the weak decay constant of the π-meson is $f_\pi = 93\,{\rm MeV}$. The transverse part in (14) is usually separated [24] by adding and subtracting the term $g_{\mu\nu} q^2 \Pi_{\rm A}$ and considering the sum rules for $\Pi_{\rm A}$ together with the sum rules for $\Pi_{\rm V}$. Strictly speaking, the obtained sum rules are the sum rules not for the axial vector current but for its divergence.

We consider the linear ansatz for the radial axial spectrum with the same value of the trajectory inclination as in the case of vector mesons. The axial analogue of correlator (8) has the form

$$\Pi_{\rm A}(Q^2) = \frac{f_\pi^2}{Q^2} + \sum_{n=0}^{\infty} \frac{F^2}{Q^2 + an + M_{\rm A}^2}. \tag{15}$$

Strictly speaking, we must consider the isosinglet η-meson instead of the π-meson. But if there are only two flavors, the difference is insignificant in the limit of large N_c. Operator expansion (15) becomes [3], [4]

$$\Pi_{\rm A}(Q^2) = -\frac{C_0}{8\pi^2}\log\frac{Q^2}{\mu^2} + \frac{1}{24Q^4}\frac{\alpha_s}{\pi}\langle (G_{\mu\nu}^a)^2\rangle + \frac{22}{9}\frac{\pi\alpha_s}{Q^6}\langle\bar{q}q\rangle^2 + \ldots. \tag{16}$$

We note that only the last terms differ in (9) and (16).

Proceeding as in the vector case and using dimensionless variables (12) ($m_a = M_{\rm A}/\sqrt{a}$), we obtain

$$\Pi_{\rm A}(Q^2) = -f^2\log\frac{Q^2}{\mu^2} + \frac{a}{Q^2}\left[\frac{f_\pi^2}{a} - f^2\left(m_a^2 - \frac{1}{2}\right)\right] + \\ + \frac{a^2}{Q^4}\frac{f^2}{2}\left(m_a^4 - m_a^2 + \frac{1}{6}\right) - \frac{a^3}{Q^6}\frac{f^2}{3} m_a^2\left(m_a^2 - \frac{1}{2}\right)(m_a^2 - 1) + \ldots. \tag{17}$$

As in the vector case, the axial sum rules can be obtained by comparing (16) and (17).

2.4. Sum rules in the vector case. As already explained, the set of vector and axial sum rules can be obtained by equating the terms with $\log Q^2$, $1/Q^2$, $1/Q^4$, and $1/Q^6$ in (9) and (13) and in (16) and (17). Our "initial" data are the constant f_π and the gluon condensate $(\alpha_s/\pi)\langle(G^a_{\mu\nu})^2\rangle$. One of the results of the calculations is the value of the condensate $\alpha_s\langle\bar{q}q\rangle^2$ of dimension six, which has a rather small but still nonzero anomalous dimension. The sum rules turn out to be consistent for some value of this condensate, which can be used to obtain the quark condensate on a certain renormalization scale. For $1/Q^6$, we therefore have only one sum rule, which follows from comparing the $1/Q^6$ terms in (13) and (17) with the multiplier $-7/11$ (just as is prescribed by operator expansions (9) and (16)). The obtained set of equations has the form

$$
\begin{aligned}
&f^2 = \frac{1}{8\pi^2},\\
&f^2\left(m_v^2 - \frac{1}{2}\right) = f_\omega^2,\\
&af^2\left(m_a^2 - \frac{1}{2}\right) = f_\pi^2,\\
&a^2\left[-f_\omega^2 m_\omega^2 + \frac{1}{2}f^2\left(m_v^4 - m_v^2 + \frac{1}{6}\right)\right] = \frac{1}{24}\frac{\alpha_s}{\pi}\langle(G^a_{\mu\nu})^2\rangle,\\
&a^2 f^2\left(m_a^4 - m_a^2 + \frac{1}{6}\right) = \frac{1}{12}\frac{\alpha_s}{\pi}\langle(G^a_{\mu\nu})^2\rangle,\\
&f_\omega^2 m_\omega^4 - \frac{1}{3}f^2 m_v^2\left(m_v^2 - \frac{1}{2}\right)(m_v^2 - 1) = \frac{7}{33}f^2 m_a^2\left(m_a^2 - \frac{1}{2}\right)(m_a^2 - 1).
\end{aligned}
\tag{18}
$$

We have a system of six polynomial equations with six variables (a, m_v^2, m_ω^2, m_a^2, f^2, and f_ω^2) and the input data $f_\pi = 93\,\text{MeV}$ and $(\alpha_s/\pi)\langle(G^a_{\mu\nu})^2\rangle = (360\pm 20\,\text{MeV})^4$. This system can be solved numerically. To show that the solutions are sensitive to the choice of the input data, we also calculate the case $f_\pi = 87\,\text{MeV}$ (the hypothetical value of f_π in the chiral limit [25]) and analyze the indeterminacy due to the inaccuracy of the gluon condensate value. The physical solutions are shown in Table 1.

Table 1

	$f_\pi = 93$ MeV	$f_\pi = 87$ MeV
a	1.43(2)	1.32(2)
M_{V}	1.60(4)	1.45(4)
M_{A}	1.31(1)	1.21(1)
M_ω	0.79(3)	0.69(3)
F	0.16	0.15
F_ω	0.14	0.13
$(-\langle\bar{q}q\rangle)^{1/3}$	0.30(1)	0.27(1)

Numerical solutions (GeV).

For the condensate $\alpha_s\langle\bar{q}q\rangle^2$ of dimension six, the self-consistent solution appears for the choice of $\alpha_s \simeq 1/\pi \simeq 0.3$, which corresponds to the scale $\mu \simeq 2\,\text{GeV}$. The obtained quark condensate value hence corresponds to this scale. Table 1 shows that the obtained values are significantly greater than $(0.23\,\text{GeV})^3$ in absolute value, which follows from the Gell-Mann–Oakes–Renner formula for the pion mass, $m_\pi^2 f_\pi^2 = -(m_u + m_d)\langle\bar{q}q\rangle$. But we must bear in mind that this is an approximate formula.

The contemporary average value of the quark condensate obtained by lattice computations is near $-(0.27\,\text{GeV})^3$ on the scale $\mu \simeq 2\,\text{GeV}$ [26]. And we reconstructed precisely this value in the chiral limit!

The data in Table 1 can be used to calculate the mass spectrum. The masses of the first three states are given in Table 2.

Table 2

Case	n	0	1	2
$f_\pi = 93\,\text{MeV}$	$M_\text{V}(n)$	0.79	1.60	2.15
	$M_\text{A}(n)$	1.31	1.93	2.41
$f_\pi = 87\,\text{MeV}$	$M_\text{V}(n)$	0.69	1.45	1.96
	$M_\text{A}(n)$	1.21	1.79	2.22

Masses of the first three predicted states (GeV).

Taking the above assumptions into account, we can conclude that the obtained solution well describes the corresponding phenomenology: the masses of ground states are close to the experimental masses of the unflavored vector $\omega(782)$-meson and axial vector $f_1(1285)$-meson [1].

There are some contradictions in interpreting the experimental data and determining real physical masses for the radially excited states [1]. If we speak about the qualitative characteristics of the model, then we note that the obtained masses are apparently near the experimental masses and hence seem reasonable.

2.5. Sum rules in the scalar case. We consider the two-point correlator of the scalar–isoscalar current $J^\text{S} = \bar{q}q$. Its resonance representation has the form

$$\Pi_\text{S}(q^2) = \langle J^\text{S}(q) J^\text{S}(-q)\rangle = \sum_n \frac{G_n^2 M_\text{S}^2(n)}{q^2 - M_\text{S}^2(n)}, \tag{19}$$

where the residues are determined by the formula $\langle 0|J^\text{S}|n\rangle = G_n M_\text{S}(n)$. As in the vector cases, we consider the linear radial spectrum with a universal inclination:

$$M_\text{S}^2(n) = an + M_\text{S}^2, \quad n = 0, 1, 2, \ldots. \tag{20}$$

Moreover, for consistency with the operator expansion, we assume that all analogues of the decay constants are equal to each other as in the vector channel: $G_n = G$.

Because we cannot state in advance that the lightest scalar meson belongs to the scalar radial trajectory, we consider two possibilities: (a) the ground state ($n = 0$) is on linear trajectory (20) or (b) the state with $n = 0$, which is further called σ, is not described by linear spectrum (20). The second assumption seems more physical (see Fig. 1). The corresponding spectral expansions in the Euclidean space have the forms

$$\Pi_\text{S}^{(\text{I})}(Q^2) = \sum_{n=0}^{\infty} \frac{G^2(an + M_\text{S}^2)}{Q^2 + an + M_\text{S}^2}, \tag{21}$$

$$\Pi_\text{S}^{(\text{II})}(Q^2) = \frac{G_\sigma^2 M_\sigma^2}{Q^2 + M_\sigma^2} + \sum_{n=1}^{\infty} \frac{G^2(an + M_\text{S}^2)}{Q^2 + an + M_\text{S}^2}. \tag{22}$$

Proceeding as in the vector case, we calculate expansions (21) and (22) for large Q^2 and compare them with the operator expansion of scalar correlator (19). Introducing the dimensionless variables $m_s = M_\text{S}/\sqrt{a}$ and

$g = G/\sqrt{a}$, we obtain expansions in the forms

$$\Pi_{\rm S}^{(\rm I)}(Q^2) = g^2 Q^2 \log\frac{Q^2}{\mu^2} - \frac{a^2}{Q^2}\frac{g^2}{2}\left(m_s^4 - m_s^2 + \frac{1}{6}\right) + \\ + \frac{a^3}{Q^4}\frac{g^2}{3} m_s^2\left(m_s^2 - \frac{1}{2}\right)(m_s^2 - 1) + \ldots, \tag{23}$$

$$\Pi_{\rm S}^{(\rm II)}(Q^2) = g^2 Q^2 \log\frac{Q^2}{\mu^2} + \frac{G_\sigma^2 M_\sigma^2}{Q^2} - \frac{a^2}{Q^2}\frac{g^2}{2}\left(m_s^4 + m_s^2 + \frac{1}{6}\right) - \\ - \frac{G_\sigma^2 M_\sigma^4}{Q^4} + \frac{a^3}{Q^4}\frac{g^2}{3} m_s^2\left(m_s^2 + \frac{1}{2}\right)(m_s^2 + 1) + \ldots. \tag{24}$$

The operator expansion of correlator (19) in the chiral limit and in the limit of large N_c [5] has the form

$$\Pi_{\rm S}(Q^2) = \frac{3C_0}{16\pi^2} Q^2 \log\frac{Q^2}{\mu^2} + \frac{1}{16Q^2}\frac{\alpha_s}{\pi}\langle (G_{\mu\nu}^a)^2\rangle - \frac{11}{3}\frac{\pi\alpha_s}{Q^4}\langle\bar{q}q\rangle^2 + \ldots, \tag{25}$$

where

$$C_0 = 1 + \frac{11\alpha_s}{3\pi}. \tag{26}$$

The contribution of the perturbative correction to the factor before the logarithm can now be greater than 30%, which is much greater than in the vector channels, and can hence be easier to take into account. Comparing the logarithmic terms, we obtain

$$g^2 = \frac{3C_0}{16\pi^2}. \tag{27}$$

We consider assumption *a*. From (23) and (25), we obtain two sum rules:

$$\begin{aligned} &\frac{3C_0}{2\pi^2} a^2\left(m_s^4 - m_s^2 + \frac{1}{6}\right) = -\frac{\alpha_s}{\pi}\langle (G_{\mu\nu}^a)^2\rangle, \\ &\frac{3C_0}{16\pi^2} a^3 m_s^2\left(m_s^2 - \frac{1}{2}\right)(m_s^2 - 1) = -11\pi\alpha_s\langle\bar{q}q\rangle^2. \end{aligned} \tag{28}$$

Substituting the numerical values of a and $\langle\bar{q}q\rangle$ from the solutions of the vector sum rules (Table 1), we obtain two independent polynomial equations. If we neglect the perturbative correction in (26) ($C_0 = 1$), then we see that for $f_\pi = 93\,\text{MeV}$,[1] Eqs. (28) have a common approximate solution $m_s^2 \simeq 0.74$, which leads to the radial scalar spectrum $M_{\rm S}(n) \simeq 1.23$, 1.89, 2.37, ... GeV. If we include the perturbative correction, then the general solution disappears.

We now study the more physical assumption *b*. Comparing (24) with operator expansion (25), we obtain the sum rules

$$G_\sigma^2 M_\sigma^2 - \frac{3C_0}{32\pi^2} a^2\left(m_s^4 + m_s^2 + \frac{1}{6}\right) = \frac{1}{16}\frac{\alpha_s}{\pi}\langle (G_{\mu\nu}^a)^2\rangle, \tag{29}$$

$$-3G_\sigma^2 M_\sigma^4 + \frac{3C_0}{16\pi^2} a^3 m_s^2\left(m_s^2 + \frac{1}{2}\right)(m_s^2 + 1) = -11\pi\alpha_s\langle\bar{q}q\rangle^2. \tag{30}$$

We have two equations with three unknowns m_s, M_σ, and G_σ. Eliminating G_σ from them, we obtain the relation for the σ-meson mass as a function of the parameter m_s^2:

$$M_\sigma^2 = \frac{(C_0/16\pi^2)a^3 m_s^2(m_s^2 + 1/2)(m_s^2 + 1) + (11/3)\pi\alpha_s\langle\bar{q}q\rangle^2}{(3C_0/32\pi^2)a^2(m_s^4 + m_s^2 + 1/6) + (\alpha_s/16\pi)\langle (G_{\mu\nu}^a)^2\rangle}. \tag{31}$$

[1] If we take $f_\pi = 87\,\text{MeV}$, then the solution $m_s^2 \simeq 0.72$ implies the spectrum $M_{\rm S}(n) \simeq 1.12$, 1.73, 2.17, ... GeV.

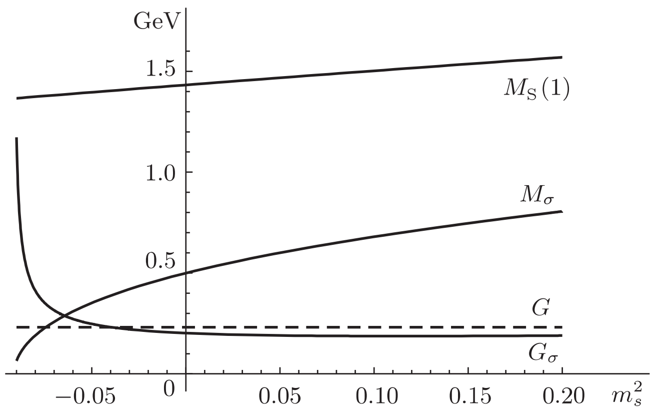

Fig. 2. Values of M_σ, G_σ, G and the masses of the first state on the scalar trajectory $M_S(1)$ (in GeV) obtained from (31) and (29) (or (30)) and (27) as functions of the dimensionless parameter m_s^2 and $M_S^2(1) = a(1 + m_s^2)$ in (20).

The "decay constant" G_σ as a function of m_s^2 can be obtained by substituting (31) in (29) or (30). Plots of M_σ, G_σ, $G = \sqrt{a}g$ (where g is defined in (27)), and the mass of the first state on the scalar trajectory are shown in Fig. 2. They are obtained using the input data given in Table 1 for $f_\pi = 93\,\mathrm{MeV}$ and $\alpha_s \simeq 1/\pi$ in (26). The parameter m_s^2 can be negative because the sum in (22) starts from $n = 1$.

Another version of input data corresponding to $f_\pi = 87\,\mathrm{MeV}$ in Table 1 and with $\alpha_s = 0$ in (26) implies that the masses are shifted by 70 to 80 MeV, which lies within the accuracy of the limit of large N_c. The general picture shown in Fig. 2 remains the same in this case. Considering the negative free term m_s^2 of the linear ansatz, we can immediately see the nonphysical behavior for relatively small values of m_s^2. The mass $M_S(1)$ is rather stable and reconstructs the value of the mass of the $a_0(1450)$-meson, $M_{a_0(1450)} = 1474 \pm 19\,\mathrm{MeV}$ [1]. Its isosinglet partner (a candidate is the $f_0(1370)$-meson) must degenerate with $a_0(1450)$ in the planar limit.

The plot in Fig. 2 demonstrates that the real prediction for M_σ is rather sensitive to the free term of the linear scalar trajectory, although M_σ was not originally described by linear spectrum (20). Conversely, the expected value of M_σ (near 0.5 GeV [1]) imposes a strong constraint on the possible values of m_s^2. The graph in Fig. 2 shows that the value of m_s^2 is close to zero.

Although both the ω-meson and the σ-meson are not on the corresponding linear trajectories (as assumed, e.g., in Fig. 1), there is still a difference between them in their analysis. In the vector case, it was important that the sum in (8) starts from $n = 0$ and we can hence compare the expansion in resonances in the vector case with expansion (15) in the axial case. If we start from $n = 0$ in scalar channel (22), then the signs of the numerator and denominator in (31) depend on the value of m_s^2:

$$M_\sigma^2 = \frac{(C_0/16\pi^2)a^3 m_s^2(m_s^2 - 1/2)(m_s^2 - 1) + (11/3)\pi\alpha_s\langle\bar{q}q\rangle^2}{(3C_0/32\pi^2)a^2(m_s^4 - m_s^2 + 1/6) + (\alpha_s/16\pi)\langle(G_{\mu\nu}^a)^2\rangle}, \tag{32}$$

which makes the prediction for M_σ extremely unstable and indeterminate. The σ-meson is not unusual in this sense, because its mass belongs to the radial scalar trajectory. Its mass is simply not described by linear ansatz (20). This interpretation can be additionally motivated by comparing the residues: G_σ is somewhat below G. Physically, this means that the external source of scalar mesons (a certain scalar current) creates the lightest state with a probability close to the probability of creating other scalar resonances. In the range of our error, the interaction between σ-mesons and this source is a bit suppressed.

It is interesting to try to assume the complete universality of the residues ($G_\sigma = G$) from the very beginning. After this substitution, sum rules (29) and (30) (or their analogues if we begin with $n = 0$

in (22); the shift arising in this case does not affect the results) are two equations with two unknowns M_σ^2 and m_s^2. This system has four numerical solutions. Two of them are not physical, because they do not lead to tachyonic masses. The third solution (in dimensionless variables, $m_\sigma = M_\sigma/\sqrt{a}$, $m_s = M_{\rm S}/\sqrt{a}$, and for $f_\pi = 93\,\text{MeV}$) is $(m_\sigma^2, m_s^2) \simeq (0.742, 0.739)$, which reproduces the solution for possibility *a* above. This solution corresponds to the branch where the condensates in the right-hand sides of Eqs. (29) and (30) can be neglected. The fourth solution $(m_\sigma^2, m_s^2) \simeq (0.074, -0.040)$ predicts the value $M_\sigma \simeq 0.39\,\text{GeV}$ and the radial spectrum $M_{\rm S}(n) \simeq 1.40,\ 2.01,\ \ldots$ GeV. These values (except the masses of higher excitations) can be seen in Fig. 2; they correspond to the point of intersection of the lines G and G_σ. This solution is most interesting: the obtained mass of the σ-meson is close to the expected domain of masses [1] and the radial spectrum seems rather reasonable.

Our prediction of the Regge trajectory with the σ-meson on it apparently contradicts the studies of the σ state on the complex Regge trajectory, which show that the corresponding state cannot belong to the usual Regge trajectory because of a very large width [2], [28]. But it is not excluded that this observation can simply indicate the restrictions of the usual methods used to describe the $\pi\pi$ scattering. These methods are based on the property of the S-matrix to be analytic and unitary and do not contain dynamical assumptions, at least not explicitly. The origin of a very large width for the $f_0(500)$-meson is most likely a certain dynamical effect. Therefore, it is necessary to use dynamical approaches to discover the true nature of the σ-meson. Such approaches must simultaneously describe all resonances on the scalar trajectory and in other channels (vector and tensor mesons). This property originally underlies the considered spectral sum rules, which makes this method conceptually different from the dispersion approaches.

Our analysis thus demonstrates that the existence of a light scalar state together with the structure of planar sum rules in the scalar channel can also be obtained naturally from the Regge phenomenology.

3. Borelized sum rules

3.1. Derivation of Borelized sum rules. The classical SVZ sum rules for light hadrons are formed in the Borelized version [3], [4]. We construct a generalization of this formalism to the case of infinitely many meson states expected in the planar limit. We first consider the case of vector mesons consisting of only u- and d-quarks in detail.

The operator expansion for the correlator of vector currents has the form

$$\Pi(Q^2) = -\frac{1}{8\pi^2}\log\frac{Q^2}{\mu^2} + \frac{1}{Q^4}\langle m\bar{q}q\rangle + \frac{1}{24Q^4}\frac{\alpha_s}{\pi}\langle (G_{\mu\nu}^a)^2\rangle - \frac{14}{9}\frac{\pi\alpha_s}{Q^6}\langle\bar{q}q\rangle^2. \tag{33}$$

This expansion without the second term was also used in Sec. 2 (see (9)). That term appears if we abandon the assumption of the chiral limit but preserve the contribution linear in the quark masses (see, e.g., [5]).

We apply the Borel transform

$$L_M(Q^2) = \lim_{\substack{Q^2,n\to\infty\\ Q^2/n=M^2}} \frac{1}{(n-1)!}(Q^2)^n\left(-\frac{d}{dQ^2}\right)^n$$

to the operator expansion of correlator (33):

$$L_M\Pi(Q^2) = \frac{1}{8\pi^2} + \frac{1}{M^4}\langle m\bar{q}q\rangle + \frac{1}{24M^4}\frac{\alpha_s}{\pi}\langle (G_{\mu\nu}^a)^2\rangle - \frac{7}{9}\frac{\pi\alpha_s}{M^6}\langle\bar{q}q\rangle^2.$$

We then consider the expansion of the correlator in an infinite sum over resonances:

$$\Pi(q^2) = \sum_n \frac{F^2}{q^2 - m_n^2 - i\varepsilon}.$$

We separate the imaginary part of this expansion using the Sokhotskii formula,

$$\operatorname{Im}\Pi(q^2) = \sum_n \pi F^2 \delta(q^2 - m_n^2).$$

The dispersion relation for this correlator has the form (we omit the inessential subtraction)

$$\Pi(q^2) = \frac{1}{\pi}\int_{s_0}^{\infty} ds \frac{\operatorname{Im}\Pi(s)}{s - q^2 - i\varepsilon}.$$

Applying the Borel transform to it, we obtain

$$L_M \Pi(q^2) = \frac{1}{\pi M^2}\int_{s_0}^{\infty} e^{-s/M^2} \operatorname{Im}\Pi(s)\, ds.$$

We substitute the imaginary part in the Borel transform for the dispersion relation and integrate:

$$\frac{1}{\pi M^2}\int ds\, e^{-s/M^2} \sum_n \pi F^2 \delta(s - m_n^2) = \frac{F^2}{M^2}\sum_n e^{-m_n^2/M^2}. \tag{34}$$

We substitute linear mass spectrum (5) and sum relation (34) as a geometric progression:

$$\sum_n e^{-m_n^2/M^2} = e^{-m_0^2/M^2}\sum_n e^{-a\cdot n/M^2} = \frac{e^{-m_0^2/M^2}}{1 - e^{-a/M^2}}.$$

After several transformations, we equate both expansions of the correlator:

$$\frac{F^2 e^{-m_0^2/M^2}}{1 - e^{-a/M^2}} = \frac{M^2}{8\pi^2}\left[1 + \frac{h_1}{M^2} + \frac{h_2}{M^4} + \frac{h_3}{M^6}\right]. \tag{35}$$

We here let h_i denote the coefficients of the corresponding powers of $1/M^2$, which in our example of vector mesons have the forms

$$h_1 = 0, \qquad h_2 = 8\pi^2\langle m\bar{q}q\rangle + \frac{\pi^2}{3}\frac{\alpha_s}{\pi}\langle (G^a_{\mu\nu})^2\rangle, \qquad h_3 = -\frac{56}{9}\frac{\pi^3\alpha_s}{M^6}\langle\bar{q}q\rangle^2.$$

We differentiate Eq. (35) with respect to $1/M^2$. The derivative of the left-hand side is

$$F^2 e^{-m_0^2/M^2}\frac{-m_0^2(1 - e^{-a/M^2}) - ae^{-a/M^2}}{(1 - e^{-a/M^2})^2}.$$

For the right-hand side, we obtain

$$-\frac{M^4}{8\pi^2}\left[1 + \frac{h_1}{M^2} + \frac{h_2}{M^4} + \frac{h_3}{M^6}\right] + \frac{M^2}{8\pi^2}\left[h_1 + \frac{2h_2}{M^2} + \frac{3h_3}{M^4}\right] = -\frac{M^4}{8\pi^2}\left[1 - \frac{h_2}{M^4} - \frac{2h_3}{M^6}\right].$$

Dividing the differentiated parts of the equation by the original parts, we obtain

$$\frac{-m_0^2(1 - e^{-a/M^2}) - ae^{-a/M^2}}{1 - e^{-a/M^2}} = -m_0^2 - \frac{ae^{-a/M^2}}{1 - e^{-a/M^2}}$$

for the left-hand side and

$$-\frac{M^2[1 - h_2/M^4 - 2h_3/M^6]}{1 + h_1/M^2 + h_2/M^4 + h_3/M^6}$$

for the right-hand side. The last two expressions must equal each other, which allows writing the expression for the squared ground-state mass m_0^2:

$$m_0^2(M^2;a) = \frac{M^2[1 - h_2/M^4 - 2h_3/M^6]}{1 + h_1/M^2 + h_2/M^4 + h_3/M^6} - \frac{a}{e^{a/M^2} - 1}. \tag{36}$$

The obtained formula can also be applied in other cases; it is only necessary to change the coefficients h_i (the coefficients for different cases are given in Table 3). For example, if we consider ϕ-mesons consisting of strange quarks, then an additional correction of the order of $1/Q^2$ arises in the operator expansion [5], which leads to the nonzero coefficient h_1. A similar situation arises if the axial vector correlator for the divergence current is considered (see Sec, 2.3): a term related to the pole of π-meson and proportional to $1/M^2$ appears.

Table 3

Case	h_1	h_2	h_3
ρ	0	$8\pi^2\langle m\bar{q}q\rangle + \frac{\pi^2}{3}\frac{\alpha_s}{\pi}\langle(G^a_{\mu\nu})^2\rangle$	$-\frac{56}{9}\pi^3\alpha_s\langle\bar{q}q\rangle^2$
ϕ	$-6m_s^2$	$8\pi^2\langle m_s\bar{s}s\rangle + \frac{\pi^2}{3}\frac{\alpha_s}{\pi}\langle(G^a_{\mu\nu})^2\rangle$	$-\frac{56}{9}\pi^3\alpha_s\langle\bar{s}s\rangle^2$
a_1 (for $\partial_\mu j^\mu_{a_1}$)	$-8\pi^2 f_\pi^2$	$-8\pi^2\langle m\bar{q}q\rangle + \frac{\pi^2}{3}\frac{\alpha_s}{\pi}\langle(G^a_{\mu\nu})^2\rangle$	$\frac{88}{9}\pi^3\alpha_s\langle\bar{q}q\rangle^2$
a_1 (for $j^\mu_{a_1}$)	0	$8\pi^2\langle m\bar{q}q\rangle - \frac{\pi^2}{3}\frac{\alpha_s}{\pi}\langle(G^a_{\mu\nu})^2\rangle$	$-\frac{176}{9}\pi^3\alpha_s\langle\bar{q}q\rangle^2$
a_1 (with s-quark)	$-6m_s^2$	$8\pi^2\langle m_s\bar{s}s\rangle - \frac{\pi^2}{3}\frac{\alpha_s}{\pi}\langle(G^a_{\mu\nu})^2\rangle$	$-\frac{176}{9}\pi^3\alpha_s\langle\bar{s}s\rangle^2$
f_0	0	$8\pi^2\langle m\bar{q}q\rangle + \frac{\pi^2}{3}\frac{\alpha_s}{\pi}\langle(G^a_{\mu\nu})^2\rangle$	$-\frac{176}{9}\pi^3\alpha_s\langle\bar{q}q\rangle^2$
f_0 (with s-quark)	$-6m_s^2$	$8\pi^2\langle m_s\bar{s}s\rangle + \frac{\pi^2}{3}\frac{\alpha_s}{\pi}\langle(G^a_{\mu\nu})^2\rangle$	$-\frac{176}{9}\pi^3\alpha_s\langle\bar{s}s\rangle^2$

Expressions for the coefficients h_i.

3.2. Scalar case. We must consider the scalar mesons separately because the calculations and the results differ noticeably from the vector analogue; the difference is related to the greater dimension of the correlator. Moreover, there are differences in the coefficients of the operator expansion:

$$\Pi_S(Q^2) = \frac{3}{8\pi^2}Q^2\log\frac{Q^2}{\mu^2} + \frac{3}{Q^2}\langle m\bar{q}q\rangle + \frac{1}{8Q^2}\frac{\alpha_s}{\pi}\langle(G^a_{\mu\nu})^2\rangle - \frac{22}{3}\frac{\pi\alpha_s}{Q^4}\langle\bar{q}q\rangle^2.$$

The expansion in resonances has the form

$$\Pi_S(p^2) = \sum_n \frac{F^2 m_n^2}{p^2 - m_n^2 - i\varepsilon}.$$

Applying the Borel transform to the dispersion relation for this correlator, we obtain an expression that can be summed as

$$\sum_n \pi F^2 m_n^2 e^{-m_n^2/M^2} = \pi F^2 e^{-m_0^2/M^2}\sum_n\left[m_0^2 e^{-a\cdot n/M^2} + ane^{-a\cdot n/M^2}\right] =$$
$$= \pi F^2 e^{-m_0^2/M^2}\left[\frac{m_0^2}{1 - e^{-a/M^2}} + \frac{ae^{a/M^2}}{(e^{a/M^2} - 1)^2}\right].$$

As a result, we have the equation

$$F^2 e^{-m_0^2/M^2}\left[\frac{m_0^2}{1-e^{-a/M^2}}+\frac{ae^{a/M^2}}{(e^{a/M^2}-1)^2}\right]=\frac{3M^4}{8\pi^2}\left[1+\frac{h_2}{M^4}+\frac{h_3}{M^6}\right].$$

where we use a notation h_i similar to that introduced in Sec. 3.1. We differentiate this equation with respect to $1/M^2$ and divide it by itself. For the right-hand side (denoted by $-\mathbf{L}$), we obtain

$$-\frac{M^2[2-h_3/M^6]}{1+h_2/M^4+h_3/M^6}\equiv-\mathbf{L}.$$

After several transformations, the same operations with the left-hand side lead to the expression

$$-\frac{a^2(e^{a/M^2}+1)+2am_0^2(e^{a/M^2}-1)+m_0^4(e^{a/M^2}-1)^2}{(e^{a/M^2}-1)(a-m_0^2+m_0^2e^{a/M^2})}.$$

As a result, we have the equation for m_0^2:

$$a^2(e^{a/M^2}+1)+2am_0^2(e^{a/M^2}-1)+m_0^4(e^{a/M^2}-1)^2=\mathbf{L}(e^{a/M^2}-1)(a-m_0^2+m_0^2e^{a/M^2}).$$

Collecting similar terms, we finally obtain the squared equation:

$$m_0^4(e^{a/M^2}-1)^2+m_0^2(e^{a/M^2}-1)[2a-\mathbf{L}(e^{a/M^2}-1)]+a^2(e^{a/M^2}+1)-\mathbf{L}a(e^{a/M^2}-1)=0. \tag{37}$$

It has two solutions:

$$\begin{aligned} m_0^2(a;M^2)&=\frac{1}{2}\left[\frac{2a}{1-e^{a/M^2}}+\mathbf{L}-\frac{\sqrt{-4a^2e^{a/M^2}+\mathbf{L}^2(e^{a/M^2}-1)^2}}{e^{a/M^2}-1}\right],\\ m_0^2(a;M^2)&=\frac{1}{2}\left[\frac{2a}{1-e^{a/M^2}}+\mathbf{L}+\frac{\sqrt{-4a^2e^{a/M^2}+\mathbf{L}^2(e^{a/M^2}-1)^2}}{e^{a/M^2}-1}\right]. \end{aligned} \tag{38}$$

We discuss an interpretation of these two solutions below.

We note that this method can also be applied in several other cases: for scalar mesons with strange quarks and also for axial vector mesons if the current itself is considered rather than the current divergence (see Sec. 3.1). This allows considering axial mesons containing the s-quark.

3.3. Mass spectra. The problem of calculating the mass spectrum reduces to the problem of determining the linear trajectory inclination and the ground state mass. We assume that the ground state masses are the initial data and prescribe them using the values obtained by the "classical" sum rules in [5].

After the ground state masses are prescribed, we must correctly choose the value of M^2 to calculate the inclination of trajectories by formula (36). In some cases, there is the so-called Borel window, i.e., the domain of values of the parameter M where the ground state mass is independent of this parameter. According to [3], [29], this domain is determined by an extremum of the function $m_0^2(M^2)$, and the position of this extremum determines the value of M^2. In the situation where the Borel window is absent or is weakly expressed, we must use the limit as $M^2\to\infty$.

Table 5 presents the calculated mass spectra for all studied cases. For comparison, we also present experimental data taken from [1]. All masses are given in MeV. The error arising in determining m_0 leads to errors in calculating the inclination and hence the mass spectrum. Taking these errors into account, we still do not need to consider the errors related to other external parameters (e.g., the condensate), because they have already been taken into account in calculating the classical sum rules.

Table 4

Case	h_1	h_2	h_3
ρ	0	0.032	−0.030
ϕ	−0.101	−0.089	−0.019
a_1 (for $\partial_\mu j^\mu_{a_1}$)	−0.674	0.046	0.048
a_1 (for $j^\mu_{a_1}$)	0	−0.046	−0.095
a_1 (with s-quark)	−0.101	−0.167	−0.061
f_0	0	0.032	−0.095
f_0 (with s-quark)	−0.101	−0.089	−0.061

Values of the coefficients h_i.

We take the values $(\alpha_s/\pi)\langle(G^a_{\mu\nu})^2\rangle = (330\,\text{MeV})^4$ and $\langle\bar{q}q\rangle = -(250\,\text{MeV})^3$ for the external parameters, the quark and gluon condensates. We note that these values differ from the condensate values in Sec. 2. This can be explained because we used condensate values that were used in calculations in the classical sum rules for self-consistency. Using the value $m_u + m_d \approx 10.7\,\text{MeV}$ for $\mu = 1\,\text{GeV}$ for the masses of light quarks, we can obtain the condensate value $\langle m\bar{q}q\rangle = -(95.6\,\text{MeV})^4$.

In the case of the ϕ-meson, we must use somewhat different condensate values because the strange quark is involved: $\langle\bar{s}s\rangle = 0.8\langle\bar{q}q\rangle = -(232\,\text{MeV})^3$ and $\langle m_s\bar{s}s\rangle = -(201\,\text{MeV})^4$. Moreover, we need the constants $f_\pi = 93\,\text{MeV}$, $m_s = 130\,\text{MeV}$, and $\alpha_s = 0.7$ (for $\mu = 1\,\text{GeV}$). These constants allow determining the coefficients h_1 and h_2. But for self-consistency, it seems reasonable to use the value of the coefficient h_3 given in [4] for the vector case and to multiply this value by an appropriate factor in the other cases. As a result, we obtain the values of the coefficients h_i given in Table 4.

Figure 3 shows the plots of Borel windows in different cases. As already mentioned, a Borel window is not always present. For example, in the case of sum rules for the current of the a_1-meson, there are two solutions, and both of them have no Borel window. In the sum rules for the divergence, the situation is somewhat different. In the "classical" sum rules, there is no extremum in M^2, and we must therefore take the limit values as $M^2 \to \infty$ in the calculations. Nevertheless, the plot shows that there is still a Borel window but related to the maximum instead of the minimum. The value at the maximum is $\approx 1250\,\text{MeV}$, which is near the experimental value. As a result, we can calculate two versions of the spectrum: one for the maximum, and the other of the limit value as $M^2 \to \infty$. But in the scalar case, there are two solutions and hence two trajectories in the plot. It can be seen that the second solution is below the first and has the value $\approx 0.62\,\text{GeV}$. This lower solution can be interpreted as the second trajectory of scalar mesons with $f_0(500)$ as the ground state.

Table 5 presents the mass spectra calculated using the trajectory inclinations obtained from Borel windows and with the experimental data for comparison. As experimental data for the ground state mass in the case of the a_1-meson with a strange quark, we use the value of the mass of the axial resonance $f_1(1510)$. For the f_0-meson with a strange quark, it is rather difficult to compare the obtained values with experimental data because it is difficult to determine which of the observed scalar mesons contain strange quarks and which do not. We note that the data for the resonance $f_0(1200-1600)$ [1] are close to the used value of the ground state mass.

4. Conclusion

In Sec. 2, we considered the spectral QCD sum rules in the limit of a large number N_c of colors under the assumption that the Regge spectrum is linear with a universal inclination for radial excitations of

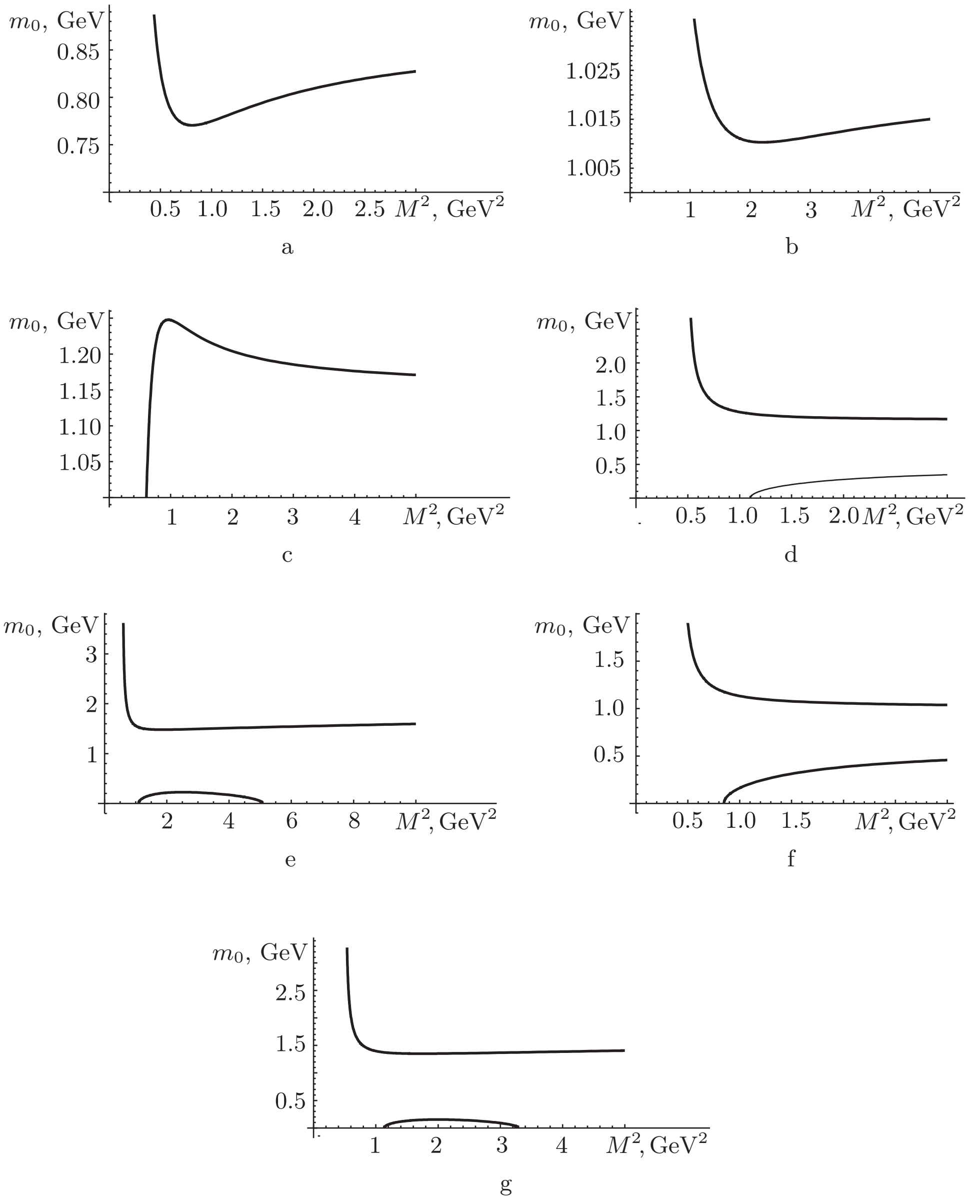

Fig. 3. Borel windows for different meson channels: (a) ρ-meson, (b) ϕ-meson, (c) a_1-meson (for the divergence), (d) a_1-meson (for the current[2]), (e) a_1-meson with s-quark,[3] (f) f_0-meson (two solutions), and (g) f_0-meson with s-quark (two solutions).

isosinglet vector, axial, and scalar mesons. The choice of the spectrum is motivated by models of hadron strings and related approaches and also by meson spectroscopy. The considered ansatz allows solving the arising sum rules with a minimum number of input parameters. Because it is impossible to use the QCD sum rules to obtain either the scale of QCD masses or the scale of spontaneous chiral symmetry breaking, the minimum number of free parameters is two. In our method, these parameters are the gluon condensate and the weak decay constant of the π-meson.

[2]The lower solution attains the value ≈ 0.487 as $M^2 \to \infty$.

[3]The lower solution takes complex values beyond the domain shown on the graph. The main solution has a barely distinguishable Borel window and diverges as $M^2 \to \infty$.

Table 5

Case	a, GeV^2	n	0	1	2	3
ρ	1.5195 ± 0.0695	Comput.	70 ± 10	1450 ± 20	1910 ± 40	2230 ± 50
		Exper.	770 ± 10	1465 ± 25	$1909 \pm 17 \pm 25$	2265 ± 40
ϕ	1.8955 ± 0.0565	Comput.	1010 ± 10	1710 ± 20	2190 ± 30	2590 ± 30
		Exper.	1010 ± 10	1680 ± 20	2188 ± 10	—
a_1 (for $\partial_\mu j^\mu_{a_1}$)	1.3 ± 0.184	Comput.	1150 ± 40	1620 ± 60	1980 ± 90	2280 ± 120
			1250	1690 ± 50	2040 ± 90	2330 ± 120
		Exper.	1010 ± 10	1680 ± 20	2188 ± 10	—
a_1 (for $j^\mu_{a_1}$)	1.561 ± 0.124	Comput.	1150 ± 40	1700 ± 40	2110 ± 60	2450 ± 80
		Exper.	1230 ± 40	1647 ± 22	1930^{+30}_{-70}	2270^{+55}_{-40}
a_1 (with s-quark)	2.3365 ± 0.1305	Comput.	1470^{+30}_{-10}	2120 ± 30	2610 ± 50	—
		Exper.	1518 ± 5	$2096 \pm 17 \pm 121$	—	—
f_0	1.384 ± 0.069	Comput.	1000 ± 30	1540 ± 20	1940 ± 40	2270 ± 50
		Exper.	990 ± 20	1504 ± 6	1992 ± 16	2189 ± 13
		Comput.	620	1330 ± 30	1780 ± 40	2130 ± 50
		Exper.	400–550	1200–1500	1723^{+6}_{-5}	2101 ± 7
f_0 (with s-quark)	1.886	Comput.	1350	1930	2370	2740

Calculated mass spectra of radial excitations of mesons for different meson channels and their comparison with experimental data: for the f_0-meson, the mass spectra are given for both trajectories.

Solving the arising equations numerically, we reconstruct the physical mass of the $\omega(782)$-meson and the value of the quark condensate that agrees well with the results of lattice computations. The radial spectrum of vector and axial states is also reasonable.

We then used the obtained values of radial trajectory inclinations and the quark condensate to analyze the scalar case. It turned out that by interpolating the scalar states by the simplest bilinear quark current, we can naturally obtain a light scalar resonance in the range $500 \pm 100\,\text{MeV}$. This could not be done previously in the framework of QCD sum rules (except where the isoscalar scalar was still interpolated by the pure gluon operator [20]). The obtained scalar state can be identified as the $f_0(500)$-meson, also called a σ-meson, which is traditionally regarded as an unusual particle [2]. Our analysis shows that at least the value for the mass of the $f_0(500)$-meson is not unusual. We demonstrated that although the mass of the lightest scalar meson is not on the scalar radial Regge trajectory, it still strongly correlates with the mass parameters of this trajectory.

In Sec. 3, we considered the Borelized sum rules, which have also been widely studied in the literature. But instead of studying finitely many esonances and the continuous spectrum after them, we used the infinite discrete spectrum. This allowed obtaining some new results. We proposed a method for calculating the inclination of linear trajectories and a related new scheme for calculating the masses of radial excitations. A very unexpected result is that in the framework of the approach developed above, we also obtained a solution predicting a light scalar meson that can be compared to the $f_0(500)$-meson.

If our conclusions are valid, then this means that the quark–antiquark component in the structure of the σ-meson strongly dominates even if it is a tetraquark state.

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