

Optimality and existence of the solution of stochastic multi-objective optimization problem

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Abstract. This article focuses on the issue of existence of the solution and its optimality for stochastic multiobjective optimization problem. The presented results make possible to assert the existence and optimality of solutions of stochastic multiobjective optimization problem with expected value from convolution function.

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1. Introduction

The problem of optimality and existence of the solution of stochastic multi-objective optimization problem is quite important and has been studied by many authors. The initial articles on this issue were published by Hanson [11] and Stoddart [21]. Further, their work was continued by the authors of the works [13, 23]. The authors of the articles [2, 19, 20, 16] studied particular cases.

However, the problem of the existence of solutions and their optimality of the stochastic multi-objective optimization problem (SMOOP) [10] has not yet been fully studied, while the use of SMOOP at application fields such as finance [1, 17], manufacturing, production planning [6, 22], supply chain management [18, 15], energy [5], transportation logistics [12], facility

location [8], humanitarian logistics [7], telecommunication and health care management [3, 4] or project management [9] is widespread.

At the present time, there is no single approach to the problem of studying the conditions of existence and optimality of SMOOP solutions. Thus, based on the results given in the works [11] that study the problem of scalar stochastic optimization, the authors generalize the results to the multiobjective case, using the approach presented in [14].

2. Stochastic multi-objective optimization problem

A stochastic multi-objective optimization problem (SMOOP) is a combination of a stochastic optimization problem (SOP) and a multi-objective optimization problem (MOOP). Consider each of this problem separately.

In the simple case, the stochastic optimization problems (SOP) contain the random parameter in the objective function. The mathematical formalization of this problem can be represented as follows:

$$\min_x f(x, \omega) \quad (1)$$

subject to

$$x \in X, \omega \in \Omega \quad (2)$$

where $f(x, \omega)$ is the objective function, X is the set of feasible solutions and $\omega \in \Omega$ is the random parameter in the probability space (Ω, A, Ψ) .

Note that in stochastic optimization, the probability model given by (Ω, A, Ψ) is always assumed as known, therefore, to simplify further, we will simply indicate that $\omega \in \Omega$.

To solve this problem, it is necessary to determine how to overcome uncertainty. Most often, to overcome an undefined service, the expectation

of a random variable is used:

$$\min_x \mathbf{E}[f(x, \omega)] \quad (3)$$

subject to

$$x \in X, \omega \in \Omega \quad (4)$$

where $f(x, \omega)$ is the objective function, $\mathbf{E}(\cdot)$ is the expected value of a random variable, X is the set of feasible solutions.

Note that there are more complex cases, including uncertainty also in the solution space:

$$\min_x \mathbf{E}[f(x, \omega)] \quad (5)$$

subject to

$$x \in X(\omega), \omega \in \Omega \quad (6)$$

where $f(x, \omega)$ is the objective function, $X(\omega)$ is the set of feasible solutions, and $\omega \in \Omega$ is the random parameter.

The general form of a multiobjective optimization problem (MOOP) is usually represented as:

$$\min_x (f(x)) \quad (7)$$

subject to

$$x \in X \quad (8)$$

where X is again a solution space, $f(x) = (f_1(x), \dots, f_m(x))$ is the vector function, $f_j : X \rightarrow R (j = 1, \dots, m)$ are objective functions ($m \geq 2$). From (7-8), it is not yet clear what is meant by minimization vector function $f = (f_1, \dots, f_m)$.

There are several approaches to determine what exactly is meant by the minimum vector of a vector function, we list some of them. These are scalarizing methods, lexicographical order, Pareto methods, etc.

Scalarization is the most common approach. This approach uses the scalarization or convolution function, which compares the values of multi-objective indicators to the scalar one, for example, by a weighted average or weighted Chebyshev distance. We will write such functions in the form $F : R^m \rightarrow R$. After scalarization, the multi-objective optimization problem is reduced to a scalar optimization problem.

We now turn to the stochastic multi-objective optimization problem (SMOOP), i.e. an optimization problem that is both a SOP and a MOOP. In particular, consider the tasks of the form:

$$\min_x f(x, \omega), \quad (9)$$

subject to

$$x \subset X(\omega) \quad \forall \omega \in \Omega. \quad (10)$$

where $X(\omega)$ is the set of feasible solutions, $f(x, \omega) = (f_1(x, \omega), \dots, f_m(x, \omega))$ is vector function of objective functions, $\omega \in \Omega$ is random parameter.

It is clear that the problem (9)-(10) is not completely defined. It is necessary to indicate in what sense the vector minimization should be interpreted and what is being done to eliminate the ambiguity of target values caused by the accidental impact of ω .

One of the methods that answers these two questions at once is the method known in the literature as *stochastic method*. The stochastic method starts with the reduction of SMOOP to SOP, to which the methods of stochastic optimization are applied.

Thus, applying first the convolution of the criteria, and then minimizing the mathematical expectation of it, we can rewrite the problem in the following form:

$$\min_x \mathbf{E}[F(x, \omega)], \quad (11)$$

subject to

$$x \subset X(\omega) \forall \omega \in \Omega. \quad (12)$$

Note. Since $F(f(x, \omega))$ is completely determined by the x and ω , we simply write $F(x, \omega)$ instead of $F(f(x, \omega))$.

In the next section, we present a result that determines the existence and optimality of the solution for such type of SMOOP methods.

3. Optimality and existence of the solution of Stochastic multi-objective optimization

In this section, we consider the SMOOP problem as follows:

$$\min_x \mathbf{E}[F(x, \omega)], \quad (13)$$

subject to

$$\mathbf{E}[g(x, \omega)] \leq 0, \quad (14)$$

$$h(x, \omega) \leq 0 \forall \omega \in \Omega, \quad (15)$$

where $F(x, \omega)$ -convolution of objective functions stochastic multi-objective optimization problem, which is a scalar function of $x \in E^n$, $g(x, \omega)$ and $h(x, \omega)$ -function vectors, $F(x, \omega)$, $g(x, \omega)$, $h(x, \omega)$ are twice continuously differentiable on

$$S = \{x\} \times \{\omega\} \subset E^n \times \Omega$$

for all variables, and distribution density $\Psi(\omega)$ is twice continuously differentiable on $\omega \in \Omega$.

For this problem the following theorem is useful:

Theorem 1. x^* is a minimizing vector for problem (13)-(15) if the following conditions are satisfied:

1. $F(x, \omega)$ - is convex in x ,

2. $g(x, \omega)$ and $h(x, \omega)$ - concave on x ,
3. there exists $\lambda \leq 0$,
4. there exists $\mu(\omega) \leq 0$,
5. $F_x(x, \omega)\Psi(\omega) + \lambda g_x(x, \omega)\Psi(\omega) + \mu(\omega)h_x(x, \omega) = 0$,
6. $\lambda g(x, \omega) = 0$,
7. $\mu(\omega)h(x, \omega) = 0$,

where $F_x = (\frac{\delta F(x_1)}{\delta x_1}, \dots, \frac{\delta F(x_n)}{\delta x_n})$, g_X , h_X -matrices of corresponding dimensions from partial derivatives, λ , $\mu(\omega)$ -r dimensional functions.

Proof. We write the Problem (13)-(15) in an equivalent form:

$$\min_x \int_{\Omega} F(x, \omega)\Psi(\omega)d\omega, \quad (16)$$

subject to

$$\int_{\Omega} g(x, \omega)\Psi(\omega)d\omega \leq 0, \quad (17)$$

$$h(x, \omega) \leq 0 \quad \forall \omega \in \Omega. \quad (18)$$

Let x^* be some feasible solution. As $F(x, \omega)$ is convex in x and is therefore continuous and differentiable and thus from the conditions of the theorem we have:

$$\begin{aligned} & \int_{\Omega} F(x^*, \omega)\Psi(\omega)d\omega \leq \int_{\Omega} \left((F(x^*, \omega) + (x^* - x)F_x(x, \omega)) \right) \Psi(\omega)d\omega \\ &= \int_{\Omega} \left(F(x, \omega)\Psi(\omega) - (x^* - x)[\lambda g_x(x, \omega)\Psi(\omega) + \mu(\omega)h_x(x, \omega)] \right) \end{aligned}$$

that, by virtue of the convexity of λ_g and μ_h , we have

$$\begin{aligned} & \leq \int_{\Omega} F(x, \omega)\Psi(\omega)d\omega \\ & - \int_{\Omega} \lambda(g(x^*, \omega) - g(x, \omega))\Psi(\omega)d\omega - \end{aligned}$$

$$\begin{aligned}
& - \int_{\Omega} \mu(\omega)(h(x^*, \omega) - h(x, \omega))d\omega \\
= & \int_{\Omega} F(x, \omega)\Psi(\omega)d\omega - \\
& - \int_{\Omega} (x^*, \omega)\Psi(\omega)d\omega - \int_{\Omega} \mu(\omega)(h(x^*, \omega)d\omega \\
\leq & \int_{\Omega} (F(x, \omega)\Psi(\omega)d\omega
\end{aligned}$$

Thus x^* is the optimal solution. \square

Hence the proved theorem gives the conditions that allow to state the existence and optimality of the solution for the problem (13)-(15).

4. Conclusion

In the present article, we have considered problems of optimality and existence of the solution of stochastic multi-optimization with normalizations introduced for them and their convolutions.

The main results of this article are presented in Section 3 that demonstrates the theorem of the existence and optimality of solutions of stochastic multiobjective optimization problem with expected value from convolution function.

The main theorem proven in this paper allows us to state optimality and existence of the solution of SMOOP depending on the fulfillment of the conditions imposed on the kind of convolution and the set of feasible solutions.

For subsequent studies, it is of interest to study them with no convolution in the case when the optimality principle is introduced on the set of the objective function.

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