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MATHEMATICS

Two-Dimensional Homogeneous Cubic Systems: Classification and Normal Forms—III

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Abstract—This article is the third in a series of works devoted to two-dimensional homogeneous cubic systems. It considers the case where the homogeneous polynomial vector on the right-hand side of the system has a quadratic common factor with real zeros. The set of such systems is divided into classes of linear equivalence, in each of which a simplest system being a third-order normal form is distinguished on the basis of appropriately introduced structural and normalization principles. In fact, this normal form is determined by the coefficient matrix of the right-hand side, which is called a canonical form (CF). Each CF is characterized by an arrangement of nonzero elements, their specific normalization, and a canonical set of admissible values of the unnormalized elements, which ensures that the given CF belongs to a certain equivalence class. In addition, for each CF, (a) conditions on the coefficients of the initial system are obtained, (b) nonsingular linear substitutions reducing the right-hand side of a system satisfying these conditions to a given CF are specified, and (c) the values of the unnormalized elements of the CF thus obtained are given.

Keywords: homogeneous cubic system, normal form, canonical form.

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INTRODUCTION

This paper is devoted to finding canonical forms of real homogeneous cubic systems having a second-degree common factor with real zeros; it consists of three sections.

In the first section, the right-hand side of the initial system, which is determined by eight coefficients, is uniquely decomposed into the product of a common factor $P_0^2(x)$ and a vector Hx, where H is a nonsingular matrix. As shown in [1], the sign of the discriminant D of the characteristic polynomial of this matrix is invariant. In each of the cases $D > 0$, $D = 0$, and $D < 0$, the system is primarily simplified by reducing the matrix H to a Jordan normal form and determining a new common factor. It is for the comparatively simple systems thus obtained that we determine conditions under which they can be reduced to various CFs by appropriate linear changes.

In the second and the third section, we consider the cases $D_0 = 0$ and $D_0 > 0$, taking into account the invariance of the sign of the discriminant D_0 of the common factor P_0^2. For each of these cases, we give lists of canonical forms together with their canonical and minimal canonical sets of admissible values of parameters introduced in [2]. We prove theorems which confirm the linear nonequivalence of the introduced CFs and give, for each CF, (a) all systems in the linear equivalence class of the given CF, (b) a linear change reducing any such system to the given CF, and (c) the values of the CF parameters in the corresponding canonical set resulting from the change.

This paper is a direct continuation of [1, 2], and we use the notation introduced in the previous papers. We often refer to formulas obtained in [1]; for brevity, we put the superscript “1” on their numbers. For example, system (2.1) in [1] is referred to as $(2.1)^1$.

1. PRIMARY SIMPLIFICATION OF THE SYSTEM FOR $l = 2$

Consider system (2.1)[1], that is, $\dot{x} = Aq^{[3]}(x)$; at $l = 2$, it can uniquely be written in the form (2.14)[1]:

$$\dot{x} = P_0^2(x)Hx, \quad \begin{matrix} P_0^2 = \alpha x_1^2 + 2\beta x_1 x_2 + \gamma x_2^2, \\ D_0 = \beta^2 - \alpha\gamma, \end{matrix} \quad H = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}, \quad \delta_{pq} = \det H \neq 0, \tag{1.1}$$

where $\alpha = 1$ or $\alpha, \gamma = 0$ and $2\beta = 1$.

According to Theorem 2.2 of [1], any nonsingular linear change (change (2.2)[1])

$$\begin{cases} x_1 = r_1 y_1 + s_1 y_2, \\ x_2 = r_2 y_1 + s_2 y_2 \end{cases} \quad \text{or} \quad x = Ly, \quad L = \begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \end{pmatrix}, \quad \delta = \det L \neq 0 \tag{1.2}$$

transforms system (1.1) into a system of the form (2.17)[1], namely, $\dot{y} = (\tilde{\alpha}, 2\tilde{\beta}, \tilde{\gamma})q^{[2]}(y)\tilde{H}y$, whose row $(\tilde{\alpha}, 2\tilde{\beta}, \tilde{\gamma})$ and matrix $\tilde{H}$ with $\delta_{\tilde{p}\tilde{q}} \neq 0$ are described by (2.18)[1].

Using notation (2.3)[1], we can write system (2.17)[1] in the form

$$\tilde{A} = \begin{pmatrix} \tilde{a}_1 & \tilde{b}_1 & \tilde{c}_1 & \tilde{d}_1 \\ \tilde{a}_2 & \tilde{b}_2 & \tilde{c}_2 & \tilde{d}_2 \end{pmatrix} = \begin{pmatrix} \tilde{\alpha}\tilde{p}_1 & \tilde{\alpha}\tilde{q}_1 + 2\tilde{\beta}\tilde{p}_1 & 2\tilde{\beta}\tilde{q}_1 + \tilde{\gamma}\tilde{p}_1 & \tilde{\gamma}q_1 \\ \tilde{\alpha}\tilde{p}_2 & \tilde{\alpha}\tilde{q}_2 + 2\tilde{\beta}\tilde{p}_2 & 2\tilde{\beta}\tilde{q}_2 + \tilde{\gamma}\tilde{p}_2 & \tilde{\gamma}q_2 \end{pmatrix}. \tag{1.3}$$

We reduce system (1.1) to various canonical forms in two steps.

At the first step, we choose a change of the form (1.2) reducing the matrix H of system (1.1) to a Jordan normal form $\tilde{H}$ in the obtained system (2.17)[1]. Of course, the form of the change depends on the sign of the discriminant $D = (p_1 + q_2)^2 - 4\delta_{pq}$ in (2.16)[1] (see [3, Appendix 3.3, p. 112]).

Here and in what follows, when referring to the appendix of [3], we mean that the program package in this appendix contains a program confirming results presented below by symbolic computations in *Maple*.

(1) Suppose that $D > 0$; then, according to (2.16)[1], the matrix H has different real eigenvalues $\lambda_1, \lambda_2 \neq 0$. To be more specific, we assume that

$$\lambda_1 = \frac{p_1 + q_2 + \sigma_0\sqrt{D}}{2}, \quad \lambda_2 = \frac{p_1 + q_2 - \sigma_0\sqrt{D}}{2}, \quad \lambda_* = p_1 - q_2 + \sigma_0\sqrt{D} \neq 0, \tag{1.4}$$

where $\sigma_0 = \{1 \text{ if } p_1 \geq q_2, -1 \text{ if } p_1 < q_2\}$; then $\sigma_0 = \operatorname{sign}\lambda_*$ and $\sigma_0\sqrt{D} = \lambda_1 - \lambda_2$.

The change $J_1^2 = \begin{pmatrix} \lambda_* & -2q_1 \\ 2p_2 & \lambda_* \end{pmatrix}$ $(\delta = 2\sigma_0\sqrt{D}\lambda_*)$, together with the expressions for $\tilde{P}_0$ in (2.18)[1], reduces system (1.1) to a system of the form (1.3) or (2.17)[1]. We have

$$\tilde{A} = \begin{pmatrix} \tilde{\alpha}\lambda_1 & 2\tilde{\beta}\lambda_1 & \tilde{\gamma}\lambda_1 & 0 \\ 0 & \tilde{\alpha}\lambda_2 & 2\tilde{\beta}\lambda_2 & \tilde{\gamma}\lambda_2 \end{pmatrix}; \quad \tilde{H} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \tilde{\alpha} = \alpha\lambda_*^2 + 4\beta p_2\lambda_* + 4\gamma p_2^2,$$
$$\tilde{\beta} = \beta\lambda_*^2 - 2(\alpha q_1 - \gamma p_2)\lambda_* - 4\beta p_2 q_1, \quad \tilde{\gamma} = \gamma\lambda_*^2 - 4\beta q_1\lambda_* + 4\alpha q_1^2. \tag{1.5}$$

(2) Suppose that $D = 0$. Then $\lambda_1, \lambda_2 = \nu = (p_1 + q_2)/2 \neq 0$ in (2.16)[1]; otherwise, $\det H = 0$.

(2_1) The change $J_{2a}^2 = \begin{pmatrix} 0 & 2q_1 \\ 2 & q_2 - p_1 \end{pmatrix}$ for $q_1 \neq 0$ (Case *a*) and the normalization $J_{2b}^2 = \begin{pmatrix} 1 & 0 \\ 0 & p_2 \end{pmatrix}$ for $q_1 = 0$ and $p_2 \neq 0$ $(p_1, q_2 = \nu)$ (Case *b*) reduce system (1.1) to (1.3) or (2.17)[1]. We have

$$\tilde{A} = \begin{pmatrix} \tilde{\alpha}\nu & 2\tilde{\beta}\nu & \tilde{\gamma}\nu & 0 \\ \tilde{\alpha} & \tilde{\alpha}\nu + 2\tilde{\beta} & 2\tilde{\beta}\nu + \tilde{\gamma} & \tilde{\gamma}\nu \end{pmatrix}; \quad \tilde{H} = \begin{pmatrix} \nu & 0 \\ 1 & \nu \end{pmatrix},$$
$$\text{a:} \quad \tilde{\alpha} = 4\gamma, \quad \tilde{\beta} = 4\beta q_1 - 2\gamma(p_1 - q_2), \quad \tilde{\gamma} = 4\alpha q_1^2 - 4\beta q_1(p_1 - q_2) + \gamma(p_1 - q_2)^2, \tag{1.6}$$
$$\text{b:} \quad \tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta p_2, \quad \tilde{\gamma} = \gamma p_2^2 \quad (\nu = p_1).$$

(2_2) If $q_1, p_2 = 0$, then the matrix H in system (1.1) is diagonal and $p_1, q_2 = \nu \neq 0$.

(3) Suppose that $D < 0$ ($\delta_{pq} > 0, p_2q_1 < 0$); then the numbers λ_1 and λ_2 in $(2.16)^1$ are complex conjugate.

The change $J_3^2 = \begin{pmatrix} \sqrt{-D} & p_1 - q_2 \\ 0 & 2p_2 \end{pmatrix}$ $(\delta = 2p_2\sqrt{-D})$, together with the expressions for $\tilde{P}_0$ in $(2.18)^1$, reduces (1.1) to a system of the form (1.3) or $(2.17)^1$. We have

$$\tilde{A} = \begin{pmatrix} \tilde{\alpha}\nu & 2\tilde{\beta}\nu - \tilde{\alpha}\mu & \tilde{\gamma}\nu - 2\tilde{\beta}\mu & -\tilde{\gamma}\mu \\ \tilde{\alpha}\mu & \tilde{\alpha}\nu + 2\tilde{\beta}\mu & 2\tilde{\beta}\nu + \tilde{\gamma}\mu & \tilde{\gamma}\nu \end{pmatrix}; \quad \tilde{H} = \begin{pmatrix} \nu & -\mu \\ \mu & \nu \end{pmatrix}, \quad \tilde{\alpha} = -\alpha D,$$
$$\tilde{\beta} = \sqrt{-D}(\alpha(p_1 - q_2) + 2\beta p_2), \quad \tilde{\gamma} = \alpha(p_1 - q_2)^2 + 4\beta p_2(p_1 - q_2) + 4\gamma p_2^2, \tag{1.7}$$

where $\nu = (p_1 + q_2)/2 (= \operatorname{Re}\lambda_{1,2})$, $\mu = \sqrt{-D}/2$ $(= |\operatorname{Im}\lambda_{1,2}|) > 0$, and $\nu^2 + \mu^2 = \delta_{pq}$.

At the second step, we make an arbitrary change of the form (1.2) in the linearly nonequivalent systems (1.5)–(1.7), which reduces each of these systems to a system of the form $(2.17)^1$; in the notation of all components of the resulting system, we replace the symbol ~ by ⌣.

As a result, taking into account $(2.18)^1$, we obtain the following system by analogy with (1.3):

$$\breve{A} = \begin{pmatrix} \breve{a}_1 & \breve{b}_1 & \breve{c}_1 & \breve{d}_1 \\ \breve{a}_2 & \breve{b}_2 & \breve{c}_2 & \breve{d}_2 \end{pmatrix} = \begin{pmatrix} \breve{\alpha}\breve{p}_1 & \breve{\alpha}\breve{q}_1 + 2\breve{\beta}\breve{p}_1 & 2\breve{\beta}\breve{q}_1 + \breve{\gamma}\breve{p}_1 & \breve{\gamma}\breve{q}_1 \\ \breve{\alpha}\breve{p}_2 & \breve{\alpha}\breve{q}_2 + 2\breve{\beta}\breve{p}_2 & 2\breve{\beta}\breve{q}_2 + \breve{\gamma}\breve{p}_2 & \breve{\gamma}\breve{q}_2 \end{pmatrix}, \tag{1.8}$$

where $\breve{\alpha} = \tilde{\alpha}r_1^2 + 2\tilde{\beta}r_1r_2 + \tilde{\gamma}r_2^2$, $\breve{\beta} = (\tilde{\alpha}s_1 + \tilde{\beta}s_2)r_1 + (\tilde{\beta}s_1 + \tilde{\gamma}s_2)r_2$, $\breve{\gamma} = \tilde{\alpha}s_1^2 + 2\tilde{\beta}s_1s_2 + \tilde{\gamma}s_2^2$, and $\breve{H} = \begin{pmatrix} \breve{p}_1 & \breve{q}_1 \\ \breve{p}_2 & \breve{q}_2 \end{pmatrix} = \delta^{-1}\begin{pmatrix} r_1\delta_{ps} + r_2\delta_{qs} & s_1\delta_{ps} + s_2\delta_{qs} \\ -r_1\delta_{pr} - r_2\delta_{qr} & -s_1\delta_{pr} - s_2\delta_{qr} \end{pmatrix}$ $(\det\breve{H} = \delta_{\breve{p}\breve{q}} = \delta_{\tilde{p}\tilde{q}} = \delta_{pq})$.

It remains to choose the coefficients of change (1.2) so that system (1.8) be simplest according to structural and normalization principles [2, Sections 1.1 and 1.2].

We implement this plan separately for each of the three classes of systems into which (1.1) is divided according to the sign of the discriminant D_0 of the common factor P_0^2, which is invariant with respect to changes (1.2) by virtue of $(2.19)^1$.

Thus, in effect, we find canonical forms separately in each of the nine linearly nonequivalent classes distinguished by the signs of the discriminants D_0 and D of system (1.1) (see [2, Corollary 1.1]).

In this paper, we consider the comparatively simple cases $D_0 = 0$ and $D_0 > 0$.

Collection 1.1. We use the following constants and changes below:

$$D = (p_1 - q_2)^2 + 4p_2q_1, \quad \sigma_0 = \{1 \text{ if } p_1 \geq q_2, -1 \text{ if } p_1 < q_2\},$$
$$\lambda_{1,2} = (p_1 + q_2 \pm \sigma_0\sqrt{D})/2, \quad \lambda_* = p_1 - q_2 + \sigma_0\sqrt{D};$$
$$\nu = (p_1 + q_2)/2, \quad \mu = \sqrt{-D}/2; \quad \tilde{\tau} = (\tilde{\beta}^2 - \tilde{\alpha}\tilde{\gamma})^{1/2}, \quad \tilde{\phi} = (2\tilde{\tau})^{-1/2},$$
$$\sigma_\alpha = \operatorname{sign}\tilde{\alpha}, \quad \sigma_\beta = \{1 \text{ if } \tilde{\beta} \geq 0, -1 \text{ if } \tilde{\beta} < 0\}, \quad \sigma_\gamma = \operatorname{sign}\tilde{\gamma},$$
$$\tilde{\eta} = \tilde{\beta} + \sigma_\beta\tilde{\tau}; \quad \tilde{\xi} = (\tilde{\alpha}\tilde{\gamma} - \tilde{\beta}^2)^{1/2};$$
$$J_1^2 = \{r_1, s_2 = \lambda_*, s_1 = -2q_1, r_2 = 2p_2\}, \quad J_{2a}^2 = \{r_1 = 0, s_1 = 2q_1, s_2 = 2, r_2 = q_2 - p_1\},$$
$$J_{2b}^2 = \{r_1 = 1, s_1, r_2 = 0, s_2 = p_2\}, \quad J_3^2 = \{r_1 = \sqrt{-D}, s_1 = p_1 - q_2, r_2 = 0, s_2 = 2p_2\}.$$

2. CONSTRUCTION OF $CF^{m,2}$ FOR P_0^2 WITH ZERO DISCRIMINANT

System (1.1) for which $P_0^2(x)$ is a full square has the form

$$\dot{x} = P_0^2(x)Hx, \quad P_0^2 = (x_1 + \beta x_2)^2 \quad (\gamma = \beta^2 \Leftrightarrow D_0 = 0, \det H = \delta_{pq} \neq 0). \tag{1.1$^=$}$$

Let us select those structural forms, up to $\mathrm{SF}_{13}^{3,2}$, in List 1.1 of [2] which refer to the case $l = 2$, $D_0 = 0$ (see [2, Statement 1.2]); there are five such forms. We normalize them by change (2.6)[1] according to the introduced normalization principles, which yields $\mathrm{NSF}^{m,2,=}$ (see [2, Definition 1.6]).

Let us prove that the list given below contains all possible canonical forms of system ($1.1^=$), and the sets specified in this list are the canonical sets described in Definition 1.10 of [2].

List 2.1. The five forms $\mathrm{CF}_i^{m,2,=}$ and the corresponding sets $cs_i^{m,2,=}$ are as follows (the matrix H and the discriminant D from (2.16)[1] ($\sigma, \kappa = \pm 1$, $u \neq 0$, $(\alpha, 2\beta, \gamma) = (1, 0, 0)$), are also specified):

$$\mathrm{CF}_3^{2,2,=,\geq} = \sigma\begin{pmatrix} u & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \sigma\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \quad (u-1)^2, \quad \begin{matrix} cs_3^{2,2,=,>} = \{u \neq 1\}, \\ cs_3^{2,2,=,=} = \{u = 1\}; \end{matrix}$$

$$\mathrm{CF}_{7.\kappa}^{2,2,=,\gtrless} = \sigma\begin{pmatrix} 0 & \kappa & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma\begin{pmatrix} 0 & \kappa \\ 1 & 0 \end{pmatrix}, \quad 4\kappa, \quad \begin{matrix} cs_{7,\kappa}^{2,2,=,>} = \{\kappa = 1\}, \\ cs_{7,\kappa}^{2,2,=,<} = \{\kappa = -1\}; \end{matrix}$$

$$\mathrm{CF}_7^{3,2,=,\geq} = \sigma\begin{pmatrix} u & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \sigma\begin{pmatrix} u & 1 \\ 0 & 1 \end{pmatrix}, \quad (u-1)^2, \quad \begin{matrix} cs_7^{3,2,=,>} = \{u \neq \pm 1\}, \\ cs_7^{3,2,=,=} = \{u = 1\}; \end{matrix}$$

$$\mathrm{CF}_{12}^{3,2,=,<} = \sigma\begin{pmatrix} 1 & u & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma\begin{pmatrix} 1 & u \\ 1 & 0 \end{pmatrix}, \quad 4u+1, \quad cs_{12}^{3,2,=,<} = \{u < -1/4\};$$

$$\mathrm{CF}_{a,13}^{3,2,=,=} = \sigma\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & u & 0 & 0 \end{pmatrix}, \quad \sigma\begin{pmatrix} 1 & 0 \\ 1 & u \end{pmatrix}, \quad (u-1)^2, \quad cs_{13}^{3,2,=,=} = \{u = 1\}.$$

Statement 2.1. The only forms in List 2.1 with parameter values specified above which reduce to structural forms preceding them according to the structural principles are as follows:

$\mathrm{NSF}_7^{3,2,=,>}$ for $u = -1$ reduces to $\mathrm{SF}_7^{2,2}$ by change (1.2) with $r_1 = r_2$ and $s_1 = 0$;

$\mathrm{NSF}_{12}^{3,2,=,>}$ ($u > -1/4$) and $\mathrm{NSF}_{12}^{3,2,=,=}$ ($u = -1/4$) reduce to $\mathrm{SF}_7^{3,2}$ by change (1.2) with $s_1 = 0$ and $r_1 = (1+\sqrt{1+4u})r_2 / 2$;

$\mathrm{NSF}_{13}^{3,2,=,>}$ ($u \neq 1$) reduces to $\mathrm{SF}_3^{2,2}$ by change (1.2) with $r_2 = (1-u)r_1$ and $s_2 = 0$.

Collection 2.1. In the rest of Section 2, we use the following constants and changes:

$$\varpi_1 = 2p_2\beta + \lambda_*, \quad \varpi_2 = 2q_1 - \lambda_*\beta, \quad \varpi_3 = (p_1 - q_2)\beta - 2q_1,$$

$$\varpi_4 = p_2\beta^2 + 2p_1\beta - q_1, \quad \varpi_5 = p_2\beta^2 + (p_1 - q_2)\beta - q_1, \quad \varpi_6 = p_1 - q_2 + 2p_2\beta;$$

$$L_3^{2,2,=,>} = \left\{r_1 = \left[0 \vee \left|\varpi_1^2\lambda_2\right|^{-1/2}\right],\ s_1 = [1 \vee 0],\ r_2 = \left[\left|\varpi_2^2\lambda_1\right|^{-1/2} \vee 0\right],\ s_2 = [0 \vee 1]\right\};$$

$$L_{7,+1}^{2,2,=,>} = \left\{r_1,\ -s_1 = \varpi_1^{-1}\varpi_2 r_2,\ r_2, s_2 = \left|4\varpi_2^2\lambda_2\right|^{-1/2}\right\};$$

$$L_7^{3,2,=,>} = \left\{r_1 = 0,\ s_1 = -\varpi_1^{-2}\varpi_2 s_2,\ r_2 = \left|\varpi_2^2\lambda_1\right|^{-1/2},\ s_2 = \lambda_1(\lambda_1 - \lambda_2)^{-1} r_2\right\};$$

$$L_3^{2,2,=,=} = \left\{r_1 = |p_1|^{-1/2},\ s_1 = -\beta,\ r_2 = 0,\ s_2 = 1\right\};$$

$$L_7^{3,2,=,=} = \left\{r_1 = 0,\ s_1 = \nu r_2,\ r_2 = \left[\left|\varpi_3^2\nu\right|^{-1/2} \vee \left|(\beta p_2)^2\nu\right|^{-1/2}\right],\ s_2 = \left[2\beta\varpi_3^{-1}\nu r_2 \vee -(\beta p_2)^{-1} r_2\right]\right\};$$

$$L_{13}^{3,2,=,=} = \left\{r_1, s_2 = 0,\ s_1 = \left[\left|4\beta^2\nu\right|^{-1/2} \vee |\nu|^{-1/2}\right],\ r_2 = \nu^{-1}s_1\right\};$$

$$L_{7,-1}^{3,2,=,<} = \left\{r_1, s_2 = (-D)^{1/4}(2^{3/2}p_2\varpi_4)^{-1},\ -s_1, r_2 = (p_1 + \beta p_2)(-D)^{-1/4}(\sqrt{2}p_2\varpi_4)^{-1}\right\};$$

$$L_{12}^{3,2,=,<} = \Big\{ r_1 = (\delta_{pq} + \nu(\beta p_2 - q_2)(-D)^{-1/2}\rho^{-1},\ s_1 = \delta_{pq}\varpi_6(-D)^{-1/2}(4\nu\rho)^{-1},$$
$$r_2 = (\beta p_2 - q_2)(2\rho)^{-1},\ s_2 = \delta_{pq}(4\nu\rho)^{-1}\Big\},$$

where $\rho = p_2\varpi_5|2\nu|^{1/2}$.

Theorem 2.1. *Any system* (2.1)[1] *with* $l = 2$ *written in the form* (1.1$^=$) *according to* (2.15)[1] *is linearly equivalent to the system generated by a representative of the corresponding canonical form in List* 2.1. *For each* $CF_i^{m,2,=,*}$, *the corresponding* (a) *conditions on* P_0^2 *and* H *in system* (1.1$^=$), (b) *changes* (1.2) *transforming the right-hand side of* (1.1$^=$) *under these conditions into the chosen form*, *and* (c) *the values of the factor* σ and *the parameter* u *in* $cs_i^{m,2,=,*}$ *obtained under these changes are as follows*:

$$\mathrm{CF}_3^{2,2,=,>}:\quad \text{(a)}\ \ D>0,\ [\varpi_1 = 0 \vee \varpi_2 = 0],\quad \text{(b)}\ \ J_1^2,\ L_3^{2,2,=,>},\quad \text{(c)}\ \ \sigma = [\operatorname{sgn}\lambda_1 \vee \operatorname{sgn}\lambda_2],$$
$$u = [\lambda_1^{-1}\lambda_2 \vee \lambda_1\lambda_2^{-1}];$$

$$\mathrm{CF}_{7,+1}^{2,2,=,>}:\quad \text{(a)}\ \ D>0,\ \varpi_1,\varpi_2 \neq 0,\ \lambda_1 = -\lambda_2,\quad \text{(b)}\ \ J_1^2,\ L_{7,+1}^{2,2,=,>},\quad \text{(c)}\ \ \sigma = \operatorname{sgn}\lambda_2;$$

$$\mathrm{CF}_7^{3,2,=,>}:\quad \text{(a)}\ \ D>0,\ \varpi_1,\varpi_2 \neq 0,\ \lambda_1 \neq -\lambda_2,\quad \text{(b)}\ \ J_1^2,\ L_7^{3,2,=,>},\quad \text{(c)}\ \ \sigma = \operatorname{sgn}\lambda_1,\ u = \lambda_1^{-1}\lambda_2;$$

$$\mathrm{CF}_3^{2,2,=,=}:\quad \text{(a)}\ \ D = 0,\ q_1 = 0,\ p_2 = 0,\quad \text{(b)}\ \ L_3^{2,2,=,=},\quad \text{(c)}\ \ \sigma = \operatorname{sgn} p_1;$$

$$\mathrm{CF}_7^{3,2,=,=}:\quad \text{(a)}\ \ D=0,\ [q_1\neq 0,\ \varpi_3\neq 0 \vee q_1 = 0,\ p_2\neq 0,\ \beta\neq 0],\quad \text{(b)}\ \ [J_{2a}^2 \vee J_{2b}^2],\ L_7^{3,2,=,=},\quad \text{(c)}\ \ \sigma=\operatorname{sgn}\nu;$$

$$\mathrm{CF}_{13}^{3,2,=,=}:\quad \text{(a)}\ \ D=0,\ [q_1\neq 0,\ \beta\neq 0,\ \varpi_3 = 0 \vee q_1 = 0,\ p_2\neq 0,\ \beta = 0],\quad \text{(b)}\ \ [J_{2a}^2 \vee J_{2b}^2],\ L_{13}^{3,2,=,=},\quad \text{(c)}\ \ \sigma=\operatorname{sgn}\nu;$$

$$\mathrm{CF}_{7,-1}^{2,2,=,<}:\quad \text{(a)}\ \ D<0,\ q_2 = -p_1,\quad \text{(b)}\ \ J_3^2,\ L_{7,-1}^{3,2,=,<},\quad \text{(c)}\ \ \sigma = 1;$$

$$\mathrm{CF}_{12}^{3,2,=,<}:\quad \text{(a)}\ \ D<0,\ \nu\neq 0,\quad \text{(b)}\ \ J_3^2,\ L_{12}^{3,2,=,<},\quad \text{(c)}\ \ \sigma = \operatorname{sgn}\nu,\ u = -\delta_{pq}(2\nu)^{-2}.$$

Proof. Depending on the sign of the discriminant D in (2.16)[1], system (1.1$^=$) with $\gamma = \beta^2$ is reduced to system (1.5), (1.6), or (1.7) with Jordan matrix $\tilde{H}$ and common factor $\tilde{P}_0^2$ by the change J_1^2, J_{2a}^2 or J_{2b}^2, J_3^2, respectively. Moreover, we have $\tilde{\alpha}, \tilde{\gamma} \geq 0$, $\tilde{\alpha} + \tilde{\gamma} > 0$, and $\tilde{\beta}^2 = \tilde{\alpha}\tilde{\gamma}$ by virtue of (2.18)[1] and (2.19)[1].

Next, in each of the obtained systems, we make an arbitrary change of the form (1.2) which transforms the given system into system (1.8), for which canonical forms will be determined.

In system (1.8), the common factor $\breve{P}_0^2$ has the following coefficients:

$$\begin{gathered}\tilde{\alpha} > 0:\quad \breve{\alpha} = \tilde{\alpha}^{-1}(\tilde{\alpha}r_1 + \tilde{\beta}r_2)^2,\quad \breve{\beta} = \tilde{\alpha}^{-1}(\tilde{\alpha}r_1 + \tilde{\beta}r_2)(\tilde{\alpha}s_1 + \tilde{\beta}s_2),\quad \breve{\gamma} = \tilde{\alpha}^{-1}(\tilde{\alpha}s_1 + \tilde{\beta}s_2)^2;\\ \tilde{\alpha} = 0\ \ (\tilde{\beta} = 0,\ \tilde{\gamma} > 0):\quad \breve{\alpha} = \tilde{\gamma}r_2^2,\quad \breve{\beta} = \tilde{\gamma}r_2s_2,\quad \breve{\gamma} = \tilde{\gamma}s_2^2.\end{gathered} \tag{2.9}$$

We can always ensure that, e.g., $\breve{\beta} = 0$ and $\breve{\gamma} = 0$ in (2.9). For this purpose, it suffices to fix the following relation between s_1 and s_2 in change (1.2):

$$\tilde{\alpha} \neq 0:\quad s_1 = -\tilde{\alpha}^{-1}\tilde{\beta}s_2,\quad \tilde{\alpha} = 0:\quad s_2 = 0, \tag{2.10}$$

as a result, the two rightmost columns of $\breve{A}$ in system (1.8) are zero.

(1) Let $D > 0$ ($\lambda_1, \lambda_2 \neq 0$, $\lambda_1 - \lambda_2 = \sigma_0\sqrt{D} \neq 0$). Applying the change J_1^2 to system (1.1$^=$), we obtain system (1.5), in which $\tilde{P}_0^2$ has $\tilde{\alpha} = \varpi_1^2$, $\tilde{\beta} = \varpi_1\varpi_2$, and $\tilde{\gamma} = \varpi_2^2$.

Suppose that change (1.2) under condition (2.10) reduces system (1.5) to system (1.8) whose coefficients $\breve{P}_0^2$ are defined by (2.9) (see [3, Appendix 3.4.1, p. 114]).

(1_1) If $\tilde{\alpha} = 0$ ($\tilde{\gamma} > 0$, $s_2 = 0$), then system (1.8) takes the form $\tilde{\gamma}r_2^2\begin{pmatrix}\lambda_2 & 0 & 0 & 0\\ (\lambda_1 - \lambda_2)r_1s_1^{-1} & \lambda_1 & 0 & 0\end{pmatrix}$. For $r_1 = 0$, $s_1 = 1$, and $r_2 = (\tilde{\gamma}|\lambda_1|)^{-1/2}$, this is $\mathrm{CF}_3^{2,2,=,>}$ with $\sigma = \operatorname{sgn}\lambda_1$ and $u = \lambda_1^{-1}\lambda_2 \neq 1$.

(1_2) If $\tilde\alpha > 0$, then system (1.8) has the form

$$\breve{A} = (r_1 + \tilde\alpha^{-1}\tilde\beta r_2)\begin{pmatrix} \tilde\alpha\lambda_1 r_1 + \tilde\beta\lambda_2 r_2 & \tilde\beta(\lambda_2 - \lambda_1)s_2 & 0 & 0 \\ \tilde\alpha(\lambda_2 - \lambda_1)r_1 r_2 s_2^{-1} & \tilde\alpha\lambda_2 r_1 + \tilde\beta\lambda_1 r_2 & 0 & 0 \end{pmatrix}. \quad (2.11)$$

(1_2^1) If $\tilde\beta = 0$ ($\tilde\gamma = 0$, $s_1 = 0$), then system (2.11) takes the form $\tilde\alpha r_1^2\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ (\lambda_2 - \lambda_1)r_2 s_2^{-1} & \lambda_2 & 0 & 0 \end{pmatrix}$. For $r_2 = 0$, $s_2 = 1$, and $r_1 = (\tilde\alpha|\lambda_2|)^{-1/2}$, this is $\mathrm{CF}_3^{2,2,=,>}$ with $\sigma = \operatorname{sgn}\lambda_2$ and $u = \lambda_\cdot\lambda_2^{-1} = 1$.

(1_2^2) Suppose that $\tilde\beta \neq 0$.

(1_2^{2a}) If $\lambda_1 = -\lambda_2 \Leftrightarrow p_1 + q_2 = 0$, then, for $r_1 = \tilde\alpha^{-1}\tilde\beta r_2$, system (2.11) has the form $4\tilde\gamma r_2^2\lambda_2\begin{pmatrix} 0 & r_2^{-1}s_2 & 0 & 0 \\ r_2 s_2^{-1} & 0 & 0 & 0 \end{pmatrix}$. For r_2, $s_2 = (4\tilde\gamma|\lambda_2|)^{-1/2}$, this is $\mathrm{CF}_{7,+1}^{2,2,=,>}$ with $\sigma = \operatorname{sgn}\lambda_2$.

(1_2^{2b}) If $\lambda_1 \neq -\lambda_2$, then, for $r_1 = 0$, system (2.11) has the form $\tilde\gamma r_2^2\begin{pmatrix} \lambda_2 & (\lambda_2 - \lambda_1)r_2^{-1}s_2 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \end{pmatrix}$. For $r_2 = (\tilde\gamma|\lambda_1|)^{-1/2}$ and $s_2 = \lambda_1(\lambda_1 - \lambda_2)^{-1}(\tilde\gamma|\lambda_1|)^{-1/2}$, this is $\mathrm{NSF}_7^{3,2,=,>}$ with $\sigma = \operatorname{sgn}\lambda_1$ and $u = \lambda_1^{-1}\lambda_2 \neq \pm 1$.

In system (2.11), we can also make $\breve{b}_2 = 0$ or $\breve{a}_1 = 0$; this results in $\mathrm{SF}_{12}^{3,2}$ or $\mathrm{SF}_{a,18}^{3,2}$, which are preceded by $\mathrm{CF}_7^{3,2,=,>}$ according to the second structural principle.

(2) Suppose that $D = (p_1 - q_2)^2 + 4p_2q_1 = 0$, i.e., $\lambda_1, \lambda_2 = \nu = (p_1 + q_2)/2 \neq 0$ in (2.16)[1].

(2_1) If $q_1 \neq 0$, then system ($1.1^=$) with $\gamma = \beta^2$ is reduced by the change J_{2a}^2 to system (1.6) with $\tilde\alpha = 4\beta^2$, $\tilde\beta = -2\beta((p_1 - q_2)\beta - 2q_1)$, and $\tilde\gamma = ((p_1 - q_2)\beta - 2q_1)^2$ according to (1.6_a); if $q_1 = 0$ and $p_2 \neq 0$ ($q_2, p_1 = \nu$), then this system is reduced by the change J_{2b}^2 to (1.6) with $\tilde\alpha = 1$, $\tilde\beta = \beta p_2$, and $\tilde\gamma = (\beta p_2)^2$ according to (1.6_b).

Suppose that change (1.2) under condition (2.10) reduces system (1.6) to system (1.8) whose coefficients $\breve{P}_0$ are defined by (2.9) (see [3, Appendix 3.4.2, p. 116]).

(2_1^1) If $\tilde\alpha = 0$ ($\tilde\gamma > 0$), then system (1.8) takes the form $\tilde\gamma r_2\begin{pmatrix} r_1 + \nu r_2 & s_1 & 0 & 0 \\ -r_1^2 s_1^{-1} & \nu r_2 - r_1 & 0 & 0 \end{pmatrix}$. For $r_1 = 0$, $r_2 = (\tilde\gamma|\nu|)^{-1/2}$, and $s_1 = \nu r_2$ ($s_2 = 0$), this is $\mathrm{CF}_7^{3,2,=,=}$ with $\sigma = \operatorname{sgn}\nu$ and $u = 1$.

In addition to $\breve\alpha_2 = 0$, we can obtain $\breve{b}_2 = 0$ or $\breve\alpha_1 = 0$ in system (1.8); this will transform the given system into one of the forms $\mathrm{SF}_{12}^{3,2}$ and $\mathrm{SF}_{a,18}^{3,2}$, which are preceded by $\mathrm{CF}_7^{3,2,=,=}$ according to the second structural principle.

(2_1^2) If $\tilde\alpha = [4\beta^2 \vee 1] > 0$, then system (1.8) has the form

$$\breve{A} = (r_1 + \tilde\alpha^{-1}\tilde\beta r_2)\begin{pmatrix} (\tilde\alpha\nu r_1 + \tilde\beta)r_1 + \tilde\beta\nu r_2 & -\tilde\alpha^{-1}\tilde\beta^2 s_2 & 0 & 0 \\ \tilde\alpha r_1^2 s_2^{-1} & (\tilde\alpha\nu r_1 - \tilde\beta)r_1 + \tilde\beta\nu r_2 & 0 & 0 \end{pmatrix}. \quad (2.12)$$

(2_1^{2a}) Suppose that $\tilde\beta = 0$ ($s_1 = 0$) $\Leftrightarrow [\varpi_3 = 0 \vee \beta = 0]$. Then (2.12) has the form $\tilde\alpha r_1^2\begin{pmatrix} \nu & 0 & 0 & 0 \\ r_1 s_2^{-1} & \nu & 0 & 0 \end{pmatrix}$. For $r_1 = (\tilde\alpha|\nu|)^{-1/2}$, $r_2 = 0$, and $s_2 = \nu^{-1}r_1$, this is $\mathrm{CF}_{a,13}^{3,2,=,=}$ with $\sigma = \operatorname{sgn}\nu$. Then, renumbering (2.7)[1] is performed.

(2_1^{2b}) If $\tilde\beta \neq 0$ ($\tilde\gamma > 0$), then, for $r_1 = 0$, system (2.12) has the form $\tilde\gamma r_2^2\begin{pmatrix} \nu & -\tilde\beta^{-1}\tilde\gamma r_2^{-1}s_2 & 0 & 0 \\ 0 & \nu & 0 & 0 \end{pmatrix}$. For $r_2 = (\tilde\gamma|\nu|)^{-1/2}$ and $s_2 = -\tilde\beta\tilde\gamma^{-1}\nu r_2$ ($s_1 = \nu r_2$), this is $\mathrm{CF}_7^{3,2,=,=}$ with $\sigma = \operatorname{sgn}\nu$ and $u = 1$.

Cases (2_1^1) and (2_1^{2b}) for $q_1 \neq 0$ ($\beta = 0$ and $\beta \neq 0$) are united in the statement of the theorem.

From system (2.12) succeeding forms $\mathrm{SF}_{12}^{3,2}$ and $\mathrm{SF}_{a,18}^{3,2}$ can also be obtained.

(2_2) Suppose that $q_1 = 0$ and $p_2 = 0$ ($q_2 = p_1$), i.e., the matrix H is diagonal in system ($1.1^=$) itself. Change (1.2) with $r_1 = |p_1|^{-1/2}$, $s_1 = -\beta$, $r_2 = 0$, and $s_2 = 1$ reduces ($1.1^=$) to $\mathrm{CF}_3^{2,2,=,=}$ ($u = 1$) with $\sigma = \mathrm{sgn}\, p_1$.

(3) Suppose that $D < 0$ ($p_2q_1 < 0$). Change J_3^2 reduces system ($1.1^=$) to system (1.7) with $\tilde\alpha = -D$, $\tilde\beta = \sqrt{-D}\varpi_6$, and $\tilde\gamma = \varpi_6^2$.

Suppose that change (1.2) under condition (2.10) reduces (1.7) to system (1.8) whose coefficients $\breve{P}_0$ are defined by (2.9) (see [3, Appendix 3.4.3, p. 118]).

In other words, system (1.8) has the form

$$(r_1 + \tilde\alpha^{-1}\tilde\beta r_2)\begin{pmatrix} (\tilde\alpha\nu + \tilde\beta\mu)r_1 + (\tilde\beta\nu - \tilde\alpha\mu)r_2 & -\tilde\alpha^{-1}(\tilde\alpha^2 + \tilde\beta^2)\mu s_2 & 0 & 0 \\ \mu(r_1^2 + r_2^2)s_2^{-1} & (\tilde\alpha\nu - \tilde\beta\mu)r_1 + (\tilde\beta\nu + \tilde\alpha\mu)r_2 & 0 & 0 \end{pmatrix}. \tag{2.13}$$

(3_1) Suppose that $\nu = 0 \Leftrightarrow q_2 = -p_1$; then $\tilde\alpha^2 + \tilde\beta^2 = -4Dp_2\varpi_4 \neq 0$, because the discriminant of ϖ_4 equals D, and $D = 4(p_1^2 + p_2q_1)$. In this case, for $r_2 = \tilde\alpha^{-1}\tilde\beta r_1$, system (2.13) takes the form $\tilde\alpha^{-3}(\tilde\alpha^2 + \tilde\beta^2)^2\mu r_1^2\begin{pmatrix} 0 & -r_1^{-1}s_2 & 0 & 0 \\ r_1 s_2^{-1} & 0 & 0 & 0 \end{pmatrix}$. For r_1, $s_2 = \tilde\alpha^{3/2}(\tilde\alpha^2 + \tilde\beta^2)^{-1}\mu^{-1/2}$, this is $\mathrm{CF}_{7,-1}^{2,2,=,<}$ with $\sigma = 1$.

(3_2) Suppose that $\nu \neq 0 \Leftrightarrow p_1 + q_2 \neq 0$; then $\tilde\alpha^2 + \tilde\beta^2 = -4Dp_2\varpi_5 \neq 0$, because the discriminant of ϖ_5 equals D. In this case, for $r_1 = \tilde\alpha^{1/2}(\tilde\alpha\mu + \tilde\beta\nu)\rho$, $r_2 = \tilde\alpha^{1/2}(\tilde\beta\mu - \tilde\alpha\nu)\rho$, and $s_2 = \tilde\alpha^{3/2}(\nu^2 + \mu^2)(2\nu)^{-1}\rho$, where $\rho = |2\nu|^{-1/2}\mu^{-1}(\tilde\alpha^2 + \tilde\beta^2)^{-1}$, we have $\breve{b}_2 = 0$ in system (2.13), and this is $\mathrm{CF}_{12}^{3,2,=,<}$ with $\sigma = \mathrm{sgn}\,\nu$ and $u = -(\nu^2 + \mu^2)(2\nu)^{-2} < -1/4$.

Making $\breve{a}_1 = 0$ in system (2.13), we obtain a SF with larger index. □

Thus, we have proved the completeness of List 2.1 of the forms $\mathrm{CF}_i^{m,2,=}$ with common factor P_0^2 having zero discriminant and the linear nonequivalence of these forms.

Below, we give nonsingular linear changes which make it possible to distinguish minimal canonical sets introduced in Definition 1.11 of [2] for the CFs in List 2.1.

Statement 2.2. The only forms in List 2.1 for which the values of the parameters of $cs_7^{m,2,=,}$ can be bounded are $\mathrm{CF}_{7,\kappa}^{2,2,=}$ and $\mathrm{CF}_7^{3,2,=,>}$: in $\mathrm{CF}_{7,\kappa}^{2,2,=}$, normalization (2.6)[1] with r_1, $-s_2 = 1$ changes the sign of σ, and in $\mathrm{CF}_7^{3,2,=,>}$ with $\tilde\sigma = \sigma$ and $\tilde u = u$, the change with $r_1 = |\tilde u|^{-1/2}$, $s_1 = 0$, $r_2 = (1 - \tilde u)|\tilde u|^{-1/2}$, and $s_2 = \tilde u|\tilde u|^{-1/2}$ yields $\sigma = \tilde\sigma\, \mathrm{sgn}\, \tilde u$ and $u = \tilde u^{-1}$.

Corollary 2.1. According to Definition 1.12 in [2], $acs_{7,\kappa}^{2,2,=} = \{\sigma = -1\}$ and $acs_7^{3,2,=,>} = \{|u| > 1\}$; for the other forms in List 2.1, $mcs^{m,2,=,*} = cs^{m,2,=,*}$.

3. CONSTRUCTION OF $\mathrm{CF}^{m,2}$ FOR P_0^2 WITH POSITIVE DISCRIMINANT

System (1.1) for a polynomial $P_0^2(x)$ with positive discriminant has the form

$$\dot{x} = P_0^2(x)Hx, \qquad \begin{matrix} P_0^2 = \alpha x_1^2 + 2\beta x_1x_2 + \gamma x_2^2, & D_0 = \beta^2 - \alpha\gamma > 0, \\ \alpha = 1 \quad \text{or} \quad \alpha, \gamma = 0, \quad 2\beta = 1 & (\det H = \delta_{pq} \neq 0). \end{matrix} \tag{$1.1^>$}$$

Consider the structural forms up to $\mathrm{SF}_{23}^{4,2}$ in List 1.1 of [2] which correspond to the case $l = 2$, $D_0 > 0$ (see [2, Statement 1.2]); there are nine such forms. We normalize them according to the normalization principles and find out which $\mathrm{NSF}^{m,2,>}$ are canonical.

Let us prove the list given below contains all canonical forms of system $(1.1^{>})$ together with their canonical sets from Definition 1.10 of [2].

List 3.1. The seven $\mathrm{CF}_i^{m,2,>}$ and their $cs_i^{m,2,>}$ are as follows (the row $(\alpha, 2\beta, \gamma)$, the matrix H, and the discriminants D_0 and D in $(2.16)^1$ for $\sigma, \kappa = \pm1$ and $u, v \neq 0$ are also specified):

$$\mathrm{CF}_4^{2,2,>,\geq} = \sigma\begin{pmatrix} 0 & u & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (0, 1, 0), \quad \sigma\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{matrix} 1/4, \\ (u-1)^2; \end{matrix}$$

$$\mathrm{CF}_{8,\kappa}^{2,2,>,\gtrless} = \sigma\begin{pmatrix} 0 & 0 & \kappa & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (0, 1, 0), \quad \sigma\begin{pmatrix} 0 & \kappa \\ 1 & 0 \end{pmatrix}, \quad \begin{matrix} 1/4, \\ 4\kappa; \end{matrix}$$

$$\mathrm{CF}_{10}^{3,2,>,\geq} = \sigma\begin{pmatrix} 0 & u & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad (0, 1, 0), \quad \sigma\begin{pmatrix} u & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{matrix} 1/4, \\ (u-1)^2; \end{matrix}$$

$$\mathrm{CF}_{16}^{3,2,>,\gtreqless} = \sigma\begin{pmatrix} 0 & 1 & u & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (0, 1, 0), \quad \sigma\begin{pmatrix} 1 & u \\ 1 & 0 \end{pmatrix}, \quad \begin{matrix} 1/4, \\ 4u+1; \end{matrix}$$

$$\mathrm{CF}_{8,-1}^{4,2,>,>} = \sigma\begin{pmatrix} u & 0 & -u & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad (1, 0, -1), \quad \sigma\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{matrix} 1, \\ (u-1)^2; \end{matrix}$$

$$\mathrm{CF}_{a,14,-1}^{4,2,>,>} = \sigma\begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & u & u & 0 \end{pmatrix}, \quad (1, 1, 0), \quad \sigma\begin{pmatrix} -1 & 1 \\ 0 & u \end{pmatrix}, \quad \begin{matrix} 1/4, \\ (u+1)^2; \end{matrix}$$

$$\mathrm{CF}_{23}^{4,2,>,\gtreqless} = \sigma\begin{pmatrix} 0 & u & v & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad (0, 1, 0), \quad \sigma\begin{pmatrix} u & v \\ 1 & 1 \end{pmatrix}, \quad \begin{matrix} 1/4, \\ (u-1)^2 + 4v; \end{matrix}$$

$$cs_4^{2,2,>,>} = \{u \neq 1\}, \quad cs_4^{2,2,>,=} = \{u = 1\}; \quad cs_{8,\kappa}^{2,2,>,>} = \{\kappa = 1\}, \quad cs_{8,\kappa}^{2,2,>,<} = \{\kappa = -1\};$$

$$cs_{10}^{3,2,>,>} = \{u \neq 1\}, \quad cs_{10}^{3,2,>,=} = \{u = 1\};$$

$$cs_{16}^{3,2,>,>} = \{u > -1/4\}, \quad cs_{16}^{3,2,>,=} = \{u = -1/4\}, \quad cs_{16}^{3,2,>,<} = \{u < -1/4\};$$

$$cs_{8,-1}^{4,2,>,>} = \{u \neq \pm1\}, \quad cs_{14,-1}^{4,2,>,>} = \{u \neq -1, -2, -3\};$$

$$cs_{23}^{4,2,>,>} = \{u \neq 1, v > -(1-u)^2/4, v \neq u, (2u-1)/4, u(2-u)/4\},$$

$$cs_{23}^{4,2,>,=} = \{u \neq -1, v = -(1-u)^2/4\}, \quad cs_{23}^{4,2,>,<} = \{v < -(1-u)^2/4\}.$$

Statement 3.1. The forms $\mathrm{NSF}_7^{4,2,>} = \sigma\begin{pmatrix} 0 & u & u & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ and $\mathrm{NSF}_{12}^{4,2,>} = \begin{pmatrix} 0 & u & 0 & -u \\ 0 & 0 & 1 & 1 \end{pmatrix}$ in List 3.1 are not $\mathrm{CF}^{4,2,>}$.

Proof. The change with $s_1 = -s_2$ and $r_2 = 0$ reduces $\mathrm{NSF}_7^{4,2,>}$ to $\mathrm{CF}_{10}^{3,2,>,>}$ or $\mathrm{CF}_4^{2,2,>,=}$ and $\mathrm{NSF}_{12}^{4,2,>}$, to $\mathrm{CF}_{10}^{3,2,>,>}$ or $\mathrm{CF}_4^{2,2,>,>}$. □

Collection 3.1. In the rest of Section 3, we use the following constants and changes:

$$\varphi_1 = \tilde{\eta}^2\lambda_1 - \tilde{\alpha}\tilde{\gamma}\lambda_2, \quad \varphi_2 = \tilde{\eta}^2\lambda_2 - \tilde{\alpha}\tilde{\gamma}\lambda_1, \quad \varphi_3^{\pm} = 2\tilde{\tau}\nu\sigma_\beta \pm \tilde{\gamma}, \quad \varphi_4^{\pm} = 2\tilde{\tau}\nu\sigma_\beta \pm (\tilde{\alpha} + \tilde{\gamma})\mu;$$

$$L_4^{2,2,>,>} = \{r_1 = 1, s_1, r_2 = 0, s_2 = (2\tilde{\beta}\lambda_2)^{-1}\};$$

$$L1_{10}^{3,2,>,>} = \{r_1 = -|\tilde{\gamma}|^{1/2} D^{1/4}(2\tilde{\beta}\lambda_2)^{-1}\sigma_0\sigma_\gamma, s_1 = -\tilde{\gamma}\tilde{\eta}^{-1}s_2, r_2 = 0, s_2 = |\tilde{\gamma}|^{-1/2} D^{-1/4}\},$$

$$L2_{10}^{3,2,>,>} = \{r_1 = 0, s_1 = |\tilde{\alpha}|^{-1/2} D^{-1/4}, r_2 = |\tilde{\alpha}|^{1/2} D^{1/4}(2\tilde{\beta}\lambda_1)^{-1}\sigma_0\sigma_\alpha, s_2 = -\tilde{\alpha}\tilde{\eta}^{-1}s_1\};$$

$$L_{8,+1}^{2,2,>,>} = \{r_1 = |4\tilde{\alpha}\lambda_1|^{-1/2}, s_1 = -\tilde{\gamma}\tilde{\eta}^{-1}s_2, r_2 = -\tilde{\alpha}\tilde{\eta}^{-1}r_1, s_2 = |4\tilde{\gamma}\lambda_1|^{-1/2}\};$$

$$L^{4,2,>,>}_{8,-1} = \{-r_1, s_1 = v^{1/4}(1+v^{1/2})^{-1/2}, r_2, s_2 = v^{-1/4}(1+v^{1/2})^{-1/2}\};$$

$$L1^{3,2,>,>}_{16} = \{r_1 = \tilde{\phi}^2\tilde{\eta}|\tilde{\alpha}\lambda_1|^{-1/2}, s_1 = -\tilde{\gamma}\tilde{\eta}^{-1}s_2, r_2 = -\tilde{\alpha}\tilde{\eta}^{-1}r_1, s_2 = \tilde{\alpha}(2\tilde{\beta})^{-1}r_1\},$$

$$L2^{3,2,>,>}_{16} = \{r_1 = -\tilde{\gamma}\tilde{\eta}^{-1}r_2, s_1 = \tilde{\gamma}(2\tilde{\beta})^{-1}r_2, r_2 = \tilde{\phi}^2\tilde{\eta}|\tilde{\gamma}\lambda_2|^{-1/2}, s_2 = -\tilde{\alpha}\tilde{\eta}^{-1}s_1\};$$

$$L1^{4,2,>,>}_{23} = \{r_1 = |2\tilde{\alpha}|^{-1/2}D^{-1/4}, s_1 = -\tilde{\gamma}\tilde{\eta}^{-1}s_2, r_2 = -\tilde{\alpha}\tilde{\eta}^{-1}r_1, s_2 = -|\tilde{\alpha}|^{1/2}D^{1/4}(-2\tilde{\alpha}\tilde{\gamma})^{-1/2}(2\nu)^{-1}\sigma_0\sigma_\alpha\},$$

$$L2^{4,2,>,>}_{23} = \{r_1 = \tilde{\phi}|\tilde{\eta}|^{1/2}|\tilde{\alpha}|^{-1/2}D^{-1/4}, s_1 = -\tilde{\gamma}\tilde{\eta}^{-1}s_2, r_2 = -\tilde{\alpha}\tilde{\eta}^{-1}r_1, s_2 = \tilde{\phi}|\tilde{\eta}|^{3/2}|\tilde{\alpha}|^{1/2}D^{1/4}\varphi_2^{-1}\sigma_0\sigma_\alpha\sigma_\beta\};$$

$$L1^{4,2,>,>}_{14,-1} = \{r_1, s_1 = 1, r_2 = 0, s_2 = -2\},$$

$$L2^{4,2,>,>}_{14,-1} = \{r_1 = 0, s_1 = (\tilde{u}-2)s_2/2, r_2, s_2 = 2|\tilde{u}(\tilde{u}-2)|^{-1/2}\};$$

$$L^{2,2,>,=}_{4} = \{r_1 = \tilde{\eta}, s_1 = \tilde{\phi}^4\tilde{\gamma}(p_1\tilde{\eta})^{-1}, r_2 = -\tilde{\alpha}, s_2 = \tilde{\phi}^4 p_1^{-1}\};$$

$$L^{3,2,>,=}_{10} = \{r_1 = -\tilde{\gamma}\tilde{\eta}^{-1}r_2, s_1 = |2\tilde{\beta}|^{-1/2}, r_2 = \nu^{-1}s_1, s_2 = -\tilde{\alpha}\tilde{\eta}^{-1}s_1\};$$

$$L1^{3,2,>,=}_{16} = \{r_1 = -\tilde{\gamma}\tilde{\eta}^{-1}r_2, s_1 = -\tilde{\phi}/2, r_2 = -\tilde{\phi}\tilde{\eta}\tilde{\gamma}^{-1}\sigma_\beta, s_2 = -\tilde{\alpha}\tilde{\eta}^{-1}s_1\};$$

$$L2^{3,2,>,=}_{16} = \{r_1 = \tilde{\phi}, s_1 = -\tilde{\gamma}\tilde{\eta}^{-1}s_2, r_2 = -\tilde{\alpha}\tilde{\eta}^{-1}r_1, s_2 = \tilde{\phi}\tilde{\eta}(2\tilde{\gamma})^{-1}\sigma_\beta\};$$

$$L^{4,2,>,=}_{23} = \{r_1 = \tilde{\phi}, s_1 = -\tilde{\gamma}\tilde{\eta}^{-1}s_2, r_2 = -\tilde{\alpha}\tilde{\eta}^{-1}r_1, s_2 = \tilde{\phi}\tilde{\eta}(\varphi_3^-)^{-1}\sigma_\beta\};$$

$$L^{2,2,>,=}_{8,-1} = \{r_1, s_2 = |\tilde{\eta}|^{1/2}(\tilde{\alpha}^2+\tilde{\beta}^2)^{-1}\mu^{-1/2}, s_1 = -\tilde{\gamma}\tilde{\eta}^{-1}s_2, r_2 = -\tilde{\alpha}\tilde{\eta}^{-1}r_1\};$$

$$L1^{3,2,>,<}_{16} = \{r_1 = -\tilde{\gamma}\tilde{\eta}^{-1}r_2, s_1 = \tilde{\phi}(\tilde{\gamma}^2+\tilde{\eta}^2)^{1/2}(\tilde{\alpha}+\tilde{\gamma})^{-1}(4\mu)^{-1/2}, r_2 = \tilde{\phi}\tilde{\eta}((\tilde{\gamma}^2+\tilde{\eta}^2)\mu)^{-1/2}, s_2 = -\tilde{\alpha}\tilde{\eta}^{-1}s_1\};$$

$$L2^{3,2,>,<}_{16} = \{r_1 = \tilde{\phi}\tilde{\eta}((\tilde{\alpha}^2+\tilde{\eta}^2)\mu)^{-1/2}, s_1 = -\tilde{\gamma}\tilde{\eta}^{-1}s_2, r_2 = -\tilde{\alpha}\tilde{\eta}^{-1}r_1, s_2 = \tilde{\phi}(\tilde{\alpha}^2+\tilde{\eta}^2)^{1/2}(\tilde{\alpha}+\tilde{\gamma})^{-1}(4\mu)^{-1/2}\};$$

$$L^{4,2,>,<}_{23} = \{r_1 = \tilde{\phi}\tilde{\eta}((\tilde{\alpha}^2+\tilde{\eta}^2)\mu)^{-1/2}, s_1 = -\tilde{\gamma}\tilde{\eta}^{-1}s_2, r_2 = -\tilde{\alpha}\tilde{\eta}^{-1}r_1, s_2 = \tilde{\phi}((\tilde{\alpha}^2+\tilde{\eta}^2)\mu)^{1/2}/\varphi_4^-\}.$$

Statement 3.2. The only forms in List 3.1 with given parameters values which can be reduced to structural forms preceding them according to the structural principles are as follows:

(1) $\mathrm{NSF}^{4,2,>,>}_{8,-1}$: (a) for $u = -1$, the change with $r_2 = -r_1$ and $s_1 = s_2$ reduces this form to $\mathrm{SF}^{2,2}_8$;

(b) $\mathrm{NSF}^{4,2,>,=}_{8,-1}$ ($u = 1$) is reduced by the same change to $\mathrm{SF}^{2,2}_4$;

(2) $\mathrm{NSF}^{4,2,>,>}_{14,-1}$: (a) for $u = -3$, the change with $r_1 = 0$ and $s_1 = -2s_2$ reduces this form to $\mathrm{SF}^{4,2}_{8,-1}$;

(b) for $u = -2$, the change with $r_2 = -r_1$ and $s_2 = 0$ reduces it to $\mathrm{SF}^{3,2}_{10}$;

(c) $\mathrm{NSF}^{4,2,>,=}_{14,-1}$ ($u = -1$) is reduced by the same change to $\mathrm{SF}^{3,2}_{16}$;

(3) $\mathrm{NSF}^{4,2,>,>}_{23}$: (a) for $\tilde{\sigma} = \sigma$ and $u = 1$ ($v > 0$, $v \neq 1$), the change $L^{4,2,>,>}_{8,-1}$ reduces this form to $\mathrm{CF}^{4,2,>,>}_{8,-1}$ with $\sigma = -\tilde{\sigma}$ and $u = (1-v^{1/2})(1+v^{1/2})^{-1} \in (-1, 1)$ ($|u| < 1$);

(b) for $\tilde{u} = u \neq 1$ and $v = (2\tilde{u}-1)/4$ ($\tilde{u} \neq \pm 1/2$), the change $L1^{4,2,>,>}_{14,-1}$ reduces it to $\mathrm{CF}^{4,2,>,>}_{14,-1}$ with $u = -2\tilde{u} - 1$ ($u \neq -1, -2, -3$);

(c) for $\tilde{\sigma} = \sigma$, $\tilde{u} = u \neq 1$, and $v = \tilde{u}(2-\tilde{u})/4$ ($\tilde{u} \neq \pm 2$), the change $L2^{4,2,>,>}_{14,-1}$ reduces it $\mathrm{CF}^{4,2,>,>}_{14,-1}$ with $\sigma = -\tilde{\sigma}\,\mathrm{sgn}(\tilde{u}(\tilde{u}-2))$ and $u = -(\tilde{u}+2)\tilde{u}^{-1}$ ($u \neq -1, -2, -3$).

Theorem 3.1. *Any system* (2.1)[1] *with* $l = 2$ *written in the form* ($1.1^>$) *according to* (2.15)[1] *is linearly equivalent to the system generated by a representative of the corresponding canonical form in List* 3.1. *For each* $\mathrm{CF}^{m,2,>,*}_i$, *the corresponding* (a) *conditions on the coefficients of system* ($1.1^>$), (b) *changes* (1.2) *transforming the right-hand side of* ($1.1^>$) *under these conditions into the chosen form*, *and* (c) *the values of the factor* σ *and the parameters* u *and* v *in* $cs^{m,2,>,*}_i$ *obtained under these change are as follows*:

$\mathrm{CF}^{2,2,>,>}_4$: (a) $D > 0$, $\tilde{\alpha} = 0$, $\tilde{\gamma} = 0$ *in* (1.5), (b) J_1^2, $L^{2,2,>,>}_4$, (c) $\sigma = 1$, $u = \lambda_1\lambda_2^{-1}$;

$\mathrm{CF}_{10}^{3,2,>,>}$: (a) $D > 0$, $[\tilde\alpha = 0,\ \tilde\gamma \neq 0\ \vee\ \tilde\alpha \neq 0,\ \tilde\gamma = 0]$ *in* (1.5), (b) J_1^2, $[L1_{10}^{3,2,>,>}\ \vee\ L2_{10}^{3,2,>,>}]$, (c) $\sigma = [-\sigma_0\sigma_\gamma \vee \sigma_0\sigma_\alpha]$, $u = [\lambda_1\lambda_2^{-1} \vee \lambda_1^{-1}\lambda_2]$;

$\mathrm{CF}_{8,+1}^{2,2,>,>}$: (a) $D > 0$, $\tilde\beta = 0$, $\nu = 0$ *in* (1.5), (b) J_1^2, $L_{8,+1}^{2,2,>,>}$, (c) $\sigma = \sigma_0\sigma_\alpha$;

$\mathrm{CF}_{8,-1}^{4,2,>,>}$: (a) $D > 0$, $\tilde\beta = 0$ and $\nu \neq 0$ *in* (1.5), (b) J_1^2, $L1_{23}^{4,2,>,>}$, $L_{8,-1}^{4,2,>,>}$ *with* $v = D(2\nu)^{-2}$, (c) $\sigma = -\sigma_0\sigma_\alpha$, $u = (|2\nu| - D^{1/2})(|2\nu| + D^{1/2})^{-1}$;

$\mathrm{CF}_{16}^{3,2,>,>}$: (a) $D > 0$, *in* (1.5), $\tilde\alpha, \tilde\beta, \tilde\gamma \neq 0$ *and* $2\tilde\tau\nu = [\sigma_0 D^{1/2}|\tilde\beta| \vee -\sigma_0 D^{1/2}|\tilde\beta|]$, (b) J_1^2, $[L1_{16}^{3,2,>,>} \vee L2_{16}^{3,2,>,>}]$, (c) $\sigma = [\sigma_\alpha \operatorname{sgn}\lambda_1 \vee \sigma_\gamma \operatorname{sgn}\lambda_2]$, $u = -\tilde\alpha\tilde\gamma(2\tilde\beta)^{-2}$;

$\mathrm{CF}_{14,-1}^{4,2,>,>}$: (a) $D > 0$, *in* (1.5), $\tilde\alpha, \tilde\beta, \tilde\gamma \neq 0$ *and* $2\tilde\tau\nu \neq \pm D^{1/2}|\tilde\beta|$, $4\tilde v = [2\tilde u - 1 \vee \tilde u(2 - \tilde u)]$, *where* $\tilde u = \varphi_1\varphi_2^{-1}$ *and* $\tilde v = -D\tilde\alpha\tilde\gamma\tilde\eta^2\varphi_2^{-2}$, (b) J_1^2, $L2_{23}^{4,2,>,>}$, $[L1_{14,-1}^{4,2,>,>} \vee L2_{14,-1}^{4,2,>,>}]$, (c) $\sigma = [\sigma_0\sigma_\alpha \vee \sigma_0\sigma_\alpha \operatorname{sgn}(\tilde u(2 - \tilde u))]$, $u = [-(1 + 2\tilde u)^{-1} \vee -\tilde u(\tilde u + 2)^{-1}]$;

$\mathrm{CF}_{23}^{4,2,>,>}$: (a) $D > 0$, *in* (1.5), $\tilde\alpha, \tilde\beta, \tilde\gamma \neq 0$, *and* $2\tilde\tau\nu \neq \pm D^{1/2}|\tilde\beta|$, $4\tilde v \neq \tilde u(2 - \tilde u)$, $(2\tilde u - 1)$, *where* $\tilde u = \varphi_1\varphi_2^{-1}$ *and* $\tilde v = -D\tilde\alpha\tilde\gamma\tilde\eta^2\varphi_2^{-2}$, (b) J_1^2, $L2_{23}^{4,2,>,>}$, (c) $\sigma = \sigma_0\sigma_\alpha$, $u = \tilde u$, $v = \tilde v$;

$\mathrm{CF}_4^{2,2,>,=}$: (a) $D = 0$, $q_1, p_2 = 0$, (b) $L_4^{2,2,>,=}$, (c) $\sigma = 1$;

$\mathrm{CF}_{10}^{3,2,>,=}$: $(1.1^>)$ *if* $D = 0$, $[q_1 \neq 0 \vee q_1 = 0,\ p_2 \neq 0]$, *and* $\tilde\gamma = 0$ *in* $[(1.6_a) \vee (1.6_b)]$, *then the changes* $[J_{2a}^2 \vee J_{2b}^2]$ *and* $L_{10}^{3,2,>,=}$ *reduce* $(1.1^>)$ *to* $CF_{10}^{3,2,>,=}$ *with* $\sigma = \sigma_\beta$;

$\mathrm{CF}_{16}^{3,2,>,=}$: (1) (a) $D = 0$, $[q_1 \neq 0 \vee q_1 = 0,\ p_2 \neq 0]$, *in* $[(1.6_a) \vee (1.6_b)]$, $\tilde\gamma \neq 0$ *and* $\varphi_3^+ = 0$, (b) $[J_{2a}^2 \vee J_{2b}^2]$, $L1_{16}^{3,2,>,=}$, (c) $\sigma = -\sigma_\beta$;

(2) (a) $D = 0$, $[q_1 \neq 0 \vee q_1 = 0,\ p_2 \neq 0]$, *in* $[(1.6_a) \vee (1.6_b)]$, $\tilde\gamma \neq 0$ *and* $\varphi_3^- = 0$, (b) $[J_{2a}^2 \vee J_{2b}^2]$, $L2_{16}^{3,2,>,=}$, (c) $\sigma = \sigma_\beta$;

$\mathrm{CF}_{23}^{4,2,>,=}$: (a) $D = 0$, $[q_1 \neq 0 \vee q_1 = 0,\ p_2 \neq 0]$, *in* $[(1.6_a) \vee (1.6_b)]$, $\tilde\gamma \neq 0$ *and* $\varphi_3^\pm \neq 0$, (b) $[J_{2a}^2 \vee J_{2b}^2]$, $L_{23}^{4,2,>,=}$, (c) $\sigma = \sigma_\beta$, $u = \varphi_3^+(\varphi_3^-)^{-1}$, $v = -\tilde\gamma^2(\varphi_3^-)^{-2}$;

$\mathrm{CF}_{8,-1}^{2,2,>,<}$: (a) $D < 0$, $\nu = 0$, $\tilde\alpha + \tilde\gamma = 0$ in (1.7), (b) J_3^2, $L_{8,-1}^{2,2,>,<}$, (c) $\sigma = \sigma_\beta$;

$\mathrm{CF}_{16}^{3,2,>,<}$: (a) $D < 0$, $\nu \neq 0$, $[\varphi_4^+ = 0 \vee \varphi_4^- = 0]$ *in* (1.7), (b) J_3^2, $[L1_{16}^{3,2,>,<} \vee L2_{16}^{3,2,>,<}]$, (c) $\sigma = [-\sigma_\beta \vee \sigma_\beta]$;

$\mathrm{CF}_{23}^{4,2,>,<}$: (a) $D < 0$, $\nu^2 + (\tilde\alpha + \tilde\gamma)^2 \neq 0$ *and* $\varphi_4^+ \neq 0$ *in* (1.7), (b) J_3^2, $L_{23}^{4,2,>,<}$, (c) $\sigma = \sigma_\beta$, $u = \varphi_4^+(\varphi_4^-)^{-1}$, $v = -\mu^2(\tilde\alpha^2 + \tilde\eta^2)(\tilde\gamma^2 + \tilde\eta^2)\tilde\eta^{-2}(\varphi_4^-)^{-2}$.

Proof. Depending on the sign of the discriminant D in $(2.16)^1$, system $(1.1^>)$ with $\beta^2 > \alpha\gamma$ is reduced to one of systems (1.5), (1.6), and (1.7) with Jordan matrix $\tilde H$ and common factor $\tilde P_0^2$ by one of the changes J_1^2, J_{2a}^2 and J_{2b}^2, J_3^2, respectively; moreover, we have $\tilde\tau, |\tilde\eta| > 0$, because $\tilde D_0 = \delta^2 D_0$ by virtue of $(2.19)^1$.

Next, for each of the systems thus obtained, we make an arbitrary change of the form (1.2), which transforms this system into system (1.8), for which canonical forms will be determined.

The coefficients $\breve\alpha$ and $\breve\gamma$ of the common factor $\breve P_0^2$ in system (1.8) can always be made zero; as a result, $\breve A$ in (1.8) will have elements $\breve a_1, \breve a_2 = 0$ and $\breve d_1, \breve d_2 = 0$.

To achieve this, it suffices to fix the following two relations in change (1.2):

$$s_1 = -\tilde\gamma\tilde\eta^{-1}s_2, \quad r_2 = -\tilde\alpha\tilde\eta^{-1}r_1 \quad (\tilde\tau = (\tilde\beta^2 - \tilde\alpha\tilde\gamma)^{1/2},\ \tilde\eta = \tilde\beta + \sigma_\beta\tilde\tau,\ \operatorname{sgn} 0 = 1), \tag{3.14}$$

which imply $\delta = r_1s_2 - r_2s_1 = 2\tilde\tau\tilde\eta^{-1}\sigma_\beta r_1s_2$ and, in system (1.8), $\breve\beta = 2\tilde\tau^2\tilde\eta^{-1}r_1s_2$. If $\tilde\alpha\tilde\gamma = 0$, then $\tilde\tau = |\tilde\beta| \neq 0$, and if $\tilde\beta = 0$, then $\tilde\tau = \tilde\eta = (-\tilde\alpha\tilde\gamma)^{1/2} > 0$.

However, no changes satisfying condition (3.14) yield $\mathrm{NSF}^{4,2,>}_{8,-1}$ and $\mathrm{NSF}^{4,2,>}_{a,14,-1}$ in List 3.1. But these forms precede only $\mathrm{NSF}^{4,2,>}_{23}$, and they will be obtained from the latter according to Statement 3.2_3 in items 1^{1b}_2 and 1^{2c}_2, respectively.

(1) Suppose that $D > 0$ ($\lambda_1, \lambda_2 \neq 0$, $\lambda_1 - \lambda_2 = \sigma_0\sqrt{D} \neq 0$). The change J^2_1 reduces system ($1.1^>$) to system (1.5) (see [3, Appendix 3.5.1, p. 119]).

An arbitrary change of the form (1.2) satisfying condition (3.14) reduces (1.5) to system (1.8) of the form

$$2\tilde{\tau}\sigma_\beta\begin{pmatrix} 0 & (\lambda_1 - \tilde{\alpha}\tilde{\gamma}\tilde{\eta}^{-2}\lambda_2)r_1s_2 & -\tilde{\gamma}\tilde{\eta}^{-1}(\lambda_1-\lambda_2)s_2^2 & 0 \\ 0 & \tilde{\alpha}\tilde{\eta}^{-1}(\lambda_1-\lambda_2)r_1^2 & (\lambda_2 - \tilde{\alpha}\tilde{\gamma}\tilde{\eta}^{-2}\lambda_1)r_1s_2 & 0 \end{pmatrix}. \tag{3.15}$$

(1_1) Consider the case $\tilde{\alpha}\tilde{\gamma} = 0$ ($\tilde{\beta} \neq 0$, $\tilde{\tau} = |\tilde{\beta}|$, $\tilde{\eta} = 2\tilde{\beta}$).

(1^1_1) If $\tilde{\alpha}, \tilde{\gamma} = 0$ (r_1, $s_2 = 0$), then (3.15) takes the form $\begin{pmatrix} 0 & 2\tilde{\beta}\lambda_1r_1s_2 & 0 & 0 \\ 0 & 0 & 2\tilde{\beta}\lambda_2r_1s_2 & 0\end{pmatrix}$. For $r_1 = 1$ and $s_2 = (2\tilde{\beta}\lambda_2)^{-1}$, this is $\mathrm{CF}^{2,2,>,>}_4$ with $\sigma = 1$ and $u = \lambda_1\lambda_2^{-1} \neq 1$.

(1^2_1) If $\tilde{\alpha} = 0$ and $\tilde{\gamma} \neq 0$, then system (3.15) takes the form $\begin{pmatrix} 0 & 2\tilde{\beta}\lambda_1r_1s_2 & -\tilde{\gamma}(\lambda_1-\lambda_2)s_2^2 & 0 \\ 0 & 0 & 2\tilde{\beta}\lambda_2r_1s_2 & 0\end{pmatrix}$. For $r_1 = -|\tilde{\gamma}(\lambda_1-\lambda_2)|^{1/2}(2\tilde{\beta}\lambda_2)^{-1}\sigma_0\sigma_\gamma$ and $s_2 = |\tilde{\gamma}(\lambda_1-\lambda_2)|^{-1/2}$, this is $\mathrm{CF}^{3,2,>,>}_{10}$ with $\sigma = -\sigma_0 \operatorname{sgn}\tilde{\gamma}$ and $u = \lambda_1\lambda_2^{-1} \neq 1$.

(1^3_1) If $\tilde{\gamma} = 0$ and $\tilde{\alpha} \neq 0$, then system (3.15) has the form $\begin{pmatrix} 0 & 2\tilde{\beta}\lambda_1r_1s_2 & 0 & 0 \\ 0 & \tilde{\alpha}(\lambda_1-\lambda_2)r_1^2 & 2\tilde{\beta}\lambda_2r_1s_2 & 0\end{pmatrix}$. For $r_1 = |\tilde{\alpha}(\lambda_1-\lambda_2)|^{-1/2}$ and $s_2 = |\tilde{\alpha}(\lambda_1-\lambda_2)|^{-1/2}(2\tilde{\beta}\lambda_1)^{-1}\sigma_0\sigma_\alpha$, this is $\mathrm{CF}^{3,2,>,>}_{a,10}$ with $u = \lambda_1^{-1}\lambda_2 \neq 1$ and $\sigma = \sigma_0\sigma_\alpha$. Renumbering $(2.7)^1$ reduces it to $\mathrm{CF}^{3,2,>,>}_{10}$ with the same σ and u.

(1_2) Suppose that $\tilde{\alpha} \neq 0$ and $\tilde{\gamma} \neq 0$ ($\tilde{\tau} \neq |\tilde{\beta}|$). In this case, we have $\breve{c}_1\breve{b}_2 \neq 0$ in system (3.15).

(1^1_2) If $\tilde{\beta} = 0$, then $\tilde{\alpha}\tilde{\gamma} < 0$ and $\tilde{\eta}, \tilde{\tau} = (-\tilde{\alpha}\tilde{\gamma})^{1/2}$. Therefore, system (3.15) has the form

$$\begin{pmatrix} 0 & 2(-\tilde{\alpha}\tilde{\gamma})^{1/2}(\lambda_1+\lambda_2)r_1s_2 & -2\tilde{\gamma}(\lambda_1-\lambda_2)s_2^2 & 0 \\ 0 & 2\tilde{\alpha}(\lambda_1-\lambda_2)r_1^2 & 2(-\tilde{\alpha}\tilde{\gamma})^{1/2}(\lambda_1+\lambda_2)r_1s_2 & 0 \end{pmatrix}. \tag{3.16}$$

(1^{1a}_2) Suppose that $\lambda_2 = -\lambda_1 \Leftrightarrow q_2 = -p_1$. Then system (3.16) with $r_1 = |4\tilde{\alpha}\lambda_1|^{-1/2}$ and $s_2 = |4\tilde{\gamma}\lambda_1|^{-1/2}$ is $\mathrm{CF}^{2,2,>,>}_{8,+1}$ with $\sigma = \operatorname{sgn}(\tilde{\alpha}\lambda_1)$ ($= \sigma_\alpha\sigma_0 = \sigma_\alpha \operatorname{sgn} p_1 = -\sigma_0\sigma_\gamma$).

(1^{1b}_2) Suppose that $\lambda_2 \neq -\lambda_1$. Then (3.16) with $s_2 = -|\tilde{\alpha}(\lambda_1-\lambda_2)|^{1/2}(-2\tilde{\alpha}\tilde{\gamma})^{-1/2}(\lambda_1+\lambda_2)^{-1}\sigma_0\sigma_\alpha$, and $r_1 = |2\tilde{\alpha}(\lambda_1-\lambda_2)|^{-1/2}$ is $\mathrm{NSF}^{4,2,>,>}_{23}$ with $\sigma = \operatorname{sgn}(\tilde{\alpha}(\lambda_1-\lambda_2))$ ($= \sigma_0\sigma_\alpha$), $u = 1$, and $v = D(\lambda_1+\lambda_2)^{-2}$ ($v > 0$, $v \neq 1$). According to Statement 3.2_3, this system is not canonical, because it reduces to $\mathrm{CF}^{4,2,>,>}_{8,-1}$.

(1^2_2) Suppose that $\tilde{\beta} \neq 0$. Then $\breve{b}_1^2 + \breve{c}_2^2 \neq 0$.

(1^{2a}_2) We have $\breve{c}_2 = 0 \Leftrightarrow \lambda_2 = \tilde{\alpha}\tilde{\gamma}\tilde{\eta}^{-2}\lambda_1 \Leftrightarrow \tilde{\tau}(\lambda_1+\lambda_2) = |\tilde{\beta}|(\lambda_1-\lambda_2)$, because $\tilde{\alpha}\tilde{\gamma} = (\tilde{\beta} - \sigma_\beta\tilde{\tau})\tilde{\eta}$. In this case, (3.15) has the form $\begin{pmatrix} 0 & 8\tilde{\beta}\tilde{\tau}^2\tilde{\eta}^{-2}\lambda_1r_1s_2 & -4\tilde{\gamma}\tilde{\tau}^2\tilde{\eta}^{-2}\lambda_1s_2^2 & 0 \\ 0 & 4\tilde{\alpha}\tilde{\tau}^2\tilde{\eta}^{-2}\lambda_1r_1^2 & 0 & 0\end{pmatrix}$. For $r_1 = \tilde{\eta}(2\tilde{\tau})^{-1}|\tilde{\alpha}\lambda_1|^{-1/2}$ and $s_2 = \tilde{\eta}\tilde{\alpha}(4\tilde{\beta}\tilde{\tau})^{-1}|\tilde{\alpha}\lambda_1|^{-1/2}$, this is $\mathrm{CF}^{3,2,>,>}_{16}$ with $\sigma = \sigma_\alpha \operatorname{sgn}\lambda_1$ and $u = -\tilde{\alpha}\tilde{\gamma}(2\tilde{\beta})^{-2} > -1/4$.

(1_2^{2b}) In the case $\breve{b}_1 = 0 \Leftrightarrow \lambda_1 = \tilde\alpha\tilde\gamma\tilde\eta^{-2}\lambda_2 \Leftrightarrow \tilde\tau(\lambda_1 + \lambda_2) = -|\tilde\beta|(\lambda_1 - \lambda_2)$, system (3.15) has the form $4\tilde\tau^2\tilde\eta^{-2}\lambda_2\begin{pmatrix}0 & 0 & \tilde\gamma s_2^2 & 0\\ 0 & -\tilde\alpha r_1^2 & 2\tilde\beta r_1 s_2 & 0\end{pmatrix}$. For $r_1 = \tilde\eta\tilde\gamma(4\tilde\beta\tilde\tau)^{-1}|\tilde\gamma\lambda_2|^{-1/2}$ and $s_2 = \tilde\eta(2\tilde\tau)^{-1}|\tilde\gamma\lambda_2|^{-1/2}$, this is $\mathrm{CF}_{a,16}^{3,2,>,>}$ with $\sigma = \sigma_\gamma \operatorname{sgn}\lambda_2$ and $u = -\tilde\alpha\tilde\gamma(2\tilde\beta)^{-2}$. Then, renumbering $(2.7)^1$ is applied.

(1_2^{2c}) In the case $\breve{b}_1, \breve{c}_2 \neq 0 \Leftrightarrow \tilde\tau(\lambda_1 + \lambda_2) \pm |\tilde\beta|(\lambda_1 - \lambda_2) \neq 0$, system (3.15) with $r_1 = |\tilde\eta|^{1/2}|\tilde\alpha(\lambda_1 - \lambda_2)|^{-1/2}\tilde\phi$ and $s_2 = |\tilde\eta|^{3/2}|\tilde\alpha(\lambda_1 - \lambda_2)|^{1/2}\tilde\phi\varphi_2^{-1}\sigma_0\sigma_\alpha\sigma_\beta$ is $\mathrm{NSF}_{23}^{4,2,>,>}$ with $\sigma = \sigma_0\sigma_\alpha$, $u = \varphi_1\varphi_2^{-1}$ $(u \neq 1)$, and $v = -\tilde\alpha\tilde\gamma\tilde\eta^2(\lambda_1 - \lambda_2)^2\varphi_2^{-2}$ $(v \neq u,\ 4v > -(1-u)^2)$.

If $v = (2u-1)/4$ or $v = u(2-u)/4$, then the obtained form $\mathrm{NSF}_{23}^{4,2,>,>}$ is not canonical, because it reduces to $\mathrm{CF}_{14,-1}^{4,2,>,>}$ by Statement 3.2_3.

If $v \neq (2u-1)/4,\ u(2-u)/4$, then $\mathrm{NSF}_{23}^{4,2,>,>} = \mathrm{CF}_{23}^{4,2,>,>}$.

(2) Suppose that $D = 0$ $(\lambda_1, \lambda_2 = \nu = (p_1 + q_2)/2 \neq 0)$ (see [3, Appendix 3.5.2, p. 128]).

(2_1) If $q_1 \neq 0$ or $q_1 = 0$ and $p_2 \neq 0$ $(q_2 = p_1 = \nu)$, then the change J_{2a}^2 or J_{2b}^2 reduces $(1.1^>)$ to system (1.6), and the latter is reduced by any change (1.2) satisfying condition (3.14) to system (1.8) of the form

$$2\tilde\tau\begin{pmatrix}0 & \varphi_3^+\tilde\eta^{-1}r_1s_2 & -\tilde\gamma^2\tilde\eta^{-2}\sigma_\beta s_2^2 & 0\\ 0 & \sigma_\beta r_1^2 & \varphi_3^-\tilde\eta^{-1}r_1s_2 & 0\end{pmatrix} \quad (\varphi_3^\pm = 2\tilde\tau\nu\sigma_\beta \pm \tilde\gamma). \tag{3.17}$$

(2_1^1) If $\tilde\gamma = 0$ $(\tilde\tau = |\tilde\beta|,\ \tilde\eta = 2\tilde\beta)$, then system (3.17) has the form $2\tilde\beta\begin{pmatrix}0 & \nu r_1s_2 & 0 & 0\\ 0 & r_1^2 & \nu r_1s_2 & 0\end{pmatrix}$. For $r_1 = |2\tilde\beta|^{-1/2}$ and $s_2 = \nu^{-1}r_1$, this is $\mathrm{CF}_{a,10}^{3,2,>,=}$ with $\sigma = \sigma_\beta$ and $u = 1$. Then, renumbering $(2.7)^1$ is applied.

(2_1^2) If $\tilde\gamma \neq 0$, then $\breve{b}_2\breve{c}_1 \neq 0$ and $\breve{b}_1^2 + \breve{c}_2^2 \neq 0$ in system (3.17).

(2_1^{2a}) Suppose that $\breve{b}_1 = 0 \Leftrightarrow \varphi_3^+ = 0$. Then system (3.17) takes the form $\begin{pmatrix}0 & 0 & -2\tilde\tau\tilde\gamma^2\tilde\eta^{-2}\sigma_\beta s_2^2 & 0\\ 0 & 2\tilde\tau\sigma_\beta r_1^2 & -4\tilde\tau\tilde\gamma\tilde\eta^{-1}r_1s_2 & 0\end{pmatrix}$. For $r_1 = -\tilde\phi/2$ and $s_2 = -\tilde\phi\tilde\eta\tilde\gamma^{-1}\sigma_\beta$, this is $\mathrm{CF}_{a,16}^{3,2,>,=}$ with $\sigma = -\sigma_\beta$ and $u = -1/4$; then, we apply $(2.7)^1$.

(2_1^{2b}) If $\breve{c}_2 = 0 \Leftrightarrow \varphi_3^- = 0$, then (3.17) has the form $\begin{pmatrix}0 & 4\tilde\tau\tilde\gamma\tilde\eta^{-1}r_1s_2 & -2\tilde\tau\tilde\gamma^2\tilde\eta^{-2}\sigma_\beta s_2^2 & 0\\ 0 & 2\tilde\tau\sigma_\beta r_1^2 & 0 & 0\end{pmatrix}$. For $r_1 = \tilde\phi$ and $s_2 = \tilde\phi\tilde\eta(2\tilde\gamma)^{-1}\sigma_\beta$, this is $\mathrm{CF}_{16}^{3,2,>,=}$ with $\sigma = \sigma_\beta$ and $u = -1/4$.

(2_1^{2c}) If $\breve{b}_1\breve{c}_2 \neq 0 \Leftrightarrow \varphi_3^\pm \neq 0$, then system (3.17) with $r_1 = \tilde\phi$ and $s_2 = \tilde\phi\tilde\eta(\varphi_3^-)^{-1}\sigma_\beta$ is $\mathrm{CF}_{23}^{4,2,>,=}$ with $\sigma = \sigma_\beta$, $u = \varphi_3^+(\varphi_3^-)^{-1}$, and $v = -\tilde\gamma^2(\varphi_3^-)^{-2}$ $(u \neq \pm 1,\ 4v = -(1-u)^2)$, because, according to Statement 3.2, $\mathrm{NSF}_{23}^{4,2,>,=}$ cannot be reduced to $\mathrm{NSF}_{8,-1}^{4,2,>,=}$ or $\mathrm{NSF}_{14,-1}^{4,2,>,=}$.

(2_2) Suppose that $q_1 = 0$ and $p_2 = 0$, i.e., the matrix H is diagonal in system $(1.1^>)$ itself, and its diagonal is (p_1, p_1). Then change (1.2) with $r_1 = \tilde\eta$, $s_1 = \tilde\phi^4\tilde\gamma(p_1\tilde\eta)^{-1}$, $r_2 = -\tilde\alpha$, and $s_2 = \tilde\phi^4p_1^{-1}$ reduces $(1.1^>)$ to $\mathrm{CF}_4^{2,2,>,=}$ with $\sigma = 1$ $(u = 1)$.

(3) Suppose that $D = (p_1 - q_2)^2 + 4p_2q_1 < 0$ $(p_2q_1 < 0)$. From system $(1.1^>)$ we obtain (1.7) (see [3, Appendix 3.5.3, p. 135]).

An arbitrary change (1.2) satisfying condition (3.14) reduces (1.7) to system (1.8) of the form

$$\frac{2\tilde\tau\sigma_\beta}{\tilde\eta^2}\begin{pmatrix}0 & \tilde\eta\varphi_4^+r_1s_2 & -(\tilde\gamma^2 + \tilde\eta^2)\mu s_2^2 & 0\\ 0 & (\tilde\alpha^2 + \tilde\eta^2)\mu r_1^2 & \tilde\eta\varphi_4^-r_1s_2 & 0\end{pmatrix} \quad (\varphi_4^\pm = 2\tilde\tau\nu\sigma_\beta + (\tilde\alpha + \tilde\gamma)\mu). \tag{3.18}$$

(3_1) If $\nu = 0$ ($\Leftrightarrow$ $p_1 + q_2 = 0$) and $\tilde\alpha + \tilde\gamma = 0$, then $\tilde\tau = (\tilde\alpha^2 + \tilde\beta^2)^{1/2}$, and (3.18) has the form $\begin{pmatrix} 0 & 0 & -4\tilde\tau^2\tilde\eta^{-1}\mu s_2^2 & 0 \\ 0 & 4\tilde\tau^2\tilde\eta^{-1}\mu r_1^2 & 0 & 0 \end{pmatrix}$. For $r_1, s_2 = \tilde\phi^2 |\tilde\eta|^{1/2} \mu^{-1/2}$, we obtain $CF_{8,-1}^{2,2,>,<}$ with $\sigma = \operatorname{sgn}\tilde\beta$.

(3_2) If $\nu^2 + (\tilde\alpha + \tilde\gamma)^2 \neq 0$, then $\breve b_1^2 + \breve c_2^2 \neq 0$.

(3_2^1) If $\breve b_1 = 0 \Leftrightarrow \varphi_4^+ = 0$ ($\nu(\tilde\alpha + \tilde\gamma) \neq 0$), then system (3.18) has the form

$$\frac{2\tilde\tau\mu\sigma_\beta}{\tilde\eta^2}\begin{pmatrix} 0 & 0 & -(\tilde\gamma^2 + \tilde\eta^2)s_2^2 & 0 \\ 0 & (\tilde\alpha^2 + \tilde\eta^2)r_1^2 & -2\tilde\eta(\tilde\alpha + \tilde\gamma)r_1 s_2 & 0 \end{pmatrix}.$$

For $r_1 = \tilde\phi(\tilde\gamma^2 + \tilde\eta^2)^{1/2}(\tilde\alpha + \tilde\gamma)^{-1}(4\mu)^{-1/2}$ and $s_2 = \tilde\phi\tilde\eta((\tilde\gamma^2 + \tilde\eta^2)\mu)^{-1/2}$, this is $CF_{a,16}^{3,2,>,<}$ with $\sigma = -\sigma_\beta$ and $u = -(\tilde\alpha^2 + \tilde\eta^2)(\tilde\gamma^2 + \tilde\eta^2)(2\tilde\eta(\tilde\alpha + \tilde\gamma))^{-2} < -1/4$. Then, renumbering $(2.7)^1$ is performed.

(3_2^2) If $\breve c_2 = 0 \Leftrightarrow \varphi_4^- = 0$ ($\nu(\tilde\alpha + \tilde\gamma) \neq 0$), then system (3.18) takes the form

$$\frac{2\tilde\tau\mu\sigma_\beta}{\tilde\eta^2}\begin{pmatrix} 0 & 2\tilde\eta(\tilde\alpha + \tilde\gamma)r_1 s_2 & -(\tilde\gamma^2 + \tilde\eta^2)s_2^2 & 0 \\ 0 & (\tilde\alpha^2 + \tilde\eta^2)r_1^2 & 0 & 0 \end{pmatrix}.$$

For $s_2 = \tilde\phi(\tilde\alpha^2 + \tilde\eta^2)^{1/2}(\tilde\alpha + \tilde\gamma)^{-1}(4\mu)^{-1/2}$ and $r_1 = \tilde\phi\tilde\eta((\tilde\alpha^2 + \tilde\eta^2)\mu)^{-1/2}$, this is $CF_{16}^{3,2,>,<}$ with $\sigma = \sigma_\beta$ and the same u as in (3_2^1).

(3_2^3) If $\breve b_1 \breve c_2 \neq 0 \Leftrightarrow \varphi_4^\pm \neq 0$, then system (3.18) with $r_1 = \tilde\phi\tilde\eta((\tilde\alpha^2 + \tilde\eta^2)\mu)^{-1/2}$ and $s_2 = \tilde\phi((\tilde\alpha^2 + \tilde\eta^2)\mu)^{-1/2}(\varphi_4^-)^{-1}$ is $CF_{23}^{4,2,>,<}$ with $\sigma = \sigma_\beta$, $u = \varphi_4^+(\varphi_4^-)^{-1}$, and $v = -\mu^2(\tilde\alpha^2 + \tilde\eta^2)(\tilde\gamma^2 + \tilde\eta^2)\tilde\eta^{-2}(\varphi_4^-)^{-2}$ ($4v < -(1-u)^2$), because, according to Statement 3.2, $NSF_{23}^{4,2,>,<}$ cannot be reduced to $NSF_{8,-1}^{4,2,>,<}$ and $NSF_{14,-1}^{4,2,>,<}$. □

Below, we give nonsingular linear changes which make it possible to distinguish minimal canonical sets introduced in Definition 1.11 of [2] for the CFs in List 3.1.

Statement 3.3. The only forms $CF^{m,2,>}$ in List 3.1 for which the values of the parameters of $cs^{m,2,>}$ can be bounded are as follows:

(1) in $CF_4^{2,2,>}$, normalization $(2.6)^1$ with $r_1, -s_2 = 1$ changes the sign of σ; for $\tilde u = u$, $|\tilde u| > 1$, the change with $r_1, s_2 = 0$, $s_1 = 1$, and $r_2 = \tilde u^{-1}$ yields $u = \tilde u^{-1}$;

(2) in $CF_{8,-1}^{2,2,>}$, renumbering $(2.7)^1$ changes the sign of σ, and in $CF_{14,-1}^{4,2,>,>}$ with $u = 1$, so does the change with $-r_1, r_2, s_2 = 3^{-1/2}$ and $s_1 = 2s_2$;

(3) in $CF_{8,-1}^{4,2,>,>}$ with $\tilde\sigma = \sigma$ and $\tilde u = u$, $|\tilde u| > 1$, the change with $r_1, s_2 = 0$ and $s_1, r_2 = |\tilde u|^{-1/2}$ yields $\sigma = -\tilde\sigma \operatorname{sign} \tilde u$ and $u = \tilde u^{-1}$;

(4) in $CF_{23}^{4,2,>,>}$ with $\tilde\sigma = \sigma$, $\tilde u = u$, $|\tilde u| > 1$, and $\tilde v = v$, the change with $r_1, s_2 = 0$, $s_1 = |\tilde v|^{3/2}(\tilde u\tilde v)^{-1}$, and $r_2 = |\tilde v|^{-1/2}$ yields $\sigma = \tilde\sigma \operatorname{sgn} \tilde v$, $u = \tilde u^{-1}$, and $v = \tilde u^{-2}\tilde v$.

Corollary 3.1. According to Definition 1.12 in [2],

$$acs_4^{2,2,>,>} = \{|u| > 1, \sigma = -1\}, \quad acs_4^{2,2,>,=} = \{\sigma = -1\}, \quad acs_{8,-1}^{2,2,>,<} = \{\sigma = -1\},$$

$$acs_{8,-1}^{4,2,>,>} = \{|u| > 1\}, \quad acs_{14,-1}^{4,2,>,>} = \{\sigma = -1 \text{ for } u = 1\}, \quad acs_{23}^{4,2,>,>} = \{|u| > 1\};$$

for the remaining canonical forms in List 3.1, $mcs^{m,2,>,*} = cs^{m,2,>,*}$.

ADDITION

In [4, 5], on the basis of real canonical linear transformations, Markeev classified the unperturbed autonomous Hamiltonians of the third and fourth orders and determined the canonical forms of such Hamiltonians (in our terminology). Thus, for Hamiltonian systems, all Hamiltonian normal forms of the second and third orders have been obtained. It is interesting to compare such forms with normal forms of the second order, which were first obtained by Sibirskii [6] and, later, by Basov and coauthors (see [12, 13] in the bibliography of [1]) on the basis of different principles, as well as with the normal forms of the third order obtained in the present cycle of papers. In the case of coincidence (coincidences do occur), it is interesting to compare the structure of the arising Hamiltonian and non-Hamiltonian generalized normal forms.

We also mention that, in [5], separate canonical Hamiltonians of the third order were used as unperturbed Hamiltonians for the purpose of a subsequent normalization of Hamiltonian perturbations of any finite order, after which a series of results on the stability or instability of an equilibrium position determined by conditions imposed on the corresponding terms of normal forms were obtained.

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