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MATHEMATICS

Two-Dimensional Homogeneous Cubic Systems: Classifications and Normal Forms – V

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Abstract—The present article is the fifth in a cycle of papers dedicated to two-dimensional homogeneous cubic systems. It considers a case when the homogeneous polynomial vector in the right-hand part of the system has a linear common factor. A set of such systems is divided into classes of linear equivalence, wherein the simplest system being a third-order normal form is distinguished based on properly introduced principles. Such a form is defined by the matrix of its right-hand part coefficients, which is called the canonical form (CF). Each CF has its own arrangement of non-zero elements, their specific normalization, and canonical set of permissible values for the unnormalized elements, which relates the CF to the selected class of equivalence. In addition to classification, each CF is provided with: (a) conditions on the coefficients of the initial system, (b) non-singular linear substitutions that reduce the right-hand part of the system under these conditions to the selected CF, (c) obtained values of CF's unnormalized elements.

Keywords: homogeneous cubic system, normal form, canonical form.

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1. INTRODUCTION

The present study is a direct continuation of papers [1, 2], preserving all the designations introduced earlier. Because of the large number of references to the formulas from [1], their numbers are labeled with a superscript "1". For example, system (2.1) from [1] is denoted as $(2.1)^1$. This work also includes the links to proofs performed in the Maple package and hosted in the database provided by https://github.com/Vladimir-Basov/DE or https://github.com/A-Cherm/DE.

This paper completes the classification of real systems $(2.1)^1$:

$$\dot{x}_1 = P_1(x_1, x_2), \quad \dot{x}_2 = P_2(x_1, x_2), \quad (P_i = a_i x_1^3 + b_i x_1^2 x_2 + c_i x_1 x_2^2 + d_i x_2^3 \not\equiv 0),$$

where polynomials P_1 and P_2 share a factor of non-zero degree l.

Set of systems $(2.1)^1$ was successfully divided into classes of linear equivalence while distinguishing a generatrix in each of them, which is the simplest system called the cubic normal form and identified with a matrix of coefficients from its right-hand part, i.e., the canonical form (CF).

The classification proposed is aimed at maximally simplifying the reduction of perturbed systems $(1.4)^1$ with various CFs in the unperturbed part to generalized normal forms. The definition of generalized normal forms, as well as the constructive method of the obtainment of their omnifarious structures, can be found in [1, Section 1.3]. Further, based on the reasoning from Section 1.4 in [2, Section 1] the structural and normalization principles were developed, which made it possible to optimally determine the CF.

Let us consider a classification of two-dimensional homogeneous cubic systems, underlied by other principles of the assignment of CFs.

Firstly, A. Cima and J. Llibre [3] classified homogeneous fourth-order polynomials in two variables with real coefficients or, for short, binary forms, finding their algebraic invariants relative to linear non-singular substitutions and distinguishing the generatrices, canonical binary forms (CBFs). For this purpose, they adapted the methods applied by G. Gurevich [4] for the classification of complex binary forms, getting ten CBFs.

Then, arbitrary system (2.1)[1] was associated with binary form $F(x_1, x_2) = x_1P_2(x_1, x_2) - x_2P_1(x_1, x_2)$ while distinguishing a three-parameter family of systems also comparable with binary form F.

It was proven that the linear non-singular substitution reducing F to any CBF transforms the initial system to a system comparable with the same CBF.

This resulted in the algebraic classification of systems (2.1)[1], allowing one to divide them into ten linearly non-equivalent classes with explicitly discharged generatrices, i.e., three-parameter families of systems, where each of them is related to its own CBF. The families of systems distinguished can be naturally called the CFs of the given classification.

These results enabled one to perform a complete topological classification of phase portraits in the case when polynomials P_1 and P_2 share no factor, which is true for nine CBFs and corresponds to $l = 0$ in the terms of the present cycle of studies. If $l = 1, 2, 3$, then CBF is identically zero.

Note that two-dimensional homogeneous quadratic systems related to real homogeneous third-order polynomials were classified in [5].

2. ASSIGNMENT OF CANONICAL FORMS AND THEIR PERMISSIBLE SETS AT l = 1

Let us distinguish the structural forms up to $SF_8^{5,1}$ inclusively from list 1.1 in [2], which refer to a case with $l = 1$ (there are 41 such forms in total) and normalize them in accordance with NP from [2, Section 1.2]. We are going to establish that normalized structural forms (*NSF*, see [2, Definition 1.6]) are canonical (*CF*, [2, Definition 1.10]). The notation explanation of a and κ subscripts may be seen in [2, Definition 1.3] and [2, Definition 1.7] respectively.

Statement 2.1. *Only* $NSF_6^{4,1} = \sigma\begin{pmatrix} u & 0 & 0 & u \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $NSF_{a,15}^{4,1} = \sigma\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & v & u & 0 \end{pmatrix}$, $NSF_{a,20}^{4,1} = \sigma\begin{pmatrix} v & 0 & 1 & 0 \\ 1 & 0 & u & 0 \end{pmatrix} (uv \neq 1)$, $NSF_{22}^{4,1} = \sigma\begin{pmatrix} u & 0 & 0 & u \\ 0 & 1 & 1 & 0 \end{pmatrix}$, $NSF_{37}^{4,1} = \sigma\begin{pmatrix} 0 & 0 & u & u \\ 1 & 1 & 0 & 0 \end{pmatrix}$, $NSF_1^{5,1} = \sigma\begin{pmatrix} u & v & v-u & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $NSF_2^{5,1} = \sigma\begin{pmatrix} u & v & 0 & u-v \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $NSF_5^{5,1} = \sigma\begin{pmatrix} u & 0 & v & u+v \\ 0 & 0 & 1 & 1 \end{pmatrix}$ *at all permissible values of the parameters can be reduced to any preceding structural form by linear substitutions* (2.2)[1] *in accordance with a SP from* [2, Section 1.1].

Proof. 1) $NSF_6^{4,1}$ by substitution with $s_1 = -s_2$, $r_2 = 0$ is reduced to $SF_5^{4,1}$;

2) $NSF_{a,15}^{4,1}$ by substitution with $r_1 = -2uv^{-1}r_2$, $s_1 = 0$ is reduced to $SF_{a,8}^{3,1}$;

3) $NSF_{a,20}^{4,1}$ $(v \neq u^{-1})$ by substitution with $s_1 = 0$, $r_2 = ur_1$ is reduced to $SF_{19}^{4,1}$;

4) $NSF_{22}^{4,1}$ by substitution with $s_1 = -s_2$, $r_2 = r_1$ is reduced to $SF_{a,20}^{4,1}$;

5) $NSF_{37}^{4,1}$ by substitution with $s_1 = -s_2$, $r_2 = r_1$ at $u = 1$ is reduced to $SF_{9,\kappa}^{3,1}$, at $u = -1$ is reduced to $SF_{a,14,\kappa}^{3,1}$, while at $u \neq \pm 1$ by substitution with $s_1 = -s_2$, $r_1 = ur_2$ is reduced to $SF_{a,27}^{4,1}$;

6) $NSF_1^{5,1}$, $NSF_2^{5,1}$, and $NSF_5^{5,1}$ by substitution with $s_1 = -s_2$, $r_2 = 0$ are reduced to $SF_5^{4,1}$.

As follows from the verification, other 33 $NSF^{m,1}$ are $CF^{m,1}$.

Note 2.1. Here and hereinafter, saying that "is reduced to any $SF^{m,1}$" means that either this form or one of its antecedents is obtained.

Let us write out the available $CF^{m,1}$ forms, their permissible sets (*ps*), and canonical sets (*cs*) from [2, Definitions 1.8 and 1.9], where $cs^{m,1}$ will be established in statements 3.1 and 3.2 below (*tps*, *tcs* mean that there are no limits on the parameters). Let us also give the decomposition of each form into line $(1, \beta)$ and matrix G, as was made in system (2.9)[1] $\dot{x} = (\alpha, \beta)xGq^{[2]}(x)$, as well as resultant $R_2 \neq 0$ (see [6]) of matrix G.

List 2.1. All $CF_i^{m,1}$ up to $CF_8^{5,1}$ inclusively with designation of coefficient β, matrix G, resultant R_2, $ps_i^{m,1}$ and $cs_i^{m,1}$ ($\sigma, \kappa = \pm 1$; $u, v, w \neq 0$, $\alpha = 1$, $R_2 \neq 0$).

I) 24 forms with $\beta = 0$ (d_1, $d_2 = 0$, G is the first three columns of the respective $CF_i^{m,1}$):

$$1)\ CF_2^{2,1} = \sigma\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = 1,\\ tps_2^{2,1};\end{matrix} \qquad CF_{a,5}^{3,1} = \sigma\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & u & 1 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = 1,\\ tps_{a,5}^{3,1};\end{matrix}$$

$$CF_{a,8}^{3,1} = \sigma\begin{pmatrix}1 & 0 & 0 & 0\\ 1 & 0 & u & 0\end{pmatrix}, \quad \begin{matrix}R_2 = u^2,\\ tps_{a,8}^{3,1};\end{matrix} \qquad 2)\ CF_3^{3,1} = \sigma\begin{pmatrix}1 & u & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = 1,\\ tps_3^{3,1};\end{matrix}$$

$$CF_{a,14,\kappa}^{3,1} = \sigma\begin{pmatrix}0 & 1 & 0 & 0\\ \kappa & 0 & u & 0\end{pmatrix}, \quad \begin{matrix}R_2 = \kappa u,\\ tps_{14,\kappa}^{3,1};\end{matrix} \qquad CF_7^{4,1} = \sigma\begin{pmatrix}u & v & 0 & 0\\ 0 & 1 & 1 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = u(u-v),\\ ps_7^{4,1} = \{v \neq u\};\end{matrix}$$

$$CF_{a,12}^{4,1} = \sigma\begin{pmatrix}1 & 1 & 0 & 0\\ v & 0 & u & 0\end{pmatrix}, \quad \begin{matrix}R_2 = u(u+v),\\ ps_{12}^{4,1} = \{v \neq -u\};\end{matrix} \qquad CF_{a,24}^{4,1} = \sigma\begin{pmatrix}0 & 1 & 0 & 0\\ v & 1 & u & 0\end{pmatrix}, \quad \begin{matrix}R_2 = uv,\\ tps_{24}^{4,1};\end{matrix}$$

$$3)\ CF_9^{2,1} = \sigma\begin{pmatrix}0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = 1,\\ tps_9^{2,1};\end{matrix} \qquad CF_6^{3,1} = \sigma\begin{pmatrix}u & 0 & 1 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = u^2,\\ tps_6^{3,1};\end{matrix}$$

$$CF_{11,\kappa}^{3,1} = \sigma\begin{pmatrix}u & 0 & \kappa & 0\\ 0 & 1 & 0 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = \kappa u,\\ tps_{11,\kappa}^{3,1};\end{matrix} \qquad CF_{17}^{3,1} = \sigma\begin{pmatrix}u & 0 & 1 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = 1,\\ tps_{17}^{3,1};\end{matrix}$$

$$CF_{a,19}^{3,1} = \sigma\begin{pmatrix}0 & 0 & 1 & 0\\ 1 & 0 & u & 0\end{pmatrix}, \quad \begin{matrix}R_2 = 1,\\ tps_{a,19}^{3,1};\end{matrix} \qquad CF_{21}^{3,1} = \sigma\begin{pmatrix}0 & u & 1 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = 1,\\ tps_{21}^{3,1};\end{matrix}$$

$$CF_{a,22}^{3,1} = \sigma\begin{pmatrix}0 & 0 & 1 & 0\\ 1 & u & 0 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = 1,\\ tps_{a,22}^{3,1};\end{matrix} \qquad CF_5^{4,1} = \sigma\begin{pmatrix}u & v & 1 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = u^2,\\ tps_5^{4,1};\end{matrix}$$

$$CF_{11}^{4,1} = \sigma\begin{pmatrix}u & 1 & v & 0\\ 0 & 1 & 0 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = uv,\\ tps_{11}^{4,1};\end{matrix} \qquad CF_{a,14}^{4,1} = \sigma\begin{pmatrix}v & 0 & 1 & 0\\ 0 & u & 1 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = v(u^2+v),\\ ps_{14}^{4,1} = \{v \neq -u^2\};\end{matrix}$$

$$CF_{19}^{4,1} = \sigma\begin{pmatrix}u & v & 1 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = 1,\\ tps_{19}^{4,1};\end{matrix} \qquad CF_{a,27}^{4,1} = \sigma\begin{pmatrix}v & 0 & 1 & 0\\ 1 & u & 0 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = u^2v+1,\\ ps_{27}^{4,1} = \{v \neq -u^{-2}\};\end{matrix}$$

$$CF_{a,29}^{4,1} = \sigma\begin{pmatrix}0 & 1 & 1 & 0\\ v & 0 & u & 0\end{pmatrix}, \quad \begin{matrix}R_2 = v(u+v),\\ ps_{29}^{4,1} = \{v \neq -u\};\end{matrix} \qquad CF_{a,30}^{4,1} = \sigma\begin{pmatrix}0 & 0 & 1 & 0\\ 1 & v & u & 0\end{pmatrix}, \quad \begin{matrix}R_2 = 1,\\ tps_{30}^{4,1};\end{matrix}$$

$$CF_{a,33}^{4,1} = \sigma\begin{pmatrix}0 & 1 & 1 & 0\\ v & u & 0 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = v(v-u),\\ ps_{33}^{4,1} = \{v \neq u\};\end{matrix} \qquad CF_8^{5,1} = \sigma\begin{pmatrix}u & v & w & 0\\ 0 & 1 & 1 & 0\end{pmatrix}, \quad \begin{matrix}R_2 = u(u-v+w),\\ ps_8^{5,1} = \{w \neq v-u\};\end{matrix}$$

II) 9 forms with $\beta = 1$ and its own G ($R_2 = u^2$ and $ps = tps$ in the first six forms):

$$CF_1^{4,1} = \sigma\begin{pmatrix}u & u & 0 & 0\\ 0 & 0 & 1 & 1\end{pmatrix}, \quad \sigma\begin{pmatrix}u & 0 & 0\\ 0 & 0 & 1\end{pmatrix}; \qquad CF_3^{4,1} = \sigma\begin{pmatrix}u & 0 & -u & 0\\ 0 & 0 & 1 & 1\end{pmatrix}, \quad \sigma\begin{pmatrix}u & -u & 0\\ 0 & 0 & 1\end{pmatrix};$$

$$CF_{13}^{4,1} = \sigma\begin{pmatrix}u & 0 & 0 & u\\ 0 & 1 & 0 & -1\end{pmatrix}, \quad \sigma\begin{pmatrix}u & -u & u\\ 0 & 1 & -1\end{pmatrix}; \qquad CF_{28}^{4,1} = \sigma\begin{pmatrix}0 & u & 0 & -u\\ 1 & 0 & 0 & 1\end{pmatrix}, \quad \sigma\begin{pmatrix}0 & u & -u\\ 1 & -1 & 1\end{pmatrix};$$

$$CF_{32}^{4,1} = \sigma\begin{pmatrix}0 & 0 & u & u\\ 1 & 0 & 0 & 1\end{pmatrix}, \quad \sigma\begin{pmatrix}0 & 0 & u\\ 1 & -1 & 1\end{pmatrix}; \qquad CF_{36}^{4,1} = \sigma\begin{pmatrix}0 & 0 & u & u\\ 1 & 0 & -1 & 0\end{pmatrix}, \quad \sigma\begin{pmatrix}0 & 0 & u\\ 1 & -1 & 0\end{pmatrix};$$

$$CF_3^{5,1} = \sigma\begin{pmatrix}u & v & v-u & 0\\ 0 & 1 & 0 & -1\end{pmatrix}, \quad \sigma\begin{pmatrix}u & v-u & 0\\ 0 & 1 & -1\end{pmatrix}, \quad \begin{matrix}R_2 = uv,\\ ps_3^{5,1} = \{v \neq u\};\end{matrix}$$

$$CF_6^{5,1} = \sigma\begin{pmatrix} u & v & 0 & u-v \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad \sigma\begin{pmatrix} u & v-u & u-v \\ 0 & 1 & -1 \end{pmatrix}, \quad \begin{matrix} R_2 = u^2, \\ ps_6^{5,1} = \{v \neq u\}; \end{matrix}$$

$$CF_7^{5,1} = \sigma\begin{pmatrix} u & v & v-u & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma\begin{pmatrix} u & v-u & 0 \\ 1 & -1 & 1 \end{pmatrix}, \quad \begin{matrix} R_2 = u^2 - uv + v^2, \\ ps_7^{5,1} = \{v \neq u\}; \end{matrix}$$

$$1)\ tcs_2^{2,1}; \quad cs_5^{3,1} = \{u \neq 2\}, \quad cs_8^{3,1} = \{u > 1/4\};$$

$$2)\ tcs_3^{3,1}, \quad cs_{14,\kappa}^{3,1} = \{(\kappa, u) \neq (1, 1/2)\}; \quad cs_7^{4,1} = \{v \neq u, 2 - u^{-1}, 2u(u+1)^{-1}\},$$

$$cs_{12}^{4,1} = \{u \neq -v, 1/2; 4v(u-1) > 1\}, \quad cs_{24}^{4,1} = \{u = 1/2, v < -1/2\};$$

$$3)\ tcs_9^{2,1}; tcs_6^{3,1}, tcs_{11,\kappa}^{3,1}, tcs_{17}^{3,1}, tcs_{19}^{3,1}, \quad cs_{21}^{3,1} = \{u \neq 2\}, \quad tcs_{22}^{3,1};$$

$$cs_1^{4,1} = \{u \neq \pm 1\}, \quad cs_3^{4,1} = \{u \neq -1/2, -2\};$$

$$cs_5^{4,1} = \{u \neq v(v-2)/4; (u,v) \neq (1,-2), (-1/9, 1)\}; \quad cs_{11}^{4,1} = \{v \neq u(2u-1)^{-2}\},$$

$$cs_{13}^{4,1} = \{u \neq -1/3, 2/3\}, \quad cs_{14}^{4,1} = \{v \neq u, -u^2; v \neq u/2 \text{ at } u > -1/2\},$$

$$cs_{27}^{4,1} = \{v \neq -u^{-2}, (u^{3/2} \pm 2^{3/2})u^{-1/2}/2; (u,v) \neq 4^{-2/3} \cdot (3,1)\},$$

$$cs_{19}^{4,1} = \{u \neq v^2/4, (v^3 - 8)(4v)^{-1}\}, \quad cs_{28}^{4,1} = \{u \neq -3, -3/4, 3/2, 6, \vartheta_1\},$$

$$cs_{29}^{4,1} = \{u \neq -1/2; v \neq -u, u^2, (1-2u)/8, (1-2u)^2/8; (u,v) \neq (\vartheta_3, \vartheta_4)\},$$

$$cs_{30}^{4,1} = \{u \neq -v^{-1}, (v^3 - 8)(4v)^{-1}; (u,v) \neq (2,3), (3,-3)\},$$

$$cs_{32}^{4,1} = \{u \neq -3, -3/4, 3/8, 6\},$$

$$cs_{33}^{4,1} = \{u \neq 1; v \neq u, (4u+1)/2, (6u + 1 \pm (2u+1)(8u+1)^{1/2})/16\},$$

$$cs_{36}^{4,1} = \{u \neq -2, -1/8, 1 \pm 3\sqrt{2}/4, 1/4, 4\}; \quad cs_3^{5,1} = \{v \neq u, 3.2_1\},$$

$$cs_6^{5,1} = \{v \neq u, 3.2_2\}, \quad cs_7^{5,1} = \{v \neq u, 3.2_3\}, \quad cs_8^{5,1} = \{w \neq v - u, 3.2_4\}.$$

Here $\{..., 3.2_i\}$ means that the values of parameters do not meet the conditions from item i ($i = 1, 2, 3, 4$) of Statement 3.2 below. The values of ϑ_j are presented in the following Collection 2.1.

Since List 2.1 is composed of $CF^{m,1}$ with only $\beta = 0$ or $\beta = \alpha$, let us clarify the circumstances at which the forms with these β can be transformed into each other.

Statement 2.2. *Let system* $(2.9)^1$ *with* $P_0^1 = \alpha x_1 + \beta x_2$ *by linear non-singular substitution* $(2.2)^1$ *be reduced to system* $(2.11)^1$ $\dot{y} = \tilde{P}_0^1(y)\tilde{G}q^{[2]}(y)$ *with* $\tilde{P}_0^1 = \tilde{\alpha}x_1 + \tilde{\beta}x_2$, *then* 1) *at* $\alpha = 1$: $\tilde{\beta} = 0 \Leftrightarrow s_1 = -\beta s_2$, 2) *at* $\beta = 0$: $\tilde{\beta} = 0 \Leftrightarrow s_1 = 0$, 3) *at* $\beta = 0$: $\tilde{\alpha} = \tilde{\beta} \Leftrightarrow r_1 = s_1 \neq 0$, 4) *at* $\alpha = \beta$: $\tilde{\beta} = 0 \Leftrightarrow s_2 = -s_1 \neq 0$.

Proof. In accordance with Theorem 2.1 from [1], $\tilde{\alpha} = \alpha r_1 + \beta r_2$, $\tilde{\beta} = \alpha s_1 + \beta s_2$.

Collection 2.1. The numerical constants used below:

$$\vartheta_1 = \rho + 20\rho^{-1} + 5, \quad \vartheta_2 = ((\sqrt{29} + 27)\rho^2 - (10\sqrt{29} - 130)\rho + 1000)/600, \quad \rho = (4\sqrt{29} + 92)^{1/3};$$

$$\vartheta_3 = ((3\sqrt{29} - 17)\rho^2 + (4\sqrt{29} - 24)\rho - 16)/24, \quad \vartheta_4 = ((72 - 13\sqrt{29})\rho^2 - (9\sqrt{29} - 59)\rho + 72)/36,$$

$$\vartheta_5 = (\rho + 4\rho^{-1})/6, \quad \vartheta_6 = 2(2\rho^2 + 9\rho + 8)/(\rho^2 - 18\rho + 4), \quad \rho = (20\sqrt{29} + 108)^{1/3};$$

$$\vartheta_7 = (8\rho^2 + (3\sqrt{57} - 1)\rho + 68)/12, \quad \vartheta_8 = ((\sqrt{57} + 85)\rho^2 + 32(\sqrt{57} - 1)\rho + 640)/96,$$

$$\vartheta_9 = (8\rho^{-1} - \rho - 1)/3, \quad \vartheta_{10} = ((11 - \sqrt{57})\rho^2 + 4(\sqrt{57} + 5)\rho + 32)/96, \quad \rho = (3\sqrt{57} + 1)^{1/3};$$

$$\vartheta_{11} = ((\sqrt{17} - 9)\rho^2 - 4(\sqrt{17} + 1)\rho - 40)/8, \quad \vartheta_{12} = -\rho + 4\rho^{-1}, \quad \rho = (2\sqrt{17} + 2)^{1/3};$$

$$\vartheta_{13} = (\rho^2 - (\sqrt{77} - 9)\rho - 16)/4, \quad \vartheta_{14} = -3((\sqrt{77} - 9)\rho^2 - 4\rho + 24)/8,$$

$$\vartheta_{15} = \rho/6 + 2(3\rho)^{-1}, \quad \vartheta_{16} = ((3\sqrt{77} - 25)\rho^2 - (2\sqrt{77} - 6)\rho - 8)/24, \quad \rho = (4\sqrt{77} + 36)^{1/3}.$$

3. ASSIGNMENT OF CANONICAL AND MINIMAL SETS FOR $CF^{m,1}$

Statement 3.1. *Only the following forms with $m \leq 4$ from List* 2.1 *at the given values of parameters can be reduced to the antecedent structural forms*:

1) $NSF_5^{3,1}$ *at* $u = 2$ *by substitution with* $r_1 = -r_2$, $s_2 = 0$*—to* $SF_2^{2,1}$;

2) $NSF_8^{3,1}$ *at* $u \leq 1/4$ *by substitution with* $s_2 = (1 + (1 - 4u)^{1/2})s_1/2$, $r_2 = 0$*—to* $SF_5^{3,1}$;

3) $NSF_{14,\kappa}^{3,1}$ *at* $\kappa = 1$, $u = 1/2$ *by substitution with* $r_1 = 2^{1/2}r_2$, $s_2 = 0$*—to* $SF_3^{3,1}$;

4) $NSF_{21}^{3,1}$ *at* $u = 2$ *by substitution with* $s_1 = 0$, $r_2 = -r_1$*—to* $SF_6^{3,1}$;

5) $NSF_1^{4,1}$: *a) at* $u = -1$ *by substitution with* $r_2 = 0$, $s_2 = -s_1$*—to* $SF_3^{3,1}$;

b) at $u = 1$ *by substitution with* $r_1 = r_2$, $s_2 = -s_1$*—to* $SF_{11}^{3,1}$;

6) $NSF_3^{4,1}$: *a) at* $u = -1/2$ *by substitution with* $r_2 = 0$, $s_2 = -s_1$*—to* $SF_3^{3,1}$;

b) at $u = -2$ *by substitution with* $r_2 = 0$, $s_2 = 2s_1$*—to* $SF_1^{4,1}$;

7) $NSF_5^{4,1}$: *a) at* $u = 1$, $v = -2$ *by substitution with* $r_1 = 0$, $s_2 = s_1$*—to* $SF_{22}^{3,1}$;

b) at $u = v(v - 2)/4$ *by substitution with* $r_2 = 0$, $s_2 = (1 - v/2)s_1$*—to* $SF_1^{4,1}$;

c) at $u = v(2v - 3)/9$ *by substitution with* $r_2 = 0$, $s_2 = (3 - 2v)s_1/3$*—to* $SF_3^{4,1}$;

8) $NSF_7^{4,1}$ ($v \neq u$): *a) at* $v = 2 - u^{-1}$ *by substitution with* $r_1 = -u^{-1}r_2$, $s_1 = 0$*—to* $SF_3^{3,1}$;

b) at $v = 2u(u + 1)^{-1}$ *by substitution with* $r_1 = 0$, $s_1 = 2(u + 1)^{-1}s_2$*—to* $SF_{14,\kappa}^{3,1}$;

9) $NSF_{11}^{4,1}$ *at* $v = u(2u - 1)^{-2}$ *by substitution with* $s_1 = 0$, $r_2 = (1 - 2u)r_1$*—to* $SF_5^{4,1}$;

10) $NSF_{12}^{4,1}$ ($v \neq -u$): *a) at* $u = 1/2$ *by substitution with* $s_1 = -s_2$, $r_2 = 0$*—to* $SF_{14,\kappa}^{3,1}$;

b) at $4v(u - 1) \leq 1$ *by substitution with* $r_2 = (1 + (1 - 4v(u - 1))^{1/2})(2v)^{-1}r_1$, $s_2 = 0$*—to* $SF_7^{4,1}$;

11) $NSF_{13}^{4,1}$: *a) at* $u = 2/3$ *by substitution with* $r_1 = 2r_2$, $s_2 = -s_1$*—to* $SF_3^{3,1}$;

b) at $u = -1/3$ *by substitution with* $r_1 = r_2/2$, $s_2 = -s_1$*—to* $SF_{11}^{4,1}$;

12) $NSF_{14}^{4,1}$ ($v \neq -u^2$): *a) at* $v = u/2$, $u > -1/2$ *by substitution with* $r_1 = (1 - (2u + 1)^{1/2})r_2/2$, $s_1 = (1 + (2u + 1)^{1/2})s_2/2$*—to* $SF_1^{4,1}$;

b) at $v = u$ *by substitution with* $r_2 = r_1$, $s_2 = 0$*—to* $SF_{11}^{4,1}$;

c) at $u = -1/4$, $v = -1/12$ *by substitution with* $r_1 = 0$, $s_2 = 2s_1$*—to* $SF_{13}^{4,1}$;

13) $NSF_{19}^{4,1}$: *a) at* $u = v^2/4$ *by substitution with* $r_1 = 0$, $s_2 = -vs_1/2$*—to* $SF_{19}^{3,1}$;

b) at $u = (v^3 - 8)(4v)^{-1}$ *by substitution with* $s_1 = 0$, $r_2 = -vr_1/2$*—to* $SF_6^{3,1}$;

14) $NSF_{24}^{4,1}$: *a) at* $u \neq 1/2$ *by substitution with* $r_2 = 0$, $s_2 = (1 - 2u)s_1$*—to* $SF_{12}^{4,1}$;

b) at $u = 1/2$, $v \geq -1/2$ *by substitution with* $r_1 = (1 + (2v + 1)^{1/2})r_2$, $s_2 = 0$*—to* $SF_7^{4,1}$;

15) $NSF_{27}^{4,1}$ ($v \neq -u^{-2}$): *a) at* $v = u/2 \pm (u/2)^{-1/2}$ *by substitution with* $r_1 = \pm(u/2)^{1/2}r_2$, $s_2 = 0$*—to* $SF_5^{4,1}$;

b) at $u = 3/4^{2/3}$, $v = 4^{-2/3}$ *by substitution with* $r_1 = 2^{1/3}r_2$, $s_1 = -3 \times 2^{-2/3}s_2$*—to* $SF_{13}^{4,1}$;

16) $NSF_{28}^{4,1}$: *a) at* $u = -3$ *by substitution with* $s_1 = 2s_2$, $r_2 = -r_1$*—to* $SF_{22}^{3,1}$;

b) at $u = 6$ *by substitution with* $r_1 = 2r_2$, $s_2 = -s_1$*—to* $SF_5^{4,1}$;

c) at $u = -3/4$ *by substitution with* $r_1 = r_2/2$, $s_2 = -s_1$*—to* $SF_{11}^{4,1}$;

d) at $u = 3/2$ *by substitution with* $s_1 = 3s_2/2$, $r_2 = -r_1$*—to* $SF_{12}^{4,1}$;

e) at $u = \vartheta_1$ *by substitution with* $s_1 = \vartheta_2 s_2$, $r_2 = -r_1$*—to* $SF_{14}^{4,1}$;

17) $NSF_{29}^{4,1}$ $(v \neq -u)$: *a) at* $v = u^2$ *by substitution with* $r_1 = ur_2$, $s_2 = 0$*—to* $SF_{11}^{4,1}$;

b) at $v = (1 - 2u)^2/8$ *by substitution with* $r_1 = (u - 1/2)r_2$, $s_2 = 0$*—to* $SF_5^{4,1}$;

c) at $v = (1 - 2u)/8$ *by substitution with* $r_2 = 0$, $s_2 = -2s_1$*—to* $SF_{14}^{4,1}$;

d) at $u = \vartheta_3$, $v = \vartheta_4$ *by substitution with* $r_1 = \vartheta_5 r_2$, $s_1 = \vartheta_6 s_2$*—to* $SF_{13}^{4,1}$;

e) at $u = -1/2$ *by substitution with* $s_1 = -s_2/2$, $r_2 = 0$*—to* $SF_{27}^{4,1}$;

18) $NSF_{30}^{4,1}$: *a) at* $u = -v^{-1}$ *by substitution with* $r_1 = -v^{-1}r_2$, $s_2 = 0$*—to* $SF_{11}^{4,1}$;

b) at $u = (v^3 - 8)(4v)^{-1}$ *by substitution with* $r_2 = -vr_1/2$, $s_2 = 0$*—to* $SF_5^{4,1}$;

c) at $u = 3$, $v = -3$ *by substitution with* $r_1 = r_2$, $s_1 = 0$*—to* $SF_{13}^{4,1}$;

d) at $u = 2$, $v = 3$ *by substitution with* $r_1 = -r_2$, $s_1 = 0$*—to* $SF_{28}^{4,1}$;

19) $NSF_{32}^{4,1}$: *a) at* $u = -3$ *by substitution with* $s_1 = 2s_1$, $r_2 = -r_1$*—to* $SF_{14}^{3,1}$;

b) at $u = 3/8$ *by substitution with* $r_2 = 2r_1$, $s_2 = -s_1$*—to* $SF_5^{4,1}$;

c) at $u = 6$ *by substitution with* $r_1 = 2r_2$, $s_2 = -s_1$*—to* $SF_{11}^{4,1}$;

d) at $u = -3/4$ *by substitution with* $r_2 = -r_1$, $s_2 = 2s_1$*—to* $SF_{30}^{4,1}$;

20) $NSF_{33}^{4,1}$ $(v \neq u)$: *a) at* $u = 1$ *by substitution with* $r_2 = 0$, $s_2 = -s_1$*—to* $SF_{29}^{4,1}$;

b) at $v = (4u + 1)/8$ *by substitution with* $r_2 = 0$, $s_2 = -2s_1$*—to* $SF_{14}^{4,1}$;

c) at $v = (6u + 1 \pm (2u + 1)(8u + 1)^{1/2})/16$ *by substitution with* $r_1 = -(1 \pm (8u + 1)^{1/2})r_2/4$, $s_2 = 0$*—to* $SF_5^{4,1}$;

21) $NSF_{36}^{4,1}$: *a) at* $u = -1/8$ *by substitution with* $r_2 = 2r_1$, $s_2 = -s_1$*—to* $SF_5^{4,1}$;

b) at $u = 4$ *by substitution with* $r_1 = 2r_2$, $s_2 = -s_1$*—to* $SF_{11}^{4,1}$;

c) at $u = -2$ *by substitution with* $s_1 = 4s_2/3$, $r_2 = -r_1$*—to* $SF_{12}^{4,1}$;

d) at $u = 1 \pm 3\sqrt{2}/4$ *by substitution with* $s_1 = (1 \pm 1/\sqrt{2})s_2$, $r_2 = -r_1$*—to* $SF_{14}^{4,1}$;

e) at $u = 1/4$ *by substitution with* $r_2 = -r_1$, $s_2 = 2s_1$*—to* $SF_{30}^{4,1}$.

The proof can be found in the file statement1.mw in the database (see Introduction).

Here and hereinafter: 1) "...$\zeta = [\varsigma_1 \vee \upsilon_1]$...$\eta = [\varsigma_2 \vee \upsilon_2]$..." means that either $\zeta = \varsigma_1$, $\eta = \varsigma_2$, or $\zeta = \upsilon_1$, $\eta = \upsilon_2$; 2) "θ_*: $P(\theta)$" means that θ_* is any real zero of polynomial P; 3) when reducing $NSF_i^{m,1}$ from List 2.1 to antecedent forms, substitutions are chosen with taking Statement 2.2 into consideration.

Statement 3.2. *Only at the given values of parameters*, $NSF_i^{5,1}$ *from List* 2.1 *are reduced to antecedent structural forms in accordance with one of SPs*:

1) $NSF_3^{5,1}$ $(u \neq v)$: *a) at* $v = [u - 3 \vee 3u - 1 \vee u + 1, u \neq 3]$ *by substitution with* $r_2 = -r_1$, $[s_1 = 0 \vee s_2 = 0 \vee s_2 = (1 - u)s_1/2]$*—to* $SF_{14}^{4,1}$;

b) at $v = (u - 1)^2u^{-1}$, $u \neq -1$ *by substitution with* $s_1 = -s_2$, $r_2 = ur_1$*—to* $SF_5^{4,1}$;

c) at $v = 2(u - 1)$ *by substitution with* $r_1 = 0$, $s_2 = -s_1$*—to* $SF_7^{4,1}$;

d) $[u = -1, v \neq -4 \vee v = 2u]$ *by substitution with* $r_1 = [(v/2 + 1)r_2 \vee 0]$, $s_2 = -s_1$—*to* $SF_{11}^{4,1}$;

e) *at* $u = \vartheta_7$, $v = \vartheta_8$ *by substitution with* $r_1 = \vartheta_9 r_2$, $s_1 = \vartheta_{10} s_2$—*to* $SF_{13}^{4,1}$;

f) *at* $v = 4u$, $u \neq -1$ *by substitution with* $r_1 = ur_2$, $s_2 = -s_1$—*to* $SF_{19}^{4,1}$;

g) *at* $v = 3(u + 1)$, $u \neq -5$ *by substitution with* $s_1 = (u + 3)s_2/2$, $r_2 = -r_1$—*to* $SF_{27}^{4,1}$;

h) *at* $v = (2u^2 + 1 \pm (2u + 1)(5 - 4u)^{1/2})(2u + 2)^{-1}$, $(u, v) \neq (-5,-12)$ *by substitution with* $r_2 = -r_1$, $s_2 = (3 \pm (5 - 4u)^{1/2})s_1/2$—*to* $SF_{29}^{4,1}$;

i) $v = u - 1 \pm 2\sqrt{-u}$, $u \neq -1$ *by substitution with* $r_2 = -r_1$, $s_2 = \pm\sqrt{-u}s_1$—*to* $SF_{30}^{4,1}$;

j) *at* $u = -(352\theta_*^5 + 396\theta_*^4 + 839\theta_*^3 + 1005\theta_*^2 - 1297\theta_* - 105)/46$, $v = -(328\theta_*^5 + 438\theta_*^4 + 844\theta_*^3 + 1098\theta_*^2 - 1046\theta_* - 366)/23$ *by substitution with* $r_1 = \theta_* r_2$, $s_2 = -(4\theta_*^5 + 39\theta_*^4 + 49\theta_*^3 + 111\theta_*^2 + 31\theta_* - 60)s_1/138$—*to* $SF_{28}^{4,1}$, θ_*: $4\theta^6 + 7\theta^5 + 13\theta^4 + 18\theta^3 - 6\theta^2 - 9\theta - 3$;

k) *at* $2u = \theta_*^2 - 2\theta_* + 3$, $2v = -3\theta_*^3 + 6\theta_*^2 - 11\theta_*$ *by substitution with* $r_1 = \theta_* r_2$, $s_1 = -(\theta_*^3 - \theta_*^2 + 3\theta_* + 3)s_2/2$—*to* $SF_{32}^{4,1}$, θ_*: $\theta^4 - \theta^3 + 2\theta^2 + 3\theta + 3$;

l) $v = 2(u + 1)^2(u + 2)^{-1}$, $u \neq -3$ *by substitution with* $r_2 = -r_1$, $s_2 = (u + 2)s_1$—*to* $SF_{33}^{4,1}$;

m) *at* $6u = -2\theta_*^3 - \theta_*^2 + 4\theta_* - 15$, $3v = -4\theta_*^3 - 3\theta_*^2 + 8\theta_* - 21$ *by substitution with* $r_1 = \theta_* r_2$, $s_1 = -(2\theta_*^3 + 3\theta_*^2 + 9)s_2/6$—*to* $SF_{36}^{4,1}$, θ_*: $2\theta^4 + 3\theta^3 - 3\theta^2 + 9\theta + 9$;

2) $NSF_6^{5,1}$ $(u \neq v)$: *a*) *at* $v = 2 - 3u$ *by substitution with* $r_1 = 2r_2$, $s_2 = -s_1$—*to* $SF_5^{4,1}$;

b) *at* $v = (3u - 2)/2$ *by substitution with* $r_2 = 0$, $s_2 = -s_1$—*to* $SF_7^{4,1}$;

c) *at* $v = (3u + 1)/2$ *by substitution with* $r_2 = 2r_1$, $s_2 = -s_1$—*to* $SF_{11}^{4,1}$;

d) *at* $v = [3u - 1 \vee 1 - u \pm 2(u^2 - u + 1)^{1/2}$, $(u, v) \neq (8/3, 3)]$ *by substitution with* $r_2 = -r_1$, $[s_2 = 0 \vee s_1 = (u - 2 \mp (u^2 - u + 1)^{1/2})(u - 1)^{-1}s_2]$—*to* $SF_{14}^{4,1}$;

e) *at* $v = 3u + 3$, $u \neq -8/3$ *by substitution with* $s_1 = 3(u + 2)s_2/2$, $r_2 = -r_1$—*to* $SF_{27}^{4,1}$;

f) *at* $v = (-u^2 - 2u \pm (2u + 1)(u^2 + u + 1)^{1/2})(u + 1)^{-1}$, $(u, v) \neq (-8/3,-5)$ *by substitution with* $r_2 = -r_1$, $s_2 = (u + 2 \pm (u^2 + u + 1)^{1/2})s_1/3$—*to* $SF_{29}^{4,1}$;

g) *at* $v = [u - 1 \vee -3u - 1]$ *by substitution with* $s_1 = [0 \vee 2s_2]$, $r_2 = -r_1$—*to* $SF_{30}^{4,1}$;

h) *at* $v = (3u^2 + 4u + 2)(2u + 2)^{-1}$, $u \neq -4/3$ *by substitution with* $s_1 = (u + 2)(2u + 2)^{-1}s_2$, $r_2 = -r_1$—*to* $SF_{33}^{4,1}$;

i) *at* $v = [(-1 \mp \sqrt{3})(3u - 1) \vee 1 - u \pm (4u^2 - 3u + 3)^{1/2}$, $(u, v) \neq ((14 \pm 4\sqrt{10})/9, (4 \pm 2\sqrt{10})/3) \vee (2\theta_*^3 - 4\theta_*^2 + 4\theta_* + 1)((\theta_* - 2)(2\theta_* - 1)\theta_*)^{-1}$, $\theta_* \neq -1]$ *by substitution with* $r_1 = [(1 \pm \sqrt{3})r_2/2 \vee -r_2 \vee \theta_* r_2]$, $s_2 = \langle 0 \vee -[4u^2 - 9u + 1 \pm 2u(4u^2 - 3u + 3)^{1/2})(15u - 3)^{-1}s_1 \vee (2\theta_*^2 - 2\theta_* - 1)(3\theta_*)^{-1}s_1]$— *to* $SF_3^{5,1}$, θ_*: $2(u - 1)\theta^3 - (5u - 7)\theta^2 + 2(u - 2)\theta - 1$;

j) *at* $u = 35/3$, $v = 12$ *by substitution with* $s_1 = 2s_2$, $r_2 = -4r_1$—*to* $SF_{13}^{4,1}$;

k) *at* $u = -35/3$, $v = -41/4$ *by substitution with* $s_1 = 2s_2$, $r_2 = -4r_1$—*to* $SF_{28}^{4,1}$;

l) *at* $u = -7/12$, $v = 3/2$ *by substitution with* $r_1 = 2r_2$, $s_2 = -4s_1$—*to* $SF_{32}^{4,1}$;

m) *at* $u = -5/9$, $v = 17/12$ *by substitution with* $r_1 = 2r_2$, $s_2 = -4s_1$—*to* $SF_{36}^{4,1}$;

3) $NSF_7^{5,1}$ $(u \neq v)$: *a*) *at* $v = 2u + 3$ *by substitution with* $r_1 = 0$, $s_2 = -s_1$—*to* $SF_7^{4,1}$;

b) *at* $u = (v - 1)(v - 3)(v - 2)^{-1}$ *by substitution with* $r_2 = (2 - v)r_1$, $s_2 = -s_1$—*to* $SF_5^{4,1}$;

c) at $v = [2u \vee 3 - u]$ *by substitution with* $r_1 = [0 \vee (u-1)r_2]$, $s_2 = -s_1$*—to* $SF_{11}^{4,1}$;

d) at $u = \vartheta_{11}$, $v = \vartheta_{11} - 3$ *by substitution with* $r_1 = \vartheta_{12}r_2$, $s_1 = -(\vartheta_{11} + 3)s_2/6$*—to* $SF_{13}^{4,1}$;

e) at $v = [u + 3 \vee 2u + 2 \pm (u^2 + 6u + 1)^{1/2}, (u, v) \neq (-6,-9)]$ *by substitution with* $r_2 = -r_1$, $s_1 = [0 \vee (u + 3 \pm (u^2 + 6u + 1)^{1/2})s_2/2]$*—to* $SF_{14}^{4,1}$;

f) at $v = 3(u-1)$, $u \neq 6$ *by substitution with* $s_1 = (3-u)s_2/3$, $r_2 = -r_1$*—to* $SF_{27}^{4,1}$;

g) at $u = -18\theta_*^4 - 24\theta_*^3 + 25\theta_*^2 - 16\theta_* + 4$, $v = -(261\theta_*^4 + 456\theta_*^3 - 187\theta_*^2 + 148\theta_* + 23)/5$ *by substitution with* $r_1 = \theta_* r_2$, $s_2 = (9\theta_*^4 + 39\theta_*^3 + 37\theta_*^2 + 2\theta_* + 7)s_1/15$*— to* $SF_{28}^{4,1}$, θ_*: $9\theta^5 + 21\theta^4 + 4\theta^3 + 3\theta^2 + 3\theta + 1$;

h) at $v = (2u^2 - 2u + 5 \mp (2u-1)(1+4u)^{1/2})(2u-4)^{-1}$, $(u, v) \neq (6, 15)$ *by substitution with* $r_2 = -r_1$, $s_2 = (3 \pm (1 + 4u)^{1/2})s_1/2$*—to* $SF_{29}^{4,1}$;

i) at $v = u + 1 \mp 2(u+1)^{1/2}$ *by substitution with* $r_2 = -r_1$, $s_2 = \pm(u+1)^{1/2}s_1$*—to* $SF_{30}^{4,1}$;

j) at $u = \vartheta_{13}$, $v = \vartheta_{14}$ *by substitution with* $r_1 = \vartheta_{15}r_2$, $s_1 = \vartheta_{16}s_2$*—to* $SF_{32}^{4,1}$;

k) $v = (2u^2 - 4u + 3)(u-2)^{-1}$, $u \neq 3$ *by substitution with* $r_2 = -r_1$, $s_2 = -(u-2)s_1$*—to* $SF_{33}^{4,1}$;

l) at $u = -\theta_*^3 - \theta_* + 2$, $v = -6\theta_*^3 - 2\theta_*^2 - 5\theta_* + 8$ *by substitution with* $r_1 = \theta_* r_2$, $s_2 = -\theta_*^2 s_1$*—to* $SF_{36}^{4,1}$, θ_*: $\theta^4 + \theta^3 + \theta^2 - \theta - 1$;

m) at $u = [v(3v - 10 \pm (v^2 + 12v - 12)^{1/2})(4v-8)^{-1} \vee (-4\theta_*^2 + 2(v-1)\theta_* + 2v - 7)/3]$ *by substitution with* $r_1 = [0 \vee (-2\theta_*^2 + v\theta_* - 2)r_2]$, $s_2 = [(v + 2 \pm (v^2 + 12v - 12)^{1/2})s_1/4 \vee \theta_* s_1]$*—to* $SF_3^{5,1}$, θ_*: $2\theta^3 - (v+2)\theta^2 + 2(v+1)\theta - 3$;

n) at $u = (\theta_*^2 - \theta_* - v + 1)(\theta_* - 1)^{-1}$ *by substitution with* $r_1 = \theta_* r_2$, $s_2 = ((2v-3)\theta_* + v)(\theta_*^2 - (v-2)\theta_* - 2)^{-1}s_1$*—to* $SF_6^{5,1}$, θ_*: $\theta^4 - (2v-3)\theta^3 + (v-3)(v+1)\theta^2 + (3v^2 - 6v + 4)\theta + v^2$;

4) $NSF_8^{5,1}$ $(w \neq v - u)$: *a) at* $v = -2$ *by substitution with* $r_1 = 0$, $s_1 = ws_2$*—to* $SF_{27}^{4,1}$;

b) at $v = [(2u-1)/2 \vee (2u-1)(3u-1)^{-1}]$, $w = [(u-2)/4 \vee -(2u-1)(3u-1)^{-2}]$ *by substitution with* $r_1 = [-r_2/2 \vee -(3u-1)^{-1}r_2]$, $s_2 = [0 \vee (3u-1)(u-1)(2u-1)^{-1}s_1]$*—to* $SF_3^{4,1}$;

c) $w = v(uv - 2u + 1)(2u-1)^{-2}$ *by substitution with* $s_1 = 0$, $r_2 = (1-2u)v^{-1}r_1$*—to* $SF_5^{4,1}$;

d) at $w = -v(u-1)^{-1}$ *by substitution with* $s_1 = 0$, $r_2 = -(u-1)v^{-1}r_1$*—to* $SF_{11}^{4,1}$;

e) at $[v = (3u-1)/2, w = (3u-2)/4 \vee u = (w^{3/2} \pm 1)(w^{1/2} \mp 2)^{-2}w^{-1/2}, v = (2w+1)(\mp w^{1/2} + 2)^{-1}]$ *by substitution with* $s_1 = [-s_2/2 \vee \mp w^{1/2}s_2]$, $r_2 = [0 \vee (-1 \pm 2w^{1/2})w^{-1/2}(w^{1/2} \mp 2)^{-1}r_1]$*—to* $SF_{13}^{4,1}$;

f) at $w = v(v-2)(4u-4)^{-1}$ *by substitution with* $r_1 = 0$, $s_1 = (2-v)(2u-2)^{-1}s_2$*—to* $SF_{14}^{4,1}$;

g) at $w = v$ *by substitution with* $r_1 = vr_2$, $s_1 = 0$*—to* $SF_{19}^{4,1}$;

h) at $u = -((16w + 18)\theta_*^2 + (4w^2 - 2w)\theta_* + w^3 + 14w^2 + 30w + 9)w^{-1}(w+6)^{-2}$, $v = (w\theta_*^2 - 2w\theta_* + w^2 + 4w - 3)(w+6)^{-1}$ *by substitution with* $s_1 = \theta_* s_2$, $r_2 = (6\theta_*^2 + 2w\theta_* + 2w + 3)(w(w+6))^{-1}r_1$*—to* $SF_{28}^{4,1}$, θ_*: $2\theta^3 + (2w+1)\theta + w$;

i) at $w = (v+2)(uv + v - 2u)(2u+1)^{-2}$ *by substitution with* $r_1 = 0$, $s_1 = -(v+2)(2u+1)^{-1}s_2$*—to* $SF_{29}^{4,1}$;

j) at $w = v^2(4u)^{-1}$ *by substitution with* $r_1 = 0$, $s_1 = -v(2u)^{-1}s_2$*—to* $SF_{30}^{4,1}$;

k) at $u = (v^2 + 2 \mp (v+1)\varrho)(3v-6)^{-1}$, $w = -(v+1)(v \mp \varrho)$ *by substitution with* $r_1 = (v \pm \varrho)r_2$, $s_1 = (-v - 2 \mp 2\varrho)s_2/3$, *where* $\varrho = (v^2 + v - 2)^{1/2}$—to $SF_{32}^{4,1}$;

l) at $w = -(v + 1)u^{-1}$ *by substitution with* $r_1 = 0$, $s_1 = -(v + 1)u^{-1}s_2$—*to* $SF_{33}^{4,1}$;

m) at $v = -(2u^2 + 4u + 1)(3u + 1)^{-1}(u + 1)^{-1}$, $w = -(5u^2 + 4u + 1)(3u + 1)^{-2}(u + 1)^{-1}$ *by substitution with* $s_1 = -(2u + 1)(3u + 1)^{-1}(u + 1)^{-1}s_2$, $r_2 = (3u + 1)r_1$—*to* $SF_{36}^{4,1}$;

n) at $[v = -(w\theta_*^2 - \theta_* + u - 1)\theta_*^{-1} \vee w = v(2uv - 3u + 1)(3u - 1)^{-2} \vee w = (2v - u - 1)/4]$ *by substitution with* $r_2 = [\theta_* r_1 \vee (1 - 3u)v^{-1}r_1 \vee 0]$, $s_2 = [(u - 1)(w\theta_*)^{-1}s_1 \vee 0 \vee -2s_1]$—*to* $SF_3^{5,1}$, θ_*: $w^2\theta^3 - w\theta^2 - w(u + 1)\theta - u + 1$;

o) at $w = [v - 3u/4 \vee -((v - 1)\theta_* + u - 1)\theta_*^{-2}]$ *by substitution with* $r_2 = [0 \vee \theta_* r_1]$, $s_2 = [-2s_1 \vee -\theta_*(v\theta_* + 3u - 1)(v\theta_* + u - 1)^{-1}s_1]$—*to* $SF_6^{5,1}$, θ_*: $v^2\theta^3 + (v^2 + 2uv - 2v)\theta^2 + (6uv - 2v - 3u^2 - 2u + 1)\theta + 5u^2 - 6u + 1$;

p) at $[w = v(3uv - 3u + 1)(3u - 1)^{-2} \vee u = (v^2 - v + 7)/9, w = 2 \vee u = ((13v - 16w - 6)\theta_*^2 + (4vw - v^2 - 2w + 2v - 3)\theta_* + 8w^2 - 2vw + 3w)(3\theta_*)^{-2}]$ *by substitution with* $r_1 = [-v(3u - 1)^{-1}r_2 \vee -r_2 \vee \theta_* r_2]$, $s_2 = [0 \vee (v - 2)s_1/3 \vee -(\theta_*^2 + (2v - 1)\theta_* + w)(3w\theta_*)^{-1}s_1]$—*to* $SF_7^{5,1}$, θ_*: $\theta^3 + (2v - 4w - 1)\theta^2 + w(v - 1)\theta + 2w^2$.

The proof can be found in the file statement2.mw in the database (see Introduction).

Corollary 3.1. *List* 1.1 *contains the CFs of* (4.1) *with their own canonical sets.*

Proof. The canonical sets for each form from List 1.1 were obtained by eliminating those values of parameters from the permissible set, at which the chosen form is reduced to antecedent one in the appropriate item of Statements 2.1 and 2.2. All canonical sets obtained are not empty, and hence each form of the list is canonical.

Statement 3.3. *The values of parameters in* $cs^{m,1}$ *can be limited only in the following* $CF^{m,1}$ *from List* 2.1, *videlicet*: 1) *in* $CF_3^{3,1}$ *at* $u = 2$ *substitution with* $r_1 = -1$, $s_1 = 0$, $r_2, s_2 = 1$, *while in* $CF_{14,\kappa}^{3,1}$ *substitution with* $-r_1$, $s_2 = -1$, $s_1, r_2 = 0$ *change a sign of* σ; 2) *in* $CF_5^{3,1}$ *at* $u_* = u < 1$ *substitution with* $r_1 = 1$, $s_1 = 1 - u_*$, $r_2 = 0$, $s_2 = 1$ *results in* $u = 2 - u_*$ $(u > 1, u \neq 2)$; 3) *in* $CF_1^{4,1}$ *at* $\sigma_* = \sigma$, $u_* = u$ *substitution with* $r_1, s_2 = 0$, $s_1, r_2 = |u_*|^{-1/2}$ *results in* $\sigma = \sigma_* \operatorname{sign} u_*$, $u = u_*^{-1}$; 4) *in* $CF_7^{4,1}$ $(v \neq 2 - u^{-1})$ *at* $\sigma_* = \sigma$, $u_* = u$ *substitution with* $r_1 = |v - 1|^{1/2}|u_*v - 2u_* + 1|^{-1/2}$, $s_1 = 0$, $r_2 = (1 - u_*)(v - 1)^{-1}r_1$, $s_2 = (v - 1)^{-1}(u_*v - 2u_* + 1)r_1$ *results in* $\sigma = \sigma_*\operatorname{sign}((v - 1)(u_*v - 2u_* + 1))$, $u = (v - u_*)(u_*v - 2u_* + 1)^{-1}$ *and the same* v.

Corollary 3.2. *Based on Definition* 1.12 *from work* [2], *one has*: $acs_3^{3,1} = \{\sigma = -1 \text{ at } u = 2\}$, $acs_5^{3,1} = \{u < 1\}$, $acs_{14,\kappa}^{3,1} = \{\sigma = -1\}$, $acs_1^{4,1} = \{|u| > 1\}$, $acs_7^{4,1} = \{u < 1 \text{ at } v \neq 1\}$; *for other canonical forms from List* 2.1 $mcs^{m,1} = cs^{m,1}$.

Collection 3.1. The constants, polynomials and substitutions used below are as follows:

$$1)\ \kappa_1 = \tilde{p}_1^2 - 4\tilde{p}_2, \quad \kappa_2 = \tilde{p}_1(1 + |\tilde{p}_1^{-1}|\kappa_1^{1/2});$$

$$2)\ \kappa_3 = \tilde{p}_2(\tilde{q}_1 - 2)^2 - \tilde{q}_2^2, \quad \kappa_4 = \tilde{q}_2^2 + 4\tilde{p}_2, \quad \kappa_5 = \tilde{q}_2(1 + |\tilde{q}_2|^{-1}\kappa_4^{1/2}),$$

$$\kappa_6 = \tilde{q}_2^2 + 4\tilde{p}_2(\tilde{q}_1 - 1), \quad \kappa_7 = -\tilde{q}_2(1 + |\tilde{q}_2|^{-1}\kappa_6^{1/2})(2\tilde{p}_2)^{-1}, \quad \kappa_8 = \tilde{q}_1 - (\kappa_6 + |\tilde{q}_2|\kappa_6^{1/2})(2\tilde{p}_2)^{-1};$$

$$3_1)\ \kappa_9 = \tilde{t}_2^2 + \tilde{p}_1\tilde{t}_1 + \tilde{q}_1\tilde{t}_2, \quad \kappa_{10} = \tilde{q}_1 + 2\tilde{t}_2, \quad \kappa_{11} = \tilde{q}_2\tilde{t}_1 - \tilde{p}_1\tilde{t}_1 - \tilde{q}_1\tilde{t}_2, \quad \kappa_{12} = \tilde{q}_2\tilde{t}_1 - \tilde{q}_1\tilde{t}_2,$$

$$\kappa_{13} = \tilde{p}_1\tilde{t}_1 - \tilde{t}_2^2, \quad \kappa_{14} = 2\tilde{t}_2^2 + \tilde{q}_2\tilde{t}_1 - \tilde{p}_1\tilde{t}_1, \quad \kappa_{15} = 2\tilde{t}_2^2 + \tilde{q}_2\tilde{t}_1, \quad \kappa_{16} = \tilde{t}_2^2 + \tilde{q}_2\tilde{t}_1, \quad \kappa_{17} = \tilde{q}_1^2 - 4\tilde{p}_1\tilde{t}_1,$$

$$\kappa_{18} = 4\tilde{p}_1\tilde{t}_1 - \tilde{q}_1^2 + 2\tilde{q}_1\tilde{t}_2 - 4\tilde{q}_2\tilde{t}_1, \quad \kappa_{19} = 4\tilde{p}_1\tilde{t}_1 - \tilde{q}_1^2 - 2\tilde{q}_1\tilde{t}_2, \quad \kappa_{20} = \tilde{q}_1\tilde{t}_2 - 2\tilde{q}_2\tilde{t}_1,$$

$$\kappa_{21}^{\pm} = -\tilde{q}_1 \pm \kappa_{17}^{1/2}, \quad \kappa_{22}^{\pm} = \tilde{q}_1^2 - 4\tilde{p}_1\tilde{t}_1 + \tilde{q}_1\tilde{t}_2 \pm (\tilde{q}_1 + \tilde{t}_2)\kappa_{17}^{1/2}, \quad \kappa_{23}^{\pm} = \kappa_{10} \pm \kappa_{17}^{1/2},$$

$$\kappa_{24}^{\pm} = 4\tilde{p}_1\tilde{t}_1 - \tilde{q}_1^2 - 2\tilde{q}_1\tilde{t}_2 + 2\tilde{q}_2\tilde{t}_1 \pm \kappa_{10}\kappa_{17}^{1/2}, \quad \kappa_{25} = (2\tilde{t}_2 - \tilde{q}_1)^2 + 8\tilde{q}_2\tilde{t}_1,$$

$$\kappa_{26}^{\pm} = \kappa_{10} \pm \kappa_{25}^{1/2}, \quad \kappa_{27}^{\pm} = 2\tilde{t}_2 - \tilde{q}_1 \pm \kappa_{25}^{1/2}, \quad \kappa_{28}^{\pm} = 8\tilde{p}_1\tilde{t}_1 - \tilde{q}_1^2 + 2\tilde{q}_1\tilde{t}_2 - 4\tilde{q}_2\tilde{t}_1 \pm \tilde{q}_1\kappa_{25}^{1/2},$$

$$\kappa_{29}^{\pm} = 8\tilde{p}_1\tilde{t}_1 - \tilde{q}_1^2 + 4\tilde{q}_2\tilde{t}_1 + 4\tilde{t}_2^2 \pm \kappa_{10}\kappa_{25}^{1/2}, \quad \kappa_{30}^{\pm} = 4\tilde{q}_2\tilde{t}_1 - \tilde{q}_1\tilde{t}_2 + 2\tilde{t}_2^2 \pm \tilde{t}_2\kappa_{25}^{1/2};$$

$$3_{\text{II}})\ \kappa_{31} = \tilde{t}_2^2 + 4\tilde{p}_1\tilde{t}_1, \quad \kappa_{32}^{\pm} = \tilde{t}_2 \pm \kappa_{31}^{1/2}, \quad \kappa_{33}^{\pm} = (\tilde{q}_2\tilde{t}_1)^{1/2} \pm 2\tilde{t}_2, \quad \kappa_{34}^{\pm} = (\tilde{q}_2\tilde{t}_1)^{1/2} \pm \tilde{t}_2,$$

$$\kappa_{35}^{\pm} = \tilde{q}_2\tilde{t}_1 \pm (\tilde{q}_2\tilde{t}_1)^{1/2}\tilde{t}_2 + \tilde{t}_2^2, \quad \kappa_{36} = 9\theta_* - 2\tilde{q}_1^2 + \tilde{q}_1\tilde{t}_2 + \tilde{t}_2^2, \quad \kappa_{37} = 3\theta_* - \tilde{q}_1^2 - \tilde{q}_1\tilde{t}_2 - \tilde{t}_2^2,$$

$$\kappa_{38} = 9\theta_*^2 - 3(\tilde{q}_1^2 - 2\tilde{q}_1\tilde{t}_2 - 2\tilde{t}_2^2)\theta_* - \tilde{t}_2(2\tilde{q}_1 + \tilde{t}_2)(2\tilde{q}_1^2 + 4\tilde{q}_1\tilde{t}_2 + 3\tilde{t}_2^2) + \tilde{q}_2\tilde{t}_1\kappa_{10}^2,$$

$$\kappa_{39} = \theta_* + \tilde{q}_1\tilde{t}_2 + \tilde{t}_2^2, \quad \kappa_{40} = \tilde{q}_1^2 + \tilde{q}_1\tilde{t}_2 - 2\tilde{t}_2^2, \quad \kappa_{41}^{\pm} = (2\tilde{q}_1 + \tilde{t}_2 \pm \kappa_{40}^{1/2})(\tilde{q}_1 + \tilde{t}_2)/3,$$

$$\kappa_{42}^{\pm} = \kappa_{41}^{\pm} + \tilde{q}_1\tilde{t}_2 + \tilde{t}_2^2, \quad \kappa_{43}^{\pm} = 3\kappa_{41}^{\pm} - \tilde{q}_1^2 - \tilde{q}_1\tilde{t}_2 - \tilde{t}_2^2, \quad \kappa_{44} = \tilde{q}_1^2 + 3\tilde{q}_1\tilde{t}_2 + 2\tilde{t}_2^2,$$

$$\kappa_{45}^{\pm} = (\tilde{q}_1 + \tilde{t}_2 \pm \kappa_{44}^{1/2})(2\tilde{q}_1 + 3\tilde{t}_2), \quad \kappa_{46}^{\pm} = \kappa_{45}^{\pm} + 2\tilde{q}_1^2 + 9\tilde{t}_2(\tilde{q}_1 + \tilde{t}_2), \quad \kappa_{47}^{\pm} = \kappa_{45}^{\pm} + \tilde{q}_1\tilde{t}_2 + \tilde{t}_2^2;$$

$$\kappa_{48} = \tilde{q}_1\tilde{q}_2 + \tilde{t}_2(\tilde{q}_2 - 3\tilde{p}_1), \quad \kappa_{49} = 3\tilde{p}_1\tilde{t}_2 - \tilde{q}_2(2\tilde{q}_1 + \tilde{t}_2), \quad \kappa_{50} = 4\tilde{p}_1\tilde{t}_1 - \tilde{q}_1^2 + \tilde{t}_2^2,$$

$$\kappa_{51} = \tilde{p}_1\tilde{t}_1 - \tilde{q}_1\tilde{t}_2 + \tilde{t}_2^2, \quad \kappa_{52} = \tilde{p}_1\tilde{t}_1 + \tilde{q}_1\tilde{t}_2 - \tilde{q}_1^2, \quad \kappa_{53} = \tilde{q}_1\tilde{t}_2 - \tilde{q}_2\tilde{t}_1, \quad \kappa_{54} = 2\tilde{q}_1\tilde{t}_2 - 4\tilde{q}_2\tilde{t}_1 - \tilde{t}_2^2,$$

$$\kappa_{55} = \tilde{t}_1^2\theta_*^4 + \tilde{t}_1(\tilde{t}_2 + \tilde{q}_1)\theta_*^3 + (5\tilde{q}_1\tilde{t}_2 - 2\tilde{t}_2^2 - 7\tilde{q}_2\tilde{t}_1 - 2\tilde{q}_1^2)\theta_*^2 + \tilde{q}_2(\tilde{t}_2 + \tilde{q}_1)\theta_* + \tilde{q}_2^2,$$

$$\kappa_{56} = 4\tilde{t}_1\theta_*^2 + (2\tilde{q}_1 - \tilde{t}_2)\theta_* + \tilde{q}_2, \quad \kappa_{57} = \tilde{t}_1\tilde{t}_2\theta_*^2 + (2\tilde{q}_1\tilde{t}_2 - 3\tilde{q}_2\tilde{t}_1 - \tilde{t}_2^2)\theta_* + \tilde{q}_2\tilde{t}_2,$$

$$\kappa_{58} = 2\tilde{t}_1^2\theta_*^3 + \tilde{t}_1(\tilde{q}_1 - \tilde{t}_2)\theta_*^2 + (2\tilde{q}_1\tilde{t}_2 - 4\tilde{q}_2\tilde{t}_1 - \tilde{t}_2^2)\theta_* + \tilde{q}_2\tilde{t}_2,$$

$$\kappa_{59} = 2\tilde{t}_1^2\theta_*^4 + 5\tilde{t}_1(\tilde{q}_1 + \tilde{t}_2)\theta_*^3 + (2\tilde{q}_1^2 + 10\tilde{q}_2\tilde{t}_1 - 5\tilde{q}_1\tilde{t}_2 + 2\tilde{t}_2^2)\theta_*^2 - \tilde{q}_2(\tilde{q}_1 + \tilde{t}_2)\theta_* - \tilde{q}_2^2,$$

$$\kappa_{60} = 3\tilde{p}_1\tilde{q}_1 + (\tilde{q}_2 - 3\tilde{p}_1)\tilde{t}_2, \quad \kappa_{61} = \tilde{q}_1\tilde{q}_2 + (\tilde{q}_2 - 3\tilde{p}_1)\tilde{t}_2,$$

$$\kappa_{62} = \tilde{t}_1\tilde{t}_2\theta_*^3 + (5\tilde{t}_1\tilde{q}_2 + 2\tilde{t}_2^2 - 4\tilde{q}_1\tilde{t}_2)\theta_*^2 - \tilde{q}_2(2\tilde{q}_1 + \tilde{t}_2)\theta_* - \tilde{q}_2^2, \quad \kappa_{63} = 2\tilde{t}_1\theta_*^2 + (\tilde{q}_1 - 2\tilde{t}_2)\theta_* - \tilde{q}_2;$$

$$S_1(\theta) = 27\theta^3 - 3(7\tilde{q}_1 + 5\tilde{t}_2)(\tilde{q}_1 - \tilde{t}_2)\theta^2 + (\tilde{q}_1^2 - 4\tilde{q}_1\tilde{t}_2 - 3\tilde{t}_2^2)(2\tilde{q}_1 + \tilde{t}_2)^2\theta$$
$$+ \tilde{t}_2(\tilde{q}_1 + \tilde{t}_2)(\tilde{q}_1^2 + \tilde{q}_1\tilde{t}_2 + \tilde{t}_2^2)(2\tilde{q}_1 + \tilde{t}_2)^2;$$

$$S_2(\theta) = \tilde{t}_1\tilde{t}_2\theta^3 + (5\tilde{q}_2\tilde{t}_1 - 4\tilde{q}_1\tilde{t}_2 + 2\tilde{t}_2^2)\theta^2 - \tilde{q}_2(2\tilde{q}_1 + \tilde{t}_2)\theta - \tilde{q}_2^2,$$

$$S_3(\theta) = 2\tilde{t}_1^2\theta^3 + (\tilde{q}_1 - \tilde{t}_2)\tilde{t}_1\theta^2 + (2\tilde{q}_1\tilde{t}_2 - 4\tilde{q}_2\tilde{t}_1 - \tilde{t}_2^2)\theta + \tilde{t}_2\tilde{q}_2;$$

$$J_0^1 = \{r_1 = 1, s_1 = -\beta, r_2 = 0, s_2 = 1\},$$

$$J_1^1 = \{r_1 = 1, s_1 = 0, r_2 = -\hat{q}_2(2\hat{t}_2)^{-1}, s_2 = \hat{t}_2^{-1}\},$$

$$J_2^1 = \{r_1 = 1, s_1 = 0, r_2 = -\hat{p}_1\hat{q}_1^{-1}, s_2 = \hat{t}_2^{-1}\}, \quad J_3^1 = \{r_1 = 1, s_1 = 0, r_2 = \theta_*, s_2 = 1\};$$

$$1)\ L_2^{2,1} = \{r_1 = |\tilde{p}_1|^{-1/2}, s_1, r_2 = 0, s_2 = \tilde{p}_1 r_1\};$$

$$L_5^{3,1} = \{r_1 = 0, s_1 = |\tilde{p}_1|^{-1/2}, r_2 = \tilde{p}_1 s_1, s_2 = \kappa_2 s_1/2\};$$

$$L_8^{3,1} = \{r_1, s_2 = 0, s_1 = |\tilde{p}_1|^{-1/2}, r_2 = \tilde{p}_2\tilde{p}_1^{-1}s_1\};$$

$$2)\ L1_3^{3,1} = \{r_1 = (\tilde{q}_1 - 2)(\tilde{q}_1\tilde{q}_2)^{-1}s_2, s_1 = 0, r_2 = \tilde{q}_1^{-1}s_2, s_2 = |\tilde{q}_1\tilde{q}_2|^{1/2}|\tilde{q}_1 - 2|^{-1/2}\},$$

$$L2_3^{3,1} = \{r_1 = (4\tilde{p}_2)^{-1/4}, s_1 = 0, r_2 = \tilde{p}_2^{1/2}r_1, s_2 = (4\tilde{p}_2)^{1/4}\};$$

$$L_{14,\kappa}^{3,1} = \{r_1, s_2 = 0, s_1 = |\tilde{p}_2|^{-1/4}|\tilde{q}_1|^{3/4}\tilde{q}_1^{-1}, r_2 = |\tilde{p}_2|^{1/4}|\tilde{q}_1|^{-3/4}\},$$

$$L1_7^{4,1} = \{r_1 = |\tilde{q}_2|^{-1/2}, s_1 = 0, r_2 = \kappa_5 r_1/2, s_2 = \tilde{q}_2 r_1\},$$

$$L2_7^{4,1} = \{r_1 = \kappa_7 r_2, s_1 = 0, r_2 = |\kappa_7\kappa_8|^{-1/2}, s_2 = \kappa_8 r_2\},$$

$$L_{12}^{4,1} = \{r_1 = 0, s_1 = |\tilde{q}_1 - 2|^{1/2}|\tilde{q}_1\tilde{q}_2|^{-1/2}, r_2, s_2 = \tilde{q}_2(\tilde{q}_1 - 2)^{-1}s_1\},$$

$$L_{24}^{4,1} = \{r_1, s_2 = 0, s_1 = |\tilde{q}_2|^{-1/2}, r_2 = \tilde{q}_2 s_1/2\},$$

$$3_{\rm I})\ L_9^{2,1} = \{r_1 = -\tilde{t}_1|\tilde{t}_1|^{-1/2}\tilde{t}_2^{-1}, s_1 = 0, r_2 = \tilde{t}_1^{-1}\tilde{t}_2 r_1, s_2 = |\tilde{t}_1|^{-1/2}\},$$

$$L1_6^{3,1} = \{r_1 = 2\tilde{t}_1\kappa_{10}^{-1}|\tilde{t}_1|^{-1/2}, s_1 = 0, r_2 = -\tilde{q}_1(2\tilde{t}_1)^{-1}r_1, s_2 = |\tilde{t}_1|^{-1/2}\},$$

$$L2_6^{3,1} = \{r_1 = \tilde{t}_1|\tilde{t}_1|^{-1/2}\tilde{t}_2^{-1}, s_1, r_2 = 0, s_2 = |\tilde{t}_1|^{-1/2}\},$$

$$L1_{11,\kappa}^{3,1} = \{r_1 = |\tilde{q}_2|^{-1/2}, s_1, r_2 = 0, s_2 = |\tilde{t}_1|^{-1/2}\},$$

$$L2_{11,\kappa}^{3,1} = \{r_1 = |\tilde{p}_1|^{-1/2}, s_1 = 0, r_2 = \tilde{t}_1^{-1}\tilde{t}_2 r_1, s_2 = |\tilde{t}_1|^{-1/2}\};$$

$$L_{17}^{3,1} = \{r_1 = -(\tilde{p}_1\tilde{t}_2)^{-1/3}|\tilde{t}_1|^{1/6}, s_1 = 0, r_2 = \tilde{t}_1^{-1}\tilde{t}_2 r_1, s_2 = |\tilde{t}_1|^{-1/2}\};$$

$$L_{19}^{3,1} = \{r_1 = 0, s_1 = -2^{2/3}|\tilde{t}_1|^{5/6}(\tilde{q}_1^2\tilde{t}_1\tilde{t}_2)^{-1/3}, r_2 = |\tilde{t}_1|^{-1/2}, s_2 = -\tilde{q}_1(2\tilde{t}_1)^{-1}s_1\};$$

$$L_{21}^{3,1} = \{r_1 = \tilde{t}_1|\tilde{t}_1|^{-1/2}(\tilde{t}_2^2(\tilde{q}_1 + \tilde{t}_2))^{-1/3}, s_1 = 0, r_2 = \tilde{t}_1^{-1}\tilde{t}_2 r_1, s_2 = |\tilde{t}_1|^{-1/2}\};$$

$$L_{22}^{3,1} = \{r_1 = 0, s_1 = \tilde{t}_1(\kappa_{16}\tilde{t}_2)^{-1/3}|\tilde{t}_1|^{-1/2}, r_2 = |\tilde{t}_1|^{-1/2}, s_2 = \tilde{t}_1^{-1}\tilde{t}_2 s_1\};$$

$$L1_5^{4,1} = \{r_1 = 4(\kappa_{26}^{\mp})^{-1}\tilde{t}_1^{-1}|\tilde{t}_1|^{-1/2}, s_1 = 0, r_2 = \kappa_{27}^{\pm}(4\tilde{t}_1)^{-1}r_1, s_2 = |\tilde{t}_1|^{-1/2}\},$$

$$L2_5^{4,1} = \{r_1 = \tilde{t}_1\tilde{t}_2^{-1}|\tilde{t}_1|^{-1/2}, s_1, r_2 = 0, s_2 = |\tilde{t}_1|^{-1/2}\};$$

$$L1_{11}^{4,1} = \{r_1 = |\tilde{q}_2|^{-1/2}, s_1, r_2 = 0, s_2 = \tilde{q}_1^{-1}\tilde{q}_2|\tilde{q}_2|^{-1/2}\},$$

$$L2_{11}^{4,1} = \{r_1 = |\kappa_{12}|^{-1/2}|\tilde{t}_1|^{1/2}, s_1 = 0, r_2 = \tilde{t}_1^{-1}\tilde{t}_2 r_1, s_2 = \kappa_{12}\tilde{t}_1^{-1}\kappa_{10}^{-1}r_1\};$$

$$L1_{14}^{4,1} = \{r_1 = 0, s_1 = 2\tilde{t}_1\kappa_{10}^{-1}|\tilde{t}_1|^{-1/2}, r_2 = |\tilde{t}_1|^{-1/2}, s_2 = -\tilde{q}_1(2\tilde{t}_1)^{-1}s_1\},$$

$$L2_{14}^{4,1} = \{r_1, s_2 = 0, s_1 = \tilde{t}_1|\tilde{t}_1|^{-1/2}\tilde{t}_2^{-1}, r_2 = |\tilde{t}_1|^{-1/2}\};$$

$$L_{19}^{4,1} = \{r_1 = -|\tilde{t}_1|^{1/6}(\tilde{p}_1\tilde{t}_2)^{-1/3}, s_1 = 0, r_2 = \tilde{t}_1^{-1}\tilde{t}_2 r_1, s_2 = |\tilde{t}_1|^{-1/2}\};$$

$$L_{27}^{4,1} = \{r_1 = 0, s_1 = |\tilde{t}_1|^{5/6}(\kappa_{14}\tilde{t}_1\tilde{t}_2)^{-1/3}, r_2 = |\tilde{t}_1|^{-1/2}, s_2 = \tilde{t}_1^{-1}\tilde{t}_2 s_1\};$$

$$L_{29}^{4,1} = \{r_1 = 0, s_1 = \pm\kappa_{17}^{-1/2}\tilde{t}_1|\tilde{t}_1|^{-1/2}, r_2 = |\tilde{t}_1|^{-1/2}, s_2 = \kappa_{21}^{\pm}(2\tilde{t}_1)^{-1}s_1\};$$

$$L_{30}^{4,1} = \{r_1 = 0, s_1 = 2|\tilde{t}_1|^{5/6}(2\kappa_{20}\tilde{t}_1\tilde{q}_1)^{-1/3}, r_2 = |\tilde{t}_1|^{-1/2}, s_2 = -\tilde{q}_1(2\tilde{t}_1)^{-1}s_1\};$$

$$L_{33}^{4,1} = \{r_1 = 0, s_1 = \tilde{t}_1\kappa_{10}^{-1}|\tilde{t}_1|^{-1/2}, r_2 = |\tilde{t}_1|^{-1/2}, s_2 = \tilde{t}_1^{-1}\tilde{t}_2 s_1\};$$

$$L_8^{5,1} = \{r_1 = |\tilde{q}_2|^{-1/2}, s_1, r_2 = 0, s_2 = \tilde{q}_2|q_2|^{-1/2}\tilde{t}_2^{-1}\};$$

$$3_{\rm II})\ L1_1^{4,1} = \{r_1 = s_1, s_1 = \kappa_{31}^{-1/4}\left|2\tilde{t}_1(\kappa_{32}^{\pm})^{-1}\right|^{1/2}, r_2 = \kappa_{32}^{\mp}(2\tilde{t}_1)^{-1}s_1, s_2 = \kappa_{32}^{\pm}(2\tilde{t}_1)^{-1}s_1\},$$

$$L2_1^{4,1} = \{r_1 = s_1, s_1 = |\tilde{p}_1|^{-1/2}, r_2 = \kappa_{32}^{\pm}(2\tilde{t}_1)^{-1}s_1, s_2 = 0\};$$

$$L1_3^{4,1} = \{r_1 = s_1, s_1 = \left|(3\tilde{q}_1 - \tilde{t}_2)(\tilde{q}_1 - \tilde{t}_2)\tilde{t}_1^{-1}\right|^{-1/2}, r_2 = \tilde{q}_1\tilde{t}_1^{-1}s_1, s_2 = (\tilde{t}_2 - 2\tilde{q}_1)\tilde{t}_1^{-1}s_1\},$$

$$L2_3^{4,1} = \{r_1 = s_1, s_1 = \sqrt{3}\left|(2\tilde{q}_1 - 3\tilde{t}_2)\tilde{t}_2\tilde{t}_1^{-1}\right|^{-1/2}, r_2 = 0, s_2 = -(2\tilde{q}_1 - 3\tilde{t}_2)(3\tilde{t}_1)^{-1}s_1\},$$

$$L3_3^{4,1} = \{r_1 = s_1, s_1 = 4\left|(2\tilde{q}_1 - 3\tilde{t}_2)(2\tilde{q}_1 + \tilde{t}_2)\tilde{t}_1^{-1}\right|^{-1/2}, r_2 = -(2\tilde{q}_1 - 3\tilde{t}_2)(4\tilde{t}_1)^{-1}s_1, s_2 = 0\};$$

$$L1_{13}^{4,1} = \{r_1 = s_1, s_1 = 2\left|(2\tilde{q}_1 - \tilde{t}_2)\tilde{t}_2\tilde{t}_1^{-1}\right|^{-1/2}, r_2 = 0, s_2 = -(2\tilde{q}_1 - \tilde{t}_2)(2\tilde{t}_1)^{-1}s_1\},$$

$$L2^{4,1}_{13} = \{r_1 = s_1, s_1 = \pm(\tilde{q}_2\tilde{t}_1)^{1/2}\tilde{q}_2^{-1}s_2, r_2 = -(2(\tilde{q}_2\tilde{t}_1)^{1/2} \pm \tilde{t}_2)(\kappa^{\pm}_{33})^{-1}s_2,$$

$$s_2 = (\tilde{q}_2\tilde{t}_1)^{1/4}\kappa^{\pm}_{33}(3\kappa^{\pm}_{35})^{-1/2}\left|\kappa^{\pm}_{34}\tilde{t}_1\right|^{-1/2}\};$$

$$L^{4,1}_{28} = \{r_1 = s_1, s_1 = \kappa_{10}\left|\kappa_{39}\kappa_{36}\tilde{t}_1^{-1}\right|^{-1/2}, r_2 = -(3\theta_* + 2\tilde{q}_1\tilde{t}_2 + \tilde{t}_2^2)(\tilde{t}_1\kappa_{10})^{-1}s_1,$$

$$s_2 = (6\theta_* - \tilde{q}_1\tilde{t}_2 - 2\tilde{q}_1^2)(\tilde{t}_1\kappa_{10})^{-1}s_1\};$$

$$L^{4,1}_{32} = \{r_1 = s_1, s_1 = \left|\tilde{t}_1\right|^{1/2}\left|\kappa^{\pm}_{42}\right|^{-1/2}, r_2 = -(\tilde{q}_1 + \tilde{t}_2)\tilde{t}_1^{-1}s_1, s_2 = (3\kappa^{\pm}_{41} - \tilde{q}_1^2 + \tilde{t}_2^2)(\tilde{t}_1\kappa_{10})^{-1}s_1\};$$

$$L^{4,1}_{36} = \{r_1 = s_1, s_1 = \left|\tilde{t}_1\right|^{1/2}\left|\kappa^{\pm}_{46}\right|^{-1/2}, r_2 = -(2\tilde{q}_1 + 3\tilde{t}_2)\tilde{t}_1^{-1}s_1, s_2 = (\kappa^{\pm}_{45} + 2\tilde{q}_1\tilde{t}_2 + 3\tilde{t}_2^2)(\tilde{t}_1\kappa_{10})^{-1}s_1\};$$

$$L1^{5,1}_3 = \{r_1, s_1 = \left|\tilde{p}_1\right|^{-1/2}, r_2 = (\tilde{q}_2 - 3\tilde{p}_1)\tilde{q}_1^{-1}s_1, s_2 = 0\},$$

$$L2^{5,1}_3 = \{r_1, s_1 = \left|\tilde{q}_2\right|^{-1/2}, r_2 = 0, s_2 = -2\tilde{q}_2\tilde{t}_2^{-1}s_1\},$$

$$L3^{5,1}_3 = \{r_1, s_1 = (\tilde{q}_1 - \tilde{t}_2)\left|\kappa_{50}\tilde{p}_1\right|^{-1/2}, r_2 = -(2\tilde{p}_1\tilde{t}_1 - \tilde{q}_1\tilde{t}_2 + \tilde{t}_2^2)(\tilde{t}_1(\tilde{q}_1 - \tilde{t}_2))^{-1}s_1,$$

$$s_2 = (2\tilde{p}_1\tilde{t}_1 + \tilde{q}_1\tilde{t}_2 - \tilde{q}_1^2)(\tilde{t}_1(\tilde{q}_1 - \tilde{t}_2))^{-1}s_1\};$$

$$L1^{5,1}_6 = \{r_1, s_1 = \left|\tilde{q}_2\right|^{-1/2}, r_2 = 0, s_2 = -2\tilde{q}_2\tilde{t}_2^{-1}s_1\},$$

$$L2^{5,1}_6 = \{r_1, s_1 = 3\theta_*\left|\kappa_{59}\tilde{t}_1^{-1}\right|^{-1/2}, r_2 = -(\tilde{t}_1\theta_*^2 + (2\tilde{q}_1 - \tilde{t}_2)\theta_* + \tilde{q}_2)(3\tilde{t}_1\theta_*)^{-1}s_1, s_2 = \theta_* s_1\};$$

$$L1^{5,1}_7 = \{r_1, s_1 = \left|\tilde{p}_1\right|^{-1/2}, r_2 = (\tilde{q}_2 - 3\tilde{p}_1)\tilde{q}_1^{-1}s_1, s_2 = 0\},$$

$$L2^{5,1}_7 = \{r_1, s_1 = \sqrt{3}\left|\kappa_{63}\right|^{-1/2}, r_2 = \theta_* s_1, s_2 = -(\tilde{t}_1\theta_*^2 + (2\tilde{q}_1 - \tilde{t}_2)\theta_* + \tilde{q}_2)(3\tilde{t}_1\theta_*)^{-1}s_1\}.$$

4. THREE CLASSES OF LINEAR EQUIVALENCY OF SYSTEMS AT $l = 1$

System (2.1)[1], $\dot{x} = Aq^{[3]}(x)$ at $l = 1$ in accordance with formula (2.10)[1] is uniquely representable in form (2.9)[1]:

$$\dot{x} = (x_1 + \beta x_2)\begin{pmatrix} p_1x_1^2 + q_1x_1x_2 + t_1x_2^2 \\ p_2x_1^2 + q_2x_1x_2 + t_2x_2^2 \end{pmatrix} = P_0^1(x)Gq^{[2]}(x), \quad G = \begin{pmatrix} p_1 & q_1 & t_1 \\ p_2 & q_2 & t_2 \end{pmatrix}, \tag{4.1}$$

where $R_2 = \delta_{pt}^2 - \delta_{pq}\delta_{qt} \neq 0$, because $l = 1$, and hence $p_1^2 + p_2^2 \neq 0$ and $t_1^2 + t_2^2 \neq 0$.

The substitution J_0^1 from Collection 2.1 transforms expression (4.1) into a much more trivial system

$$\hat{A} = \begin{pmatrix} \hat{a}_1 & \hat{b}_1 & \hat{c}_1 & \hat{d}_1 \\ \hat{a}_2 & \hat{b}_2 & \hat{c}_2 & \hat{d}_2 \end{pmatrix} = \begin{pmatrix} \hat{p}_1 & \hat{q}_1 & \hat{t}_1 & 0 \\ \hat{p}_2 & \hat{q}_2 & \hat{t}_2 & 0 \end{pmatrix}, \quad (\hat{\alpha}, \hat{\beta}) = (1, 0), \quad \hat{G} = \begin{pmatrix} \hat{p}_1 & \hat{q}_1 & \hat{t}_1 \\ \hat{p}_2 & \hat{q}_2 & \hat{t}_2 \end{pmatrix}, \tag{4.2}$$

where $\hat{p}_1 = p_1 + \beta p_2$, $\hat{q}_1 = q_1 + \beta(q_2 - 2p_1) - 2\beta^2 p_2$, $\hat{t}_1 = t_1 + \beta(t_2 - q_1) - \beta^2 q_2 + \beta^3 p_2$, $\hat{p}_2 = p_2$, $\hat{q}_2 = q_2 - 2\beta p_2$, $\hat{t}_2 = t_2 - \beta q_2 + \beta^2 p_2$ ($\hat{p}_1^2 + \hat{p}_2^2 \neq 0$, $\hat{t}_1^2 + \hat{t}_2^2 \neq 0$, $\hat{R}_2 = R_2 \neq 0$), because in accordance with expression (2.12)[1] at any substitution (2.2)[1], $x = Ly$, the matrix $\hat{G} = L^{-1}GM$, $\hat{R}_2 = \delta_{\hat{p}\hat{t}}^2 - \delta_{\hat{p}\hat{q}}\delta_{\hat{q}\hat{t}} = \delta^2 R_2$, $\hat{\alpha} = r_1 + \beta r_2$, $\hat{\beta} = s_1 + \beta s_2$ ($\hat{\alpha}^2 + \hat{\beta}^2 \neq 0$) and $\hat{A}$ from expression (2.5)[1].

Making an arbitrary substitution (2.2)[1] with $s_1 = 0$ ($r_1, s_2 \neq 0$) in the expression (4.2), one obtains:

$$\tilde{A} = \begin{pmatrix} \hat{p}_1r_1^2 + \hat{q}_1r_1r_2 + \hat{t}_1r_2^2 & (\hat{q}_1r_1 + 2\hat{t}_1r_2)s_2 & \hat{t}_1s_2^2 & 0 \\ -S_0(r_1^{-1}r_2)r_1^3s_2^{-1} & \hat{q}_2r_1^2 - (\hat{q}_1 - 2\hat{t}_2)r_1r_2 - 2\hat{t}_1r_2^2 & (\hat{t}_2r_1 - \hat{t}_1r_2)s_2 & 0 \end{pmatrix}, \tag{4.3}$$

where $S_0(\theta) = \hat{t}_1\theta^3 + (\hat{q}_1 - \hat{t}_2)\theta^2 + (\hat{p}_1 - \hat{q}_2)\theta - \hat{p}_2$, $\hat{p}_1^2 + \hat{p}_2^2$, $\hat{t}_1^2 + \hat{t}_2^2 \neq 0$, $\tilde{a}_1^2 + \tilde{a}_2^2$, $\tilde{c}_1^2 + \tilde{c}_2^2 \neq 0$.

In particular, system (4.3) at $\hat{t}_1 = 0$ ($\hat{t}_2 \neq 0$) takes the form:

$$\begin{pmatrix} (\hat{p}_1 r_1 + \hat{q}_1 r_2) r_1 & \hat{q}_1 r_1 s_2 & 0 & 0 \\ (\hat{p}_2 r_1^2 - (\hat{p}_1 - \hat{q}_2) r_1 r_2 - (\hat{q}_1 - \hat{t}_2) r_2^2) r_1 s_2^{-1} & (\hat{q}_2 r_1 + (2\hat{t}_2 - \hat{q}_1) r_2) r_1 & \hat{t}_2 r_1 s_2 & 0 \end{pmatrix}. \tag{4.4}$$

Statement 4.1. *Set of systems* (4.2) *is divided into three linearly nonequivalent classes by the following conditions*: 1) $\hat{t}_1 = 0$, $\hat{q}_1 = 0$; 2) $\hat{t}_1 = 0$, $\hat{q}_1 \neq 0$; 3) $\hat{t}_1 \neq 0$.

Indeed, the above conditions are linear invariants of system (4.2), since for any change that connects systems of form (4.2), $s_1 = 0$ according to Statement 2.2 p. 2, and therefore, in systems (4.3) and (4.4), the substitution factors are $r_1, s_2 \neq 0$.

Let us distinguish in each class the simplest system similar to (4.3) or (4.4).

1) $\hat{t}_1 = 0$, $\hat{q}_1 = 0$ ($\hat{p}_1 \neq 0$, $\hat{t}_2 \neq 0$). Then, system (4.4) can be written as follows:

$$\begin{pmatrix} \hat{p}_1 r_1^2 & 0 & 0 & 0 \\ (\hat{p}_2 r_1^2 + (\hat{q}_2 - \hat{p}_1) r_1 r_2 + \hat{t}_2 r_2^2) r_1 s_2^{-1} & (\hat{q}_2 r_1 + 2\hat{t}_2 r_2) r_1 & \hat{t}_2 r_1 s_2 & 0 \end{pmatrix}$$

and one can always obtain $\tilde{b}_2 = 0$, $\tilde{c}_2 = 1$ in it. In particular, substitution J_1^1 transforms formulas (4.2) into system

$$\tilde{A}_1 = \begin{pmatrix} \tilde{p}_1 & 0 & 0 & 0 \\ \tilde{p}_2 & 0 & 1 & 0 \end{pmatrix}, \qquad \begin{aligned} \tilde{p}_1 &= \hat{p}_1 (\neq 0), \\ \tilde{p}_2 &= (2\hat{p}_1\hat{q}_2 + 4\hat{p}_2\hat{t}_2 - \hat{q}_2^2)/4. \end{aligned} \tag{4.5}$$

When $\tilde{p}_2 \neq 0$, a system $\tilde{A}_1$ is $SF_{a,8}^{3,1}$.

2) $\hat{t}_1 = 0$, $\hat{q}_1 \neq 0$ ($\hat{t}_2 \neq 0$). Then, one can always obtain $\tilde{a}_1 = 0$, $\tilde{c}_2 = 1$ in system (4.4). In particular, the substitution J_2^1 transforms system (4.2) into

$$\tilde{A}_2 = \begin{pmatrix} 0 & \tilde{q}_1 & 0 & 0 \\ \tilde{p}_2 & \tilde{q}_2 & 1 & 0 \end{pmatrix}, \qquad \begin{aligned} \tilde{q}_1 &= \hat{q}_1 \hat{t}_2^{-1} (\neq 0), \quad \tilde{q}_2 = \hat{p}_1 + \hat{q}_2 - 2\hat{p}_1 \hat{q}_1^{-1} \hat{t}_2, \\ \tilde{p}_2 &= \hat{q}_1^{-2} \hat{t}_2 (\hat{t}_2 \hat{p}_1^2 - \hat{q}_2 \hat{q}_1 \hat{p}_1 + \hat{p}_2 \hat{q}_1^2)(\neq 0). \end{aligned} \tag{4.6}$$

At $\tilde{q}_2 \neq 0$ system $\tilde{A}_2$ is $SF_{a,24}^{4,1}$.

3) $\hat{t}_1 \neq 0$. Then, one can get $\tilde{a}_2 = 0$ in system (4.3). In particular, substitution J_3^1, where $\theta_* \in \mathbb{R}^1$ is any zero of $S_0(\theta)$, brings system (4.2) into (4.3):

$$\tilde{A}_3 = \begin{pmatrix} \tilde{p}_1 & \tilde{q}_1 & \tilde{t}_1 & 0 \\ 0 & \tilde{q}_2 & \tilde{t}_2 & 0 \end{pmatrix}; \qquad \begin{aligned} \tilde{p}_1 &= \hat{p}_1 + \hat{q}_1\theta_* + \hat{t}_1\theta_*^2 (\neq 0), \quad \tilde{q}_1 = \hat{q}_1 + 2\hat{t}_1\theta_*, \quad \tilde{t}_1 = \hat{t}_1, \\ \tilde{q}_2 &= \hat{q}_2 - (\hat{q}_1 - 2\hat{t}_2)\theta_* - 2\hat{t}_1\theta_*^2, \quad \tilde{t}_2 = \hat{t}_2 - \hat{t}_1\theta_*. \end{aligned} \tag{4.7}$$

At $\tilde{q}_1, \tilde{q}_2, \tilde{t}_2 \neq 0$, a system $\tilde{A}_3$ is $SF_8^{5,1}$. In turn, at $\tilde{q}_2 = 0$ it is $SF_5^{4,1}$, at $\tilde{t}_2 = 0$ it is $SF_{11}^{4,1}$ and at $\tilde{q}_1 = 0$ a system is $SF_{a,14}^{4,1}$.

Statement 4.2. *List* 1.1 *comprises all CFs of system* (4.1).

Proof. Any system (4.1) is reduced to system (4.5), (4.6), or (4.7), thus there are no CFs with antecedent $CF_8^{5,1}$ obtained by the normalization of system (4.7) with $\tilde{q}_1, \tilde{q}_2, \tilde{t}_2 \neq 0$.

5. REDUCTION OF THE INITIAL SYSTEM TO EACH $CF^{m,1}$

Below, the conditions for coefficients and substitutions $(2.2)^1$, reducing a system chosen at these conditions to the appropriate *CF* from List 2.1, will be found for systems (4.5), (4.6), and (4.7), and CF parameters will be evaluated.

Lemma 5.1. *Any system* (4.5) *is linearly equivalent to a representative of a certain* $CF_i^{m,1}$ *from List* 2.1_1. *Below, for each of the three* $CF_i^{m,1}$: *a*) *the conditions for coefficients* $\tilde{p}_1$ ($\neq 0$) *and* $\tilde{p}_2$ *of system* (4.5), *b*) *the substitution of* $(2.2)^1$, *transforming the right-hand part of system* (4.5) *at the given conditions into the form chosen*; *and c*) *the values of parameters* σ *and those from* $cs_i^{m,1}$ *are found to be*, *as follows*:

$CF_2^{2,1}$: *a*) $\tilde{p}_2 = 0$, *b*) $L_2^{2,1}$, *and c*) $\sigma = \operatorname{sign} \tilde{p}_1$;

$CF_5^{3,1}$: *a*) $\tilde{p}_2 \neq 0$, $\kappa_1 \geq 0$, *b*) $L_5^{3,1}$, *and c*) $\sigma = \operatorname{sign}\tilde{p}_1$, $u = \kappa_2\tilde{p}_1^{-1}$;

$CF_8^{3,1}$: *a*) $\tilde{p}_2 \neq 0$, $\kappa_1 < 0$, *b*) $L_8^{3,1}$, *and c*) $\sigma = \operatorname{sign}\tilde{p}_1$, $u = \tilde{p}_2\tilde{p}_1^{-2}$.

The proof is available in file lemma1.mw in the database (see Introduction).

Lemma 5.2. *Any system* (4.6) *is linearly equivalent to a representative of a certain* $CF_i^{m,1}$ *from List* 2.1_2. *Below for each of the five* $CF_i^{m,1}$: *a*) *the conditions for the coefficients* $\tilde{q}_1$ ($\neq 0$), $\tilde{p}_2$ ($\neq 0$) *and* $\tilde{q}_2$ *of system* (4.6), *b*) *the substitution* (2.2)[1], *transforming the right-hand part of system* (4.6) *at the given conditions into the form chosen*; *and c*) *the values of parameters* σ *and those from* $cs_i^{m,1}$ *are found to be*, *as follows*:

$CF_3^{3,1}$: 1) *a*) $\tilde{q}_1 \neq 2$, $\kappa_3 = 0$, *b*) $L1_3^{3,1}$, *c*) $\sigma = \operatorname{sign}((\tilde{q}_1 - 2)\tilde{q}_1\tilde{q}_2)$, $u = \tilde{q}_1$; 2) *a*) $\tilde{p}_2 > 0$, $\tilde{q}_1 = 2$, $\tilde{q}_2 = 0$, *b*) $L2_3^{3,1}$, *c*) $\sigma = 1$, $u = 2$;

$CF_{14,\kappa}^{3,1}$: *a*) $\tilde{p}_2 < 0$, *if* $\tilde{q}_1 = 2$, $\tilde{q}_2 = 0$, *b*) $L_{14,\kappa}^{3,1}$ *c*) $\kappa = \operatorname{sign}(\tilde{p}_2\tilde{q}_1)$, $u = \tilde{q}_1^{-1}$;

$CF_7^{4,1}$: 1) *a*) $\tilde{q}_1 = 2$, $\kappa_4 \geq 0$, $\tilde{q}_2 \neq 0$, *b*) $L1_7^{4,1}$, *c*) $\sigma = \operatorname{sign}\tilde{q}_2$, $u = \kappa_5\tilde{q}_2^{-1}$, $v = 2$; 2) *a*) $\tilde{q}_1 \neq 2$, $\tilde{q}_2 \neq 0$, $\kappa_6 \geq 0$, $\kappa_8 \neq 0$, *b*) $L2_7^{4,1}$, *c*) $\sigma = \operatorname{sign}(\kappa_7\kappa_8)$, $u = \kappa_8^{-1}\tilde{q}_1$, $v = \tilde{q}_1$;

$CF_{12}^{4,1}$: *a*) $\tilde{q}_1 \neq 2$, $\tilde{q}_2 \neq 0$, $4\kappa_3(1 - \tilde{q}_1) > \tilde{q}_1^2\tilde{q}_2^2$, *b*) $L_{12}^{4,1}$, *c*) $\sigma = \operatorname{sign}(\tilde{q}_1\tilde{q}_2(\tilde{q}_1 - 2))$, $u = \tilde{q}_1^{-1}$, $v = \kappa_3\tilde{q}_1^{-1}\tilde{q}_2^{-2}$;

$CF_{24}^{4,1}$: *a*) $\tilde{q}_1 = 2$, $\kappa_4 < 0$, $\tilde{q}_2 \neq 0$, *b*) $L_{24}^{4,1}$, *c*) $\sigma = \operatorname{sign}\tilde{q}_2$, $u = 1/2$, $v = 2\tilde{p}_2\tilde{q}_2^{-2}$.

The proof is available in the file lemma2.mw in the database (see the Introduction).

Now we are going to assume that the following conditions for coefficients $\xi_1^{\pm} = 0$, $\xi_2^{\mp} \neq 0$ mean that $\xi_1^+ = 0$, $\xi_2^- \neq 0$ or $\xi_1^- = 0$, $\xi_2^+ \neq 0$, and choosing from the first or second sign results in the same choice in the substitution coefficients given below, as well as in the values of *CF* parameters.

Lemma 5.3. *Any system* (4.7) *is linearly equivalent to a representative of a certain* $CF_i^{m,1}$ *from Lists* 2.1_3 *and* 2.1_{II}. *Below, for each of the 25* $CF_i^{m,1}$: *a*) *conditions on coefficients* $\tilde{p}_1, \tilde{t}_1, \tilde{q}_1, \tilde{q}_2, \tilde{t}_2$ ($\tilde{p}_1, \tilde{t}_1, \tilde{q}_2^2 + \tilde{t}_2^2 \neq 0$) *of system* (4.7), *b*) *substitution* (2.2)[1], *transforming the right-hand part of system* (4.7) *at the given conditions into the form chosen*, *and c*) *values of* σ *and parameters from* $cs_i^{m,1}$ *are found to be, as follows*:

$CF_9^{2,1}$: *a*) $\kappa_{10} = 0$, $\kappa_{13} = 0$, $\kappa_{15} = 0$, *b*) $L_9^{2,1}$, *and c*) $\sigma = \operatorname{sign}\tilde{t}_1$;

$CF_{22}^{3,1}$: *a*) $\kappa_{10} = 0$, $\kappa_{13} = 0$, $\kappa_{15} \neq 0$, *b*) $L_{22}^{3,1}$, *and c*) $\sigma = \operatorname{sign}\tilde{t}_1$, $u = \kappa_{15}(\kappa_{16}\tilde{t}_2)^{-2/3}$;

$CF_{17}^{3,1}$: *a*) $\kappa_{10} = 0$, $\kappa_{15} = 0$, $\kappa_{13} \neq 0$, *b*) $L1_{17}^{3,1}$, *and c*) $\sigma = \operatorname{sign}\tilde{t}_1$, $u = \kappa_{13}(\tilde{p}_1\tilde{t}_1\tilde{t}_2)^{-2/3}$;

$CF_{11,\kappa}^{3,1}$: 1) *a*) $\tilde{q}_1 = 0$, $\tilde{t}_2 = 0$, *b*) $L1_{11,\kappa}^{3,1}$, *and c*) $\sigma = \operatorname{sign}\tilde{q}_2$, $\kappa = \operatorname{sign}(\tilde{t}_1\tilde{q}_2)$, $u = \tilde{p}_1\tilde{q}_2^{-1}$; 2) *a*) $\kappa_{10} = 0$, $\tilde{t}_2 \neq 0$, $\kappa_{14} = 0$, *b*) $L2_{11,\kappa}^{3,1}$, *and c*) $\sigma = \operatorname{sign}\tilde{p}_1$, $\kappa = \operatorname{sign}(\tilde{p}_1\tilde{t}_1)$, $u = \kappa_{13}(\tilde{p}_1\tilde{t}_1)^{-1}$;

$CF_{27}^{4,1}$: *a*) $\kappa_{10} = 0$, $\tilde{t}_2 \neq 0$, $\kappa_{13} \neq 0$, $\kappa_{14} \neq 0$, $\kappa_{15} \neq 0$, 3.1_{15}, *b*) $L_{27}^{4,1}$, *and c*) $\sigma = \operatorname{sign}\tilde{t}_1$, $u = \kappa_{15}(\kappa_{14}\tilde{t}_2)^{-2/3}$, $v = \kappa_{13}(\kappa_{14}\tilde{t}_2)^{-2/3}$;

$CF_{21}^{3,1}$: *a*) $\kappa_{10} \neq 0$, $\tilde{q}_1 \neq 0$, $\kappa_{12} = 0$, $\kappa_9 = 0$, *b*) $L_{21}^{3,1}$, *c*) $\sigma = \operatorname{sign}\tilde{t}_1$, $u = \kappa_{10}(\tilde{q}_1 + \tilde{t}_2)^{-1/3}\tilde{t}_2^{-2/3}$;

$CF_{19}^{4,1}$: *a*) $\kappa_{10} \neq 0$, $\tilde{q}_1 \neq 0$, $\kappa_{12} = 0$, $\kappa_9 \neq 0$, $\kappa_{17} \neq 0$, $\kappa_{19} \neq 0$, *b*) $L_{19}^{4,1}$, *and c*) $\sigma = \operatorname{sign}\tilde{t}_1$, $u = \kappa_9(\tilde{p}_1\tilde{t}_1\tilde{t}_2)^{-2/3}$, $v = -(\tilde{q}_1 + 2\tilde{t}_2)(\tilde{p}_1\tilde{t}_1\tilde{t}_2)^{-1/3}$;

$CF_{33}^{4,1}$: *a*) $\kappa_{10} \neq 0$, $\kappa_{12} \neq 0$, $\kappa_9 = 0$, 3.1_{20}, *b*) $L_{33}^{4,1}$, *and c*) $\sigma = \operatorname{sign}\tilde{t}_1$, $u = \kappa_{12}\kappa_{10}^{-2}$, $v = \kappa_{16}\tilde{t}_2\kappa_{10}^{-3}$;

$CF_{11}^{4,1}$: 1) *a*) $\tilde{t}_2 = 0$, $\tilde{q}_1 \neq 0$, 3.1_9, *b*) $L1_{11}^{4,1}$, *and c*) $\sigma = \operatorname{sign}\tilde{q}_2$, $u = \tilde{p}_1\tilde{q}_2^{-1}$, $v = \tilde{q}_1^{-2}\tilde{q}_2\tilde{t}_1$; 2) *a*) $\kappa_{10} \neq 0$, $\tilde{t}_2 \neq 0$, $\kappa_{11} = 0$, 3.1_9, *b*) $L2_{11}^{4,1}$, *and c*) $\sigma = \operatorname{sign}(\kappa_{12}\tilde{t}_1)$, $u = \kappa_{16}\kappa_{12}^{-1}$, $v = \kappa_{12}\kappa_{10}^{-2}$;

$CF_{19}^{3,1}$: *a*) $\kappa_{10} \neq 0$, κ_{12}, $\kappa_{17} = 0$, *b*) $L_{19}^{3,1}$, *and c*) $\sigma = \operatorname{sign}\tilde{t}_1$, $u = -\kappa_{10}(2\tilde{q}_1^2\tilde{t}_2)^{-1/3}$;

$CF_6^{3,1}$: 1) *a*) $\kappa_{12} = 0$, $\kappa_{19} = 0$, *b*) $L1_6^{3,1}$, *and c*) $\sigma = \operatorname{sign}\tilde{t}_1$, $u = 2\tilde{q}_1\tilde{t}_2\kappa_{10}^{-2}$; 2) *a*) $\tilde{q}_1 = 0$, $\tilde{q}_2 = 0$, *b*) $L2_6^{3,1}$, *and c*) $\sigma = \operatorname{sign}\tilde{t}_1$, $u = \tilde{p}_1\tilde{t}_1\tilde{t}_2^{-2}$;

$CF_{30}^{4,1}$: *a)* $\kappa_{10}\neq 0$, $\kappa_{12}\neq 0$, $\kappa_{17}=0$, 3.1$_{18}$, *b)* $L_{30}^{4,1}$, *and c)* $u=\kappa_{10}(2\kappa_{20}\tilde q_1)^{-1/3}$, $v=4\kappa_{12}(2\kappa_{20}\tilde q_1)^{-2/3}$, $\sigma=\text{sign}\tilde t_1$;

$CF_{14}^{4,1}$: 1) *a)* $\kappa_{10}\neq 0$, $\tilde q_1\neq 0$, $\kappa_{12}\neq 0$, $\kappa_{18}=0$, 3.1$_{12}$, *b)* $L1_{14}^{4,1}$, *and c)* $\sigma=\text{sign}\tilde t_1$, $u=4\kappa_{12}\kappa_{10}^{-2}$, $v=-2\kappa_{20}\kappa_{10}^{-2}$;

2) *a)* $\tilde q_1=0$, $\tilde q_2\neq 0$, $\tilde t_2\neq 0$, 3.1$_{12}$, *b)* $L2_{14}^{4,1}$, *and c)* $\sigma=\text{sign}\tilde t_1$, $u=\tilde q_2\tilde t_1\tilde t_2^{-2}$, $v=\tilde p_1\tilde t_1\tilde t_2^{-2}$;

$CF_{29}^{4,1}$: *a)* $\kappa_{17}>0$, $\kappa_{24}^{\pm}=0$, $\kappa_{23}^{\mp}\neq 0$, 3.1$_{17}$, *b)* $L_{29}^{4,1}$, *and c)* $\sigma=\text{sign}\tilde t_1$, $u=\pm\kappa_{23}^{\mp}\kappa_{17}^{-1/2}/2$, $v=\pm\kappa_{21}^{\pm}\kappa_{22}^{\mp}\kappa_{17}^{-3/2}/4$;

$CF_5^{4,1}$: 1) *a)* $\tilde q_2\neq 0$, $\kappa_{25}\geq 0$, $\kappa_{28}^{\pm}=0$, 3.1$_7$, *b)* $L1_5^{4,1}$, *and c)* $\sigma=\text{sign}\tilde t_1$, $u=4\kappa_{30}^{\pm}(\kappa_{26}^{\mp})^{-2}$, $v=2\kappa_{26}^{\pm}(\kappa_{26}^{\mp})^{-1}$;

2) *a) at* $\tilde q_1\neq 0$, $\tilde q_2=0$, 3.1$_7$, *b)* $L2_5^{4,1}$, *and c)* $\sigma=\text{sign}\tilde t_1$, $u=\tilde p_1\tilde t_1\tilde t_2^{-2}$, $v=\tilde q_1\tilde t_2^{-1}$;

$CF_8^{5,1}$: *a)* $\tilde q_1\neq 0$, $\tilde q_2\neq 0$, $\tilde t_2\neq 0$, 3.2$_4$, *b) normalization* $L_8^{5,1}$, *and c)* $\sigma=\text{sign}\tilde q_2$, $u=\tilde p_1\tilde q_2^{-1}$, $v=\tilde q_1\tilde t_2^{-1}$, $w=\tilde q_2\tilde t_1\tilde t_2^{-2}$;

$CF_1^{4,1}$: 1) *a)* $\tilde q_1=0$, $\tilde q_2=2\tilde p_1$, $\tilde t_2\neq 0$, $\kappa_{31}>0$, 3.1$_5$, *b)* $L1_1^{4,1}$, *and c)* $\sigma=\pm\text{sign}(\kappa_{32}^{\pm}\tilde t_1)$, $u=-\kappa_{32}^{\mp}(\kappa_{32}^{\pm})^{-1}$;

2) *a)* $\tilde q_2=0$, $\tilde t_2\neq 0$, $\kappa_{31}\geq 0$, $\tilde q_1=\kappa_{32}^{\mp}$, 3.1$_5$, *b)* $L2_1^{4,1}$, *and c)* $\sigma=\text{sign}\,\tilde p_1$, $u=\kappa_{32}^{\pm}\tilde t_2(2\tilde p_1\tilde t_1)^{-1}$;

$CF_3^{4,1}$: 1) *a)* $\tilde p_1=\tilde q_1(\tilde q_1-\tilde t_2)\tilde t_1^{-1}$, $\tilde q_2=\tilde q_1(3\tilde q_1-2\tilde t_2)\tilde t_1^{-1}$, $\tilde t_2\neq 3\tilde q_1$, $\tilde q_2^2+\tilde t_2^2\neq 0$, 3.1$_6$, *b)* $L1_3^{4,1}$, *and c)* $\sigma=\text{sign}(\tilde t_1(3\tilde q_1-\tilde t_2)(\tilde q_1-\tilde t_2))$, $u=\tilde q_1(\tilde q_1-\tilde t_2)^{-1}$;

2) *a)* $\tilde p_1=\tilde q_1(2\tilde q_1-3\tilde t_2)(9\tilde t_1)^{-1}$, $\tilde q_2=0$, $\tilde t_2\neq 0$, 3.1$_6$, *b)* $L2_3^{4,1}$, *and c)* $\sigma=-\text{sign}(\tilde t_1\tilde t_2(2\tilde q_1-3\tilde t_2))$, $u=-\tilde q_1(3\tilde t_2)^{-1}$;

3) *a)* $\tilde p_1=(2\tilde q_1-3\tilde t_2)(2\tilde q_1+\tilde t_2)(16\tilde t_1)^{-1}$, $\tilde q_2=\tilde t_2(2\tilde q_1-3\tilde t_2)(8\tilde t_1)^{-1}$, $\tilde t_2\neq 0$, 3.1$_6$, *b)* $L3_3^{4,1}$, *and c)* $\sigma=\text{sign}((2\tilde q_1-3\tilde t_2)(2\tilde q_1+\tilde t_2)\tilde t_1)$, $u=-2\tilde t_2(2\tilde q_1+\tilde t_2)^{-1}$;

$CF_{13}^{4,1}$: 1) *a)* $\tilde p_1=(4\tilde q_1^2-\tilde t_2^2)(12\tilde t_1)^{-1}$, $\tilde t_2\neq 0$, $\tilde q_2=(2\tilde q_1-\tilde t_2)\tilde t_2(4\tilde t_1)^{-1}$, 3.1$_{11}$, *b)* $L1_{13}^{4,1}$, *c)* $\sigma=\text{sign}((2\tilde q_1-\tilde t_2)\tilde t_2\tilde t_1)$, $u=(2\tilde q_1+\tilde t_2)(3\tilde t_2)^{-1}$;

2) *a)* $\tilde t_2\neq 0$, $\tilde q_2\tilde t_1>0$, $\tilde p_1=(\tilde q_2\tilde t_1)^{1/2}\kappa_{34}^{\mp}\kappa_{35}^{\pm}(\kappa_{33}^{\pm})^{-2}\tilde t_1^{-1}$, *if* $\kappa_{33}^{\pm}\neq 0$, $\tilde q_1=\pm(2\tilde q_2\tilde t_1+\tilde t_2^2)(\kappa_{33}^{\pm})^{-1}$, $\kappa_{34}^{\pm}\neq 0$, 3.1$_{11}$, *b)* $L2_{13}^{4,1}$, *c)* $\sigma=-\text{sign}(\kappa_{34}^{\pm}\tilde t_1)$, $u=-\kappa_{34}^{\mp}\kappa_{33}^{\pm}(3\kappa_{35}^{\pm})^{-1}$;

$CF_{28}^{4,1}$: *a)* $\tilde q_1\neq 0, -\tilde t_2, -2\tilde t_2$, $\tilde q_2\neq 0$, $\tilde t_2\neq 0$, $\tilde p_1=\theta_*\tilde t_1^{-1}$, $\kappa_{38}=0$, $\kappa_{36}, \kappa_{37}, \kappa_{39}\neq 0$, *where* $\theta_*\in\mathbb{R}^1$ *is any zero of* $S_1(\theta)$, 3.1$_{16}$, *b)* $L_{28}^{4,1}$, *c)* $\sigma=\text{sign}(\kappa_{36}\kappa_{39}\tilde t_1)$, $u=-\kappa_{37}\kappa_{39}^{-1}$;

$CF_{32}^{4,1}$: *a)* $\tilde q_1\neq 0, -\tilde t_2, -2\tilde t_2$, $\tilde q_2\neq 0$, $\tilde t_2\neq 0$, $\kappa_{40}\geq 0$, $\kappa_{41}^{\pm}\neq 0$, $\tilde p_1=\kappa_{41}^{\pm}\tilde t_1^{-1}$, $\tilde q_2=((\tilde q_1+\tilde t_2)^2-3\kappa_{41}^{\pm})\tilde t_1^{-1}$, $\kappa_{42}^{\pm}\neq 0$, $\kappa_{43}^{\pm}\neq 0$, 3.1$_{19}$, *b)* $L_{32}^{4,1}$, *c)* $\sigma=\text{sign}(\kappa_{42}^{\pm}\tilde t_1)$, $u=3\kappa_{43}^{\pm}\kappa_{10}^{-2}$;

$CF_{36}^{4,1}$: *a)* $\tilde q_1\neq 0, -\tilde t_2, -3\tilde t_2/2, -2\tilde t_2$, $\tilde q_2\neq 0$, $\tilde t_2\neq 0$, $\kappa_{44}\geq 0$, $\kappa_{45}^{\pm}\neq 0$, $\tilde p_1=\kappa_{45}^{\pm}\tilde t_1^{-1}$, $\tilde q_2=-(3\kappa_{45}^{\pm}+2\tilde q_1\tilde t_2+3\tilde t_2^2)\tilde t_1^{-1}$, $\kappa_{46}^{\pm}\neq 0$, $\kappa_{47}^{\pm}\neq 0$, 3.1$_{21}$, *b)* $L_{36}^{4,1}$, *c)* $\sigma=\text{sign}(\kappa_{46}^{\pm}\tilde t_1)$, $u=\kappa_{47}^{\pm}\kappa_{10}^{-2}$;

$CF_3^{5,1}$: 1) *a)* $\tilde q_2\neq 0, 3\tilde p_1$, $\tilde t_1=\tilde q_1(2\tilde p_1\tilde q_1+\tilde t_2\tilde q_2-3\tilde t_2\tilde p_1)(\tilde q_2-3\tilde p_1)^{-2}$, $\tilde t_2\neq 0$, $\kappa_{48}\neq 0, -\tilde q_1\tilde q_2$, 3.2$_1$, *b)* $L1_3^{5,1}$, *c)* $\sigma=-\text{sign}\,\tilde p_1$, $u=-\kappa_{48}(\tilde p_1\tilde q_1)^{-1}$, $v=-(\tilde q_1\tilde q_2+\kappa_{48})(\tilde p_1\tilde q_1)^{-1}$; 2) *a)* $\tilde q_2\neq 0$, $\tilde t_1=-\tilde t_2(\tilde p_1\tilde t_2-2\tilde q_1\tilde q_2+\tilde q_2\tilde t_2)(2\tilde q_2)^{-2}$, $\tilde q_1\neq 0$, $\tilde t_2\neq 0$, $\kappa_{49}\neq 0, \tilde p_1\tilde t_2$, 3.2$_1$, *b)* $L2_3^{5,1}$, *c)* $\sigma=\text{sign}\tilde q_2$, $u=\tilde p_1\tilde q_2^{-1}$, $v=\kappa_{49}(\tilde q_2\tilde t_2)^{-1}$;

3) *a)* $\tilde q_1\neq 0, \tilde t_2$, $\tilde q_2=(\tilde t_2\tilde q_1^3-(\tilde p_1\tilde t_1+2\tilde t_2^2)\tilde q_1^2-\tilde t_2(2\tilde p_1\tilde t_1-\tilde t_2^2)\tilde q_1+\tilde p_1\tilde t_1(4\tilde p_1\tilde t_1+3\tilde t_2^2))\tilde t_1^{-1}(\tilde q_1-\tilde t_2)^{-2}\neq 0$, $\tilde t_2\neq 0$, $\kappa_{50}\neq 0$, $\kappa_{51}\neq 0$, $\kappa_{52}\neq 0$, 3.2$_1$, *b)* $L3_3^{5,1}$, *c)* $\sigma=-\text{sign}(\kappa_{50}\tilde p_1)$, $u=-\kappa_{51}(\tilde p_1\tilde t_1)^{-1}$, $v=-(\tilde q_1-\tilde t_2)^2(\tilde p_1\tilde t_1)^{-1}$;

$CF_6^{5,1}$: 1) *a)* $\tilde q_1\neq 0$, $\tilde t_2\neq 0$, $\tilde p_1=4\kappa_{53}\tilde q_2\tilde t_2^{-2}/3$, $\kappa_{54}\neq 0$, $4\kappa_{53}\neq 3\kappa_{54}$, 3.2$_2$, *b)* $L1_6^{5,1}$, *c)* $\sigma=\text{sign}\tilde q_2$, $u=4\kappa_{53}(3\tilde t_2^2)^{-1}$, $v=\kappa_{54}\tilde t_2^{-2}$;

2) *a)* $\tilde q_1\neq 0$, $\tilde q_2\neq 0$, $\tilde t_2\neq 0$, $\tilde p_1=-\kappa_{55}(9\tilde t_1\theta_*^2)^{-1}$, $\kappa_{56}\neq 0$, $\kappa_{57}\neq 0$, $\kappa_{58}\neq 0$, $\kappa_{59}\neq 0$, $\kappa_{57}\neq\kappa_{58}$, *where* $\theta_*\in\mathbb{R}^1$*—any zero of* $S_2(\theta)$, 3.2$_2$, *b)* $L2_6^{5,1}$, *c)* $\sigma=-\text{sign}(\kappa_{59}\tilde t_1)$, $u=3\kappa_{57}\theta_*\kappa_{59}^{-1}$, $v=3\kappa_{58}\theta_*\kappa_{59}^{-1}$;

$CF_7^{5,1}$: 1) *a)* $\tilde q_2\neq 0, 3\tilde p_1$, $\tilde t_2\neq 0$, $\tilde t_1=\kappa_{60}\tilde q_1(\tilde q_2-3\tilde p_1)^{-2}$, $\kappa_{61}\neq 0, -\tilde q_1\tilde q_2$, 3.2$_3$, *b)* $L1_7^{5,1}$, *c)* $\sigma=\text{sign}\,\tilde p_1$, $u=\kappa_{61}(\tilde p_1\tilde q_1)^{-1}$, $v=(\kappa_{61}+\tilde q_1\tilde q_2)(\tilde p_1\tilde q_1)^{-1}$;

2) *a*) $\tilde{q}_1 \neq 0$, $\tilde{q}_2 \neq 0$, $\tilde{t}_2 \neq 0$, $\tilde{p}_1 = -\kappa_{55}(9\tilde{t}_1\theta_*^2)^{-1}$, $\kappa_{56} \neq 0$, $\kappa_{59} \neq 0$, $3\kappa_{62}$, $\kappa_{62} \neq 0$, $\kappa_{63} \neq 0$, *where* $\theta_* \in \mathbb{R}^1$ *is any zero of* $S_3(\theta)$, 3.2_3, *b*) $L2_7^{5,1}$, *c*) $\sigma = \mathrm{sign}\kappa_{63}$, $u = \kappa_{59}(3\kappa_{63}\tilde{t}_1\theta_*^2)^{-1}$, $v = \kappa_{62}(\kappa_{63}\tilde{t}_1\theta_*^2)^{-1}$.

Here 3.1_i *means that elements of system* (4.7) *are so that the values of parameters composing its CF do not satisfy the conditions from item i of Statement* 3.1; *label* 3.2_j *has a similar mean*; *constants* ϑ, κ, *polynomials* $S(\theta)$ *and linear substitutions of* J, L *are given in Collections* 2.1 *and* 3.1.

The proof is available in the file lemma3.mw stored in the database (see Introduction).

Theorem 5.1. *At* $l = 1$, *any system* $(2.1)^1$, *written in the form* (4.1) *in accordance with formulas* $(2.10)^1$, *is linearly equivalent to a representative of a certain* $CF_i^{m,1}$ *from List* 2.1. *With that, if coefficients* β, p_k, q_k, *and* t_k $(k = 1, 2)$ *of system* (4.1) *are so that*:

1) *in system* (4.2) $\hat{t}_1 = 0$, $\hat{q}_1 = 0$ *and for each of three* $CF_i^{m,1}$ *from List* 2.1_1, *coefficients* $\tilde{p}_1$ *and* $\tilde{p}_2$ *of system* (4.5) *meet the conditions provided by Lemma* 5.1, *then composition of substitutions* J_0^1, J_1^1 *and* $L_i^{m,1}$ *transforms the right-hand part of system* (4.1) *into* $CF_i^{m,1}$ *chosen with values of the parameters from Lemma* 5.1;

2) *in system* (4.2) $\hat{t}_1 = 0$, $\hat{q}_1 \neq 0$ *and for each of five* $CF_i^{m,1}$ *from List* 2.1_2, *coefficients* $\tilde{q}_1$, $\tilde{p}_2$, *and* $\tilde{q}_2$ *of system* (4.6) *meet the conditions formulated in Lemma* 5.2, *then composition of substitutions* J_0^1, J_2^1, *and* $L_i^{m,1}$ *transforms the right-hand part of system* (4.1) *into* $CF_i^{m,1}$ *chosen with values of the parameters from Lemma* 5.2;

3) *in system* (4.2) $\hat{t}_1 \neq 0$, *and for* 25 $CF_i^{m,1}$ *from Lists* 2.1_3 *and* 2.1_{II}, *coefficients* $\tilde{p}_1$, $\tilde{t}_1$, $\tilde{q}_1$, $\tilde{q}_2$, *and* $\tilde{t}_2$ *of system* (4.7) *meet the conditions provided by Lemma* 5.3, *then composition of substitutions* J_0^1, J_3^1, *and* $L_i^{m,1}$ *transforms the right-hand part of system* (4.1) *into* $CF_i^{m,1}$ *chosen with values of the parameters from Lemma* 5.3.

Proof. It is shown in Section 4 that the initial system (4.1) can be reduced by means of two linear substitutions to either (4.5), (4.6) or (4.7).

In turn, according to Lemmas 5.1, 5.2, and 5.3, any of the above systems can be reduced to the appropriate representative of the corresponding $CF_i^{m,1}$ from List 2.1.

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