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MATHEMATICS

Two-Dimensional Homogeneous Cubic Systems: Classification and Normal Forms. I

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Abstract—This work is the first in a series of papers devoted to classifying of two-dimensional homogeneous cubic systems based on partitioning into classes of linear equivalence. Principles have been developed that are capable of constructively distinguishing the structure of a simplest system in each class and a canonical set that defines the admissible values that can be assumed by its coefficients. The polynomial vector in the right-hand part of this system identified with a 2×4 matrix is called the canonical form (CF) and the system itself is called the normal cubic form. One of the main objectives of this series of papers is to maximally simplify the reduction of a system with a homogeneous cubic polynomial in the unperturbed part to the various structures of a generalized normal form (GNF). *Generalized normal form* refers to a system in which the perturbed part has the simplest form in some sense. The constructive implementation of the normalization process depends on the ability to explicitly specify the conditions of compatibility and possible solutions of the so-called bonding system, which is understood to be a countable set of linear algebraic equations that specify the normalizing transformations of the perturbed system. The above principles are based on the idea of the maximum possible simplification of the bonding system. This will allow one to first reduce the initial perturbed system by an invertible linear substitution of variables to a system with some CF in the unperturbed part, then reduce the resulting system, which is optimal for normalization, by almost identical substitutions to various structures of the GNF. In this paper, the following tasks are carried out: (1) the general problem is set, close problems are formulated, and the available results are described; (2) a bonding system is derived that is capable of determining the equivalence of any two perturbed systems with the same homogeneous cubic part, the possibilities of its simplification are discussed, the GNF is defined, and the method of resonant equations is given allowing one to constructively obtain all its structures; (3) special forms of recording homogeneous cubic systems in the presence of a common homogeneous factor in their right-hand parts with a degree of 1–3 are introduced, and the linear equivalence of these systems, as well as of systems without a common factor is studied, and key linear invariants are offered.

Keywords: homogeneous cubic system, normal form, canonical form.

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INTRODUCTION

1.1. Statement of the Problem

The present series of papers is concerned with a real two-dimensional nondegenerate homogeneous cubic system of ODE

$$\dot{x} = P(x), \tag{1.1}$$

where $x = (x_1, x_2)$, $P = (P_1(x), P_2(x))$, $P_i = a_i x_1^3 + b_i x_1^2 x_2 + c_i x_1 x_2^2 + d_i x_2^3$, $P_1, P_2 \not\equiv 0$.

We assume that the real nonsingular linear substitution

$$x = Ly \quad (y = (y_1, y_2),\ \det L \neq 0) \tag{1.2}$$

transforms system (1.1) into the system

$$\dot{y} = \tilde{P}(y) \quad (\tilde{P}_i = \tilde{a}_i y_1^3 + \tilde{b}_i y_1^2 y_2 + \tilde{c}_i y_1 y_2^2 + \tilde{d}_i y_2^3,\ i = 1, 2). \tag{1.3}$$

The main problems that will be posed and solved in this series of papers are as follows.

(1) Classify the set of systems (1.1) by partitioning the vector polynomials $P(x)$ into classes of linear equivalence.

(2) For each class, develop structural and normalization principles that enable one to perfectly order the polynomials $\tilde{P}$ obtained as a result of substitution (1.2).

(3) Based on the selected principles to highlight in each class a generator, which is the simplest polynomial $\tilde{P}$ (called the canonical form (CF)).

It will be shown that any CF can be identified with a 2×4 matrix of coefficients of the polynomial $\tilde{P}$ with fixed zero elements, while for nonzero elements the canonical sets are specified describing all their admissible values.

A system with CF in the right-hand side is naturally called a cubic normal form.

In parallel with the principal problems, we shall solve the four accompanying auxiliary numerical problems, which enables one to efficiently employ the classification in practice. For each CF, we will explicitly write the following:

(a) the conditions on the coefficients of the vector polynomial $P(x)$;

(b) the substitution (1.2) that transforms the polynomial $P(x)$ under the specified conditions into the selected CF;

(c) the values of the elements of the CF from the canonical set;

(d) the minimal canonical set in which there are no values of the elements from which one may get rid by substitutions (1.2), which conserve the structure of the CF.

Aside from their intrinsic interest related to the development of classification of homogeneous cubic systems, the results obtained aim primarily to facilitate the normalization of perturbed systems by making a preliminary reduction of their unperturbed part by substitutions (1.2) to canonical forms, which is succeeded by the normalization of perturbations in the systems thus obtained.

This purpose will be the underlying motif for the principles that enable one to single out the CF.

It is worth noting that a good deal of symbolic calculations related to various linear transforms of homogeneous cubic systems, their normalization and singling out common factors of various degrees, as well as solutions of various algebraic systems and equations would be impossible without the machinery of symbolic mathematics. For these purposes, the Maple analytical software package is available. There is a set of standard subroutines, based on which Maple software packages can be utilized to justify almost each assertion.

1.2. Formal Equivalence of Perturbed Systems

We consider a two-dimensional real perturbed formal (analytic at zero) system

$$\dot{x}_i = P_i(x) + X_i(x) \quad (i = 1, 2), \tag{1.4}$$

where the polynomials P_i are taken from system (1.1), $X_i = \sum_{p=4}^{\infty} X_i^{(p)}(x)$ are perturbations. Here and in what follows, $Z_i^{(p)}(z) = \sum_{s=0}^{p} Z_i^{(s,p-s)} z_1^s z_2^{p-s}$ is a homogeneous polynomial of order p.

Assume that a formal real almost identical substitution of variables

$$x_i = y_i + h_i(y) \tag{1.5}$$

with $h_i = \sum_{p=2}^{\infty} h_i^{(p)}(y)$ transforms (1.4) into the system

$$\dot{y}_i = P_i(y) + Y_i(y), \tag{1.6}$$

which has a similar structure.

Differentiating the substitution (1.5) in t by virtue of systems (1.4) and (1.6) and setting $H_i(y, h) = P_i(h) + \sum_{j=1}^{2} y_j \partial P_i(h)/\partial h_j$ $(i = 1, 2)$, we get two identities

$$\sum_{j=1}^{2}\left(\frac{\partial h_i(y)}{\partial y_j} P_j(y) - \frac{\partial P_i(y)}{\partial y_j} h_j(y)\right) + Y_i(y) = X_i(y+h) + H_i(y,h) - \sum_{j=1}^{2} \frac{\partial h_i(y)}{\partial y_j} Y_j(y).$$

Singling out in them, for each $p \geqslant 4$, the homogeneous polynomials of order p, we prove that the homogeneous polynomials $h_j^{(p-2)}$ and $Y_i^{(p)}$ satisfy the recurrence relations

$$(a_1y_1^3 + b_1y_1^2y_2 + c_1y_1y_2^2 + d_1y_2^3)\frac{\partial h_i^{(p-2)}}{\partial y_1} + (a_2y_1^3 + b_2y_1^2y_2 + c_2y_1y_2^2 + d_2y_2^3)\frac{\partial h_i^{(p-2)}}{\partial y_2} -$$
$$- (3a_iy_1^2 + 2b_iy_1y_2 + c_iy_2^2)h_1^{(p-2)} - (b_iy_1^2 + 2c_iy_1y_2 + 3d_iy_2^2)h_2^{(p-2)} + Y_i^{(p)} = \tilde{Y}_i^{(p)},$$

in which $\tilde{Y}_i^{(p)} = \{X_i(y + h) + H_i(y, h) - Y_1\partial h_i/\partial y_1 - Y_2\partial h_i/\partial y_2\}^{(p)}$, $(i = 1, 2)$.

Clearly, for a successive (with respect to $p \geqslant 4$) definition of $h_i^{(p-2)}$ and $Y_i^{(p)}$ the homogeneous polynomials $\tilde{Y}_i^{(p)}(y)$ become known, since they depend only on $h_j^{(r-2)}$ and $Y_j^{(r)}$ with $2 \leqslant r \leqslant p - 1$ $(j = 1, 2)$.

Equating the coefficients with $y_1^sy_2^{p-s}$ $(p \geqslant 4, s = \overline{0, p})$, we obtain a linear bonding system of $2p + 2$ equations with $2p - 2$ unknowns $h_i^{(0,p-2)}, \ldots, h_i^{(p-2,0)}$,

$$\begin{aligned}
&a_2(p - s + 1)h_1^{(s-3,p-s+1)} + (a_1(s - 2) + b_2(p - s) - 3a_1)h_1^{(s-2,p-s)} + \\
&+ (b_1(s - 1) + c_2(p - s - 1) - 2b_1)h_1^{(s-1,p-s-1)} + (c_1s + d_2(p - s - 2) - c_1)h_1^{(s,p-s-2)} + \\
&+ d_1(s + 1)h_1^{(s+1,p-s-3)} - b_1h_2^{(s-2,p-s)} - 2c_1h_2^{(s-1,p-s-1)} - 3d_1h_2^{(s,p-s-2)} = \hat{Y}_1^{(s,p-s)}, \\
&a_2(p - s + 1)h_2^{(s-3,p-s+1)} + (a_1(s - 2) + b_2(p - s) - b_2)h_2^{(s-2,p-s)} + \\
&+ (b_1(s - 1) + c_2(p - s - 1) - 2c_2)h_2^{(s-1,p-s-1)} + (c_1s + d_2(p - s - 2) - 3d_2)h_2^{(s,p-s-2)} + \\
&+ d_1(s + 1)h_2^{(s+1,p-s-3)} - 3a_2h_1^{(s-2,p-s)} - 2b_2h_1^{(s-1,p-s-1)} - c_2h_1^{(s,p-s-2)} = \hat{Y}_2^{(s,p-s)},
\end{aligned} \tag{1.7}$$

in which $\hat{Y}_i^{(s,p-s)} = \tilde{Y}_i^{(s,p-s)} - Y_i^{(s,p-s)}$ $(i = 1, 2)$.

Thus, systems (1.4) and (1.6) are equivalent if there is a substitution (1.5), the coefficients of which satisfy the bonding system (1.7).

Clearly, a bonding system (4) in [1] is a particular case of (1.7).

1.3. The Method of Resonant Equations and the Determination of the Generalized Normal Form

Compatibility conditions for the bonding system with any $\forall p \geqslant 4$ can be written as a system of n_p linearly independent linear equations $(n_p \geqslant 4)$ that relate the coefficients of the polynomials $Y_i^{(p)}$ in system (1.6) as follows:

$$\sum_{s=0}^{p}(\alpha_{\mu s}^pY_1^{(s,p-s)} + \beta_{\mu s}^pY_2^{(s,p-s)}) = \sum_{s=0}^{p}(\alpha_{\mu s}^p\tilde{Y}_1^{(s,p-s)} + \beta_{\mu s}^p\tilde{Y}_2^{(s,p-s)}) \quad (\mu = 1, \ldots, n_p). \tag{1.8}$$

These equations are called resonant.

Here, two countable families of constant vectors α_μ^p and β_μ^p that specify equations (1.8) are determined only by the coefficients of $P(x)$ and are independent of the perturbations. They enable one to establish a formal equivalence between any two systems with the same unperturbed part.

For system (1.4), we present a brief account the concept (see references in [1]) of a resonant family and the definition of the GNF and recall the existence theorem for the GNF.

Definition 1.1. The coefficients of the polynomials $Y_i^{(p)}$ in (1.6) that enter into at least one of the equations (1.8), and the coefficients of the polynomials $h_i^{(p-2)}$ in (1.8) that remain free in solving system (1.7) will be called *resonant*; the remaining coefficients will be called *nonresonant*.

To any n_p different resonant coefficients $Y^{p,k} = Y_{i_k}^{(s_k,p-s_k)}$ of homogeneous polynomials $Y_1^{(p)}$, $Y_2^{(p)}$, where $k = \overline{1, n_p}$, $i_k \in \{1,2\}$, $0 \leqslant s_k \leqslant p$, we associate the matrix of factors $\Upsilon^p = \{v_{\mu k}^p\}_{\mu,k=1}^{n_p}$, in which $v_{\mu k}^p = \{\alpha_{\mu s_k}^p$ for $i_k = 1$, $\beta_{\mu s_k}^p$ for $i_k = 2\}$.

Definition 1.2. For any $p \geqslant 4$, the family of resonant coefficients $\mathcal{Y}^p = \{Y_k^p\}_{k=1}^{n_p}$ will be called a *resonant p-family* if $\det \Upsilon^p \neq 0$.

So, for any $p \geqslant 4$, resonant equations are uniquely solvable with respect to the coefficients from any $\mathcal{Y}^p$.

Definition 1.3. For any $\mathcal{Y}^4$, $\mathcal{Y}^5$, ..., the family $\mathcal{Y} = \bigcup_{p=4}^{\infty} \mathcal{Y}^p$ is called a resonant family.

Definition 1.4. A system (1.6) is called a *GNF* if, for any $p \geqslant 4$, all of the coefficients $Y_i^{(p)}$ (both resonant and nonresonant) are zero, except for the coefficients from some resonant p-family $\mathcal{Y}^p$, which are allowed to have arbitrary values.

In this way, the structure of any GNF is generated by some resonant family $\mathcal{Y}$. The knowledge of the resonant equations (1.8) makes the following theorems clear.

Theorem 1.1. *System* (1.6) *is formally equivalent to the original system* (1.4) *if and only if, for all* $k \geqslant 2$, *the coefficients of its homogeneous polynomials* $Y_1^{[k]}$, $Y_2^{[k]}$ *satisfy the resonant equations* (1.8).

Theorem 1.2. *For any system* (1.4) *and for any resonant family* $\mathcal{Y}$ *chosen from its unperturbed part there is an almost identical substitution that transforms* (1.4) *into the GNF* (1.6), *the structure of which is generated by* $\mathcal{Y}$.

We note that there are various definitions of GNF (see, e.g., [2–5]), which depend both on the choice of the terms that pertain to the unperturbed part of the original system and the required degree of simplification. It is worth noting that not all definitions are constructing, and certain efforts are required in order to verify their well-posedness and establish the form of the GNF. Thus, a nontrivial example of a complete Belitskii NF only appeared in [4] 20 years after this normal form was introduced in [2]. The definition of a GNF given in the present paper corresponds to the definition of a first-order GNF from [3].

Clearly, the constructive utilization of the above method for the explicit generation of all possible structures of GNF for system (1.4), which the author calls the *method of resonant equations*, depends solely on a possibility of writing down the compatibility conditions of the bonding system that specifies for each order p the number n_p of resonant equations (1.8) and, what is much more difficult, on the possibility of finding α^p and β^p in explicit form for (1.8), which enables one to write all resonant families.

For the successful solution of this problem, the bonding system should have the simplest form and, hence, the vector polynomial P should have the largest possible number of zero coefficients located (if possible) at optimal places, and the nonzero coefficients should be optimally normalized. Hence, a CF will be introduce to best fit the above requirements.

1.4. On the Possibility of Simplifying the Bonding System

The matrix of the linear bonding system (1.7) depends on eight coefficients of the polynomials P_1 and P_2 of system (1.4). The number and form of the constraints imposed on the right-hand sides of system (1.7) described by the resonant equations (1.8), as well as the possibility of constructively ascertaining these bonds, depend on which of these coefficients are zero and the number of zero coefficients.

Thus, we shall study the structure of the bonding system in order to be able to correctly formulate the principles of selection of the coefficients of P, which one should try by a linear nonsingular substitution of variables to make zero in the first head.

We take as a basis the principle of the maximality of the number of zero coefficients in the vector polynomial P and consider their various arrangement (for example, in (1.7_1)).

The presence of the zero coefficient a_1 has almost no effect on the structure of the bonding system.

The most favorable situation occurs when $b_1, c_1, d_1 = 0$. In this case (1.7_1) is an independent linear system, which, in the worst case, has the four-diagonal matrix.

After a study of its compatibility, finding the coefficients of the polynomial $h_1^{(p)}$ and substitution of these coefficients in (1.7_2), a scrutable linear system with a matrix that has at most four diagonals will again appear.

There will also be no principally new problems in the case when only one of the coefficients b_1, c_1, d_1 is nonzero.

We assume that, for example, $b_1 = c_1 = 0$, $d_1 \neq 0$. Now we again may confine ourselves to solving linear systems with diagonal matrices with a bounded number of diagonals. Indeed, for any $p \geqslant 2$, the subsystem

(1.7_1) can be solved with respect to coefficients of the polynomial $h_2^{(p)}$, after the substitution of which in (1.7_2) we obtain a linear system with respect to the coefficients $h_1^{(p)}$ in which the number of diagonals is independent of p.

At the same time, if only one of the coefficients b_1, c_1, d_1 is zero, then an attempt to solve the subsystem (1.7_1) with respect to the coefficients $h_2^{(p)}$ with their successive substitution into (1.7_2) makes the resulting matrix nondiagonal, and makes it impossible to find a constructive solution to the bonding system.

Hence, in the study of the compatibility of (1.7), it is essential whether or not the following condition on the coefficients of the unperturbed part of system (1.4) is satisfied:

$$(b_1^2c_1^2 + c_1^2d_1^2 + d_1^2b_1^2)(a_2^2b_2^2 + b_2^2c_2^2 + c_2^2a_2^2) = 0. \tag{1.9}$$

When satisfied, this condition means that, among the coefficients b_1, c_1, d_1 or a_2, b_2, c_2, only one coefficient may be different from zero.

In addition to condition (1.9), we also give a number of reasons for the simplification of the study of the compatibility and solution of the bonding system.

1. The less weakly related the equations of a unperturbed system (that is, the less maximal degree of the variable x_2 in P_1 and x_1 in P_2), the smaller the number of diagonals that will have the matrix of the bonding system.

2. If it is possible to choose only one zero coefficient, then (1.7) admits a maximal simplification for $d_1 = 0$ ($a_2 = 0$) because two terms will disappear in the left-hand side of (1.7_1) and one term will disappear in (1.7_2).

3. If it is possible to choose two zero coefficients, then it is better to take the pairs c_1, d_2 or a_1, b_2 as these coefficients. This has the same effect as is achieved with $d_1 = 0$. The most optimal case we have is $b_1 = c_2 = 0$; in this case, in each equation of system (1.7), two terms will disappear.

The above arguments will underlie the hierarchic structural principles capable of splitting the set of unperturbed parts of system (1.4) into equivalence classes with respect to substitutions (1.2) upon singling out the best representative in each class (a canonical form) for the purpose of initial reduction of an arbitrary system (1.4) by a linear nonsingular substitution to system (1.6) with a CF in the unperturbed part, which is followed by a reduction of (1.6) by an almost identical substitution to a generalized normal form. This provides much greater possibilities for constructively obtaining the resonant terms of each order and write down all the resonant families.

1.5. Survey of the Available Results under a More General Setting of the Problem

This series of papers finishes solving the following, much more general problem: to single out all nondegenerate CFs, the degree of which is at most three and, if possible, to constructively obtain all GNF systems with these CF in the unperturbed parts.

In the case when the degrees of the unperturbed exceed three, the technical difficulties make it impossible, with rare exception (see, e.g., [6, 7]), to solve the problem in the same generality.

So, let us consider what has been done for the real formal system

$$\dot{x}_1 = Q_1^{(k)}(x) + X_1(x), \quad \dot{x}_2 = Q_2^{(m)}(x) + X_2(x) \quad (1 \le k \le m \le 3), \tag{1.10}$$

in which $Q_i^{(l)}$ are homogeneous polynomial of degree l and $Q_1^{(k)}(x)$, $Q_2^{(k)}(x) \neq 0$, and all the terms of the perturbation X have degrees that are greater in a sense than $Q = (Q_1^{(k)}, Q_2^{(m)})$.

1. Case $(k, m) = (1, 1)$. In this case, we have $Q(x) = Ax$, and Jz is a unique CF, where J is the Jordan form of A. The most comprehensive treatment of the theory of normal forms for systems (1.10) of arbitrary dimension is given by Bryuno under the condition that not all eigenvalues of A are zero [8].

The canonical form becomes much more complicated if one assumes that the unperturbed part Ax of system(1.10) of arbitrary dimension is Hamiltonian (see [9]). The normalization of Hamiltonian systems with simplest CF in the unperturbed part was already considered in [10].

The canonical forms of contact systems, which in a sense extend the Hamiltonian systems, were obtained by Lychagin in [11, Ch. 3, Sec. 2].

2. The case $(k, m) = (2, 2)$. In this case $Q_i = a_ix_1^2 + 2b_ix_1x_2 + c_ix_2^2$ $(i = 1, 2)$. In [12], the set of homogeneous quadratic systems was first partitioned into 19 equivalence classes with respect to linear nonsingular

substitutions and, in each class, based on principles that are capable of delivering a maximally simple bonding system, each own CF was singled out and the admissible ranges of variation of its elements were identified. Next, for each CF, the conditions on the six coefficients of the polynomial Q and a substitution (1.2) that transform Q into a selected CF were explicitly specified. Finally, in papers [13–15] for systems (1.10) with eleven different CFs in the unperturbed part, all structures of the GNF were explicitly given. For linear invariants, the highlighting of which is based on different principles, the reader is referred to [16].

3. The case $(k, m) = (3, 3)$. This case is clearly the subject of the present study, which in a certain sense continues and develops the ideas and methods employed in the study of the case $(k, m) = (2, 2)$.

There are already some applications of the theory being developed. In [1] as P the CF $(x_2^3, -x_1^3)$ was chosen, for which system (1.1) is conservative, the method of resonant equations was employed to obtain all the structures of the GNF that are formally equivalent to the perturbed system (1.4).

Convention 1.1. An unperturbed system with some CF on the right-hand side in cases 1)–3) is naturally called a linear, quadratic, or cubic normal form, correspondingly.

In the cases when $k < m$, singling out of canonical forms for unperturbed system $\dot{x} = Q(x)$ in the previous understanding is not always possible, but only for those values of the coefficients for which Q can be written as a normalized quasi-homogeneous polynomial of certain generalized degree and weight of variables (and which is called the canonical quasi-homogeneous form (CQHF).

4. Case $(k, m) = (1, 2)$. In the reference 8 of [12] two CQHFs are singled out; moreover, all possible structures of the GNF were found in the unperturbed part for systems (1.10) for each of these CQHFs.

5. Case $(k, m) = (1, 3)$. In references 1 and 2 of [6], two CQHFs are singled out; for each of these CQHFs, the same problems were solved.

6. Case $(k, m) = (2, 3)$. In [17], seven CQHFs are singled out; for each of these CQHFs, the same problems were solved.

The normalization of systems with a degenerated unperturbed part when, e.g., $Q_2^{(m)} \equiv 0$ in system (1.10), is of special importance. The first serious results in this direction were obtained in [18, 19]. In [12], five quadratic degenerated CFs were singled out; for systems with each of them in the unperturbed part all the GNFs were written down which can be obtained by almost identical formal substitutions. Cubic degenerated CFs will be singled out and studied in the present paper.

LINEAR EQUIVALENCE OF HOMOGENEOUS CUBIC SYSTEMS

2.1. Form of the System and the Resultant

Consider a real two-dimensional homogeneous cubic system (1.1), which is written in the form

$$\dot{x} = P(x) \quad \text{or} \quad \dot{x} = Aq^{[3]}(x), \tag{2.1}$$

where $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} a_1x_1^3 + b_1x_1^2x_2 + c_1x_1x_2^2 + d_1x_2^3 \\ a_2x_1^3 + b_2x_1^2x_2 + c_2x_1x_2^2 + d_2x_2^3 \end{pmatrix}, A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}, x = \text{colon}(x_1, x_2), q^{[3]}(x) =$ $\text{colon}(x_1^3, x_1^2x_2, x_1x_2^2, x_2^3), A_1, A_2 \neq 0$.

Convention 2.1. Below, for brevity, the matrix coefficients A will be identified with system (2.1) (or say that the matrix A generates system (2.1)).

If desired, both matrix A and system (2.1) may be called nondegenerate because it is assumed that both of its rows are nonzero and $A_1, A_2 \neq 0 \Leftrightarrow P_1(x), P_2(x) \not\equiv 0$.

Definition 2.1. We let P_0 denote any homogeneous polynomial with real coefficients which is a common factor of P_1 and P_2. A common factor P_0 of maximal degree l ($l = 1, 2, 3$) will be denoted by P_0^l. If there is no common factor, then we shall assume that $l = 0$.

For vectors $r = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}, s = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$, we consider the function $\delta_{rs} = \begin{vmatrix} r_1 & s_1 \\ r_2 & s_2 \end{vmatrix} = r_1s_2 - r_2s_1$. The function $R = R(P_1, P_2)$, which is called the resultant,

$$R = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & 0 & 0 \\ 0 & a_1 & b_1 & c_1 & d_1 & 0 \\ 0 & 0 & a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 & 0 & 0 \\ 0 & a_2 & b_2 & c_2 & d_2 & 0 \\ 0 & 0 & a_2 & b_2 & c_2 & d_2 \end{vmatrix} = \delta_{ad}^3 + \delta_{ac}^2\delta_{cd} + \delta_{ab}\delta_{bd}^2 - 2\delta_{ab}\delta_{ad}\delta_{cd} - \delta_{ab}\delta_{bc}\delta_{cd} - \delta_{ac}\delta_{ad}\delta_{bd}.$$

is capable of testing the existence or absence of a common factor for any two polynomials.

Assertion 2.1 (see [20, Sec. 50]). *Polynomials P_1, P_2 have a real common factor P_0 of nonzero degree if and only if $R(P_1, P_2) = 0$.*

2.2. Linear Transformations of a System

To simplify system (2.1), we shall employ the nonsingular linear substitutions

$$\begin{cases} x_1 = r_1y_1 + s_1y_2, \\ x_2 = r_2y_1 + s_2y_2 \end{cases} \quad \text{or} \quad x = Ly, \quad L = \begin{pmatrix} r_1 & s_1 \\ r_2 & s_2 \end{pmatrix}, \quad \delta = \det L \neq 0. \tag{2.2}$$

Assume that substitution (2.2) transforms system (2.1) into the system

$$\dot{y} = \tilde{P}(y) \quad \text{or} \quad \dot{y} = \tilde{A}q^{[3]}(y), \tag{2.3}$$

where $\tilde{P} = \begin{pmatrix} \tilde{P}_1 \\ \tilde{P}_2 \end{pmatrix} = \begin{pmatrix} \tilde{a}_1y_1^3 + \tilde{b}_1y_1^2y_2 + \tilde{c}_1y_1y_2^2 + \tilde{d}_1y_2^3 \\ \tilde{a}_2y_1^3 + \tilde{b}_2y_1^2y_2 + \tilde{c}_2y_1y_2^2 + \tilde{d}_2y_2^3 \end{pmatrix}$, $\tilde{A} = \begin{pmatrix} \tilde{a}_1 & \tilde{b}_1 & \tilde{c}_1 & \tilde{d}_1 \\ \tilde{a}_2 & \tilde{b}_2 & \tilde{c}_2 & \tilde{d}_2 \end{pmatrix}$.

For the polynomials $\tilde{P}_1$, $\tilde{P}_2$, in analogy with R, we introduce the resultant $\tilde{R} = R(\tilde{P}_1, \tilde{P}_2)$.

Differentiating (2.2) by virtue of systems (2.1) and (2.3), we obtain $P(Ly) = L\tilde{P}(y)$; thus,

$$\tilde{P}(y) = L^{-1}P(Ly) = L^{-1}Aq^{[3]}(Ly). \tag{2.4}$$

It follows that

$$\begin{pmatrix} \tilde{a}_1y_1^3 + \tilde{b}_1y_1^2y_2 + \tilde{c}_1y_1y_2^2 + \tilde{d}_1y_2^3 \\ \tilde{a}_2y_1^3 + \tilde{b}_2y_1^2y_2 + \tilde{c}_2y_1y_2^2 + \tilde{d}_2y_2^3 \end{pmatrix} = \delta^{-1}\begin{pmatrix} \delta_{as} & \delta_{bs} & \delta_{cs} & \delta_{ds} \\ -\delta_{ar} & -\delta_{br} & -\delta_{cr} & -\delta_{dr} \end{pmatrix}\times$$
$$\times\,\mathrm{colon}\Big((r_1y_1 + s_1y_2)^3, (r_1y_1 + s_1y_2)^2(r_2y_1 + s_2y_2), (r_1y_1 + s_1y_2)(r_2y_1 + s_2y_2)^2, (r_2y_1 + s_2y_2)^3\Big).$$

In this identity, equating the coefficients of $y_1^s y_2^{3-s}$ ($s = \overline{0,3}$) and repositioning the terms, we obtain eight equalities in the matrix form

$$\tilde{A} = \delta^{-1}\begin{pmatrix} \delta_{P(r)s} & s_1\delta_{\frac{\partial P(r)}{\partial r_1}s} + s_2\delta_{\frac{\partial P(r)}{\partial r_2}s} & r_1\delta_{\frac{\partial P(s)}{\partial s_1}s} + r_2\delta_{\frac{\partial P(s)}{\partial s_2}s} & \delta_{P(s)s} \\ -\delta_{P(r)r} & -s_1\delta_{\frac{\partial P(r)}{\partial r_1}r} - s_2\delta_{\frac{\partial P(r)}{\partial r_2}r} & -r_1\delta_{\frac{\partial P(s)}{\partial s_1}r} - r_2\delta_{\frac{\partial P(s)}{\partial s_2}r} & -\delta_{P(s)r} \end{pmatrix}, \tag{2.5}$$

where, e.g., in $\tilde{b}_2$, we have the expression $\delta_{\frac{\partial P(r)}{\partial r_1}r} = (\partial P_1(r_1, r_2)/\partial r_1)r_2 - (\partial P_2(r_1, r_2)/\partial r_1)r_1 = (3a_1r_1^2 + 2b_1r_1r_2 + c_1r_2^2)r_2 - (3a_2r_1^2 + 2b_2r_1r_2 + c_2r_2^2)r_1$, while $\tilde{d}_2 - \delta\tilde{d}_2 = -\delta^{-1}((a_1s_1^3 + b_1s_1^2s_2 + c_1s_1s_2^2 + d_1s_2^3)r_2 - (a_2s_1^3 + b_2s_1^2s_2 + c_2s_1s_2^2 + d_2s_2^3)r_1)$.

Assertion 2.2. *For systems* (2.1) *and* (2.3), *the formula $\tilde{R} = \delta^6R$ holds.*

Thus, the sign of the resultant is invariant under any substitution (2.2).

Among the substitutions (2.2), which transform (2.1) into (2.3), we single out the two following special substitutions:

$$\begin{pmatrix} r_1 & 0 \\ 0 & s_2 \end{pmatrix}\text{—normalization,} \quad \tilde{A} = \begin{pmatrix} a_1r_1^2 & b_1r_1s_2 & c_1s_2^2 & d_1s_2^3/r_1 \\ a_2r_1^3/s_2 & b_2r_1^2 & c_2r_1s_2 & d_2s_2^2 \end{pmatrix}; \tag{2.6}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\text{—relabeling,} \quad \tilde{A} = \begin{pmatrix} d_2 & c_2 & b_2 & a_2 \\ d_1 & c_1 & b_1 & a_1 \end{pmatrix}. \tag{2.7}$$

Remark 2.1. Normalization (2.6) has the following peculiarities:

(1) a_2, b_1, c_2, d_1 will be called elements of an odd zigzag and a_1, b_2, c_1, d_2 are called elements of an even zigzag. Then, for all elements of an odd zigzag, one may simultaneously change the sign, whereas one cannot reverse the sign for any element of an even zigzag.

(2) None of the relations a_1/b_2, b_1/c_2, c_1/d_2 can be changed on diagonals.

Remark 2.2. In the system obtained after a substitution $L = (r, s)$, if there is a need for relabeling, then in the original system, one must make the substitution $L = (s, r)$.

At the same time, relabeling (2.7) allows one to achieve the following agreement.

Convention 2.2. In what follows, we shall assume without a loss of generality that, in system (2.1) with $l \geqslant 1$,

$$a_1^2 + a_2^2 \neq 0, \quad \text{if} \quad a_1^2 + a_2^2 + d_1^2 + d_2^2 \neq 0, \tag{2.8}$$

that is, when there is a common factor.

2.3. Form and Linear Equivalence of Systems with $l = 1$

For system (2.1) of the form $\dot{x} = P(x)$ with $l = 1$ we have $a_1^2 + a_2^2 \neq 0$ by Convention 2.2, for otherwise $P_0 = x_1x_2$ and $l \geqslant 2$, and hence it can be put in the form

$$\dot{x} = P_0^1(x)Gq^{[2]}(x), \tag{2.9}$$

where $P_0^1 = x_1 + \beta x_2$ ($\beta \in \mathbb{R}^1$), $G = \begin{pmatrix} p_1 & q_1 & t_1 \\ p_2 & q_2 & t_2 \end{pmatrix}$, $q^{[2]} = \text{colon}(x_1^2, x_1x_2, x_2^2)$. Besides, $p_1^2 + p_2^2 \neq 0$, $t_1^2 + t_2^2 \neq 0$, for otherwise $l > 1$, and the resultant $R_2 = \delta_{pt}^2 - \delta_{pq}\delta_{qt} \neq 0$, as constructed from the polynomials $p_iz^2 + q_iz + t_i$ ($i = 1, 2$), is nonzero (see, e.g., [20], Sec. 50).

The number β and the elements of H in system (2.9) are uniquely expressed in terms of the elements A using the equality $\begin{pmatrix} p_1 & q_1 + \beta p_1 & t_1 + \beta q_1 & \beta t_1 \\ p_2 & q_2 + \beta p_2 & t_2 + \beta q_2 & \beta t_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}$ with ($a_1^2 + a_2^2 \neq 0$) as follows:

$$\beta = \theta_*, \quad p_i = a_i, \quad q_i = b_i - a_i\theta_*, \quad t_i = c_i - b_i\theta_* + a_i\theta_*^2 (= d_i\theta_*^{-1}), \tag{2.10}$$

where $\theta_* \in \mathbb{R}^1$ is the common zero of the polynomials $P_i^{(1)}(\theta) = a_i\theta^3 - b_i\theta^2 + c_i\theta - d_i = 0$ ($i = 1, 2$).

The polynomials $P_1^{(1)}$, $P_2^{(1)}$ have a unique real common zero because, if P_1, P_2 has a zero, then any zero of P_i with the opposite sign will be a zero of $P_i^{(1)}$.

Theorem 2.1. *For $l = 1$, the substitution* (2.2) *of the form $x = Ly$ transforms system* (2.1) *of the form* (2.9) *with $P_0^1 = \alpha x_1 + \beta x_2$ into system* (2.3) *of the form*

$$\dot{y} = \tilde{P}_0^1(y)\tilde{G}q^{[2]}(y). \tag{2.11}$$

Here, $\tilde{P}_0^1(y) = \tilde{\alpha}y_1 + \tilde{\beta}y_2$ is the common factor, the matrix $\tilde{G} = \begin{pmatrix} \tilde{p}_1 & \tilde{q}_1 & \tilde{t}_1 \\ \tilde{p}_2 & \tilde{q}_2 & \tilde{t}_2 \end{pmatrix}$ and the resultant $\tilde{R}_2 = \delta_{\tilde{p}\tilde{t}}^2 - \delta_{\tilde{p}\tilde{q}}\delta_{\tilde{q}\tilde{t}}$ are calculated from the following formulas:

$$(\tilde{\alpha}, \tilde{\beta}) = (\alpha, \beta)L \neq 0, \quad \tilde{G} = L^{-1}GM, \quad M = \begin{pmatrix} r_1^2 & 2r_1s_1 & s_1^2 \\ r_1r_2 & \delta_* & s_1s_2 \\ r_2^2 & 2r_2s_2 & s_2^2 \end{pmatrix}, \quad \begin{matrix} \delta_* = r_1s_2 + r_2s_1, \\ \det M = \delta^3; \end{matrix}$$

$$\tilde{R}_2 = \delta^2 R_2 \neq 0 \quad \text{or} \quad \tilde{\alpha} = \alpha r_1 + \beta r_2, \quad \tilde{\beta} = \alpha s_1 + \beta s_2, \tag{2.12}$$

$$\delta\tilde{G} = \begin{pmatrix} r_1^2\delta_{ps} + r_1r_2\delta_{qs} + r_2^2\delta_{ts} & 2r_1s_1\delta_{ps} + \delta_*\delta_{qs} + 2r_2s_2\delta_{ts} & s_1^2\delta_{ps} + s_1s_2\delta_{qs} + s_2^2\delta_{ts} \\ r_1^2\delta_{rp} + r_1r_2\delta_{rq} + r_2^2\delta_{rt} & 2r_1s_1\delta_{rp} + \delta_*\delta_{rq} + 2r_2s_2\delta_{rt} & s_1^2\delta_{rp} + s_1s_2\delta_{rq} + s_2^2\delta_{rt} \end{pmatrix}.$$

Proof. We first show that the following formula holds:

$$q^{[2]}(Ly) = Mq^{[2]}(y). \tag{2.13}$$

So, we have

$$q^{[2]}(Ly) = \begin{pmatrix} (r_1y_1 + s_1y_2)^2 \\ (r_1y_1 + s_1y_2)(r_2y_1 + s_2y_2) \\ (r_2y_1 + s_2y_2)^2 \end{pmatrix} = \begin{pmatrix} r_1^2 & 2r_1s_1 & s_1^2 \\ r_1r_2 & \delta_* & s_1s_2 \\ r_2^2 & 2r_2s_2 & s_2^2 \end{pmatrix}\begin{pmatrix} y_1^2 \\ y_1y_2 \\ y_2^2 \end{pmatrix} = Mq^{[2]}(y).$$

Now formula (2.11) follows from the following chain of equalities:

$$\tilde{P}(y) \overset{(2.4)}{=} L^{-1}P(Ly) \overset{(2.9)}{=} L^{-1}((\alpha,\beta)Ly)Gq^{[2]}(Ly) \overset{(2.13)}{=} ((\alpha,\beta)Ly)L^{-1}GMq^{[2]}(y) \overset{(2.12)}{=} (\tilde{\alpha},\tilde{\beta})y\tilde{G}q^{[2]}(y). \quad \square$$

2.4. Form and Linear Equivalence of Systems with $l = 2$

System (2.1) of the form $\dot{x} = P(x)$ with $l = 2$ can be written in view of Convention 2.2 in the form

$$\dot{x} = P_0^2(x)Hx, \tag{2.14}$$

where $H = \begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix}$ and $\det H = \delta_{pq} \neq 0$, the real common factor $P_0^2 = \alpha x_1^2 + 2\beta x_1x_2 + \gamma x_2^2 = p_0^2q^{[2]}(x)$ with $p_0^2 = (\alpha, 2\beta, \gamma)$ has the discriminant $D_0 = \beta^2 - \alpha\gamma$. Furthermore, either $\alpha = 1$ ($D_0 = \beta^2 - \gamma$) or $\alpha, \gamma = 0$, $2\beta = 1$ ($D_0 = 1$) because (2.8) enables one to exclude the case $\alpha = 0, \gamma \neq 0$ (substitution (2.7) reduces p_0 to $(\gamma, 2\beta, \alpha)$).

Indeed, the row p_0^2 and elements of H in (2.14) are uniquely expressed in terms of the elements of A from the equality $\begin{pmatrix} \alpha p_1 & \alpha q_1 + 2\beta p_1 & 2\beta q_1 + \gamma p_1 & \gamma q_1 \\ \alpha p_2 & \alpha q_2 + 2\beta p_2 & 2\beta q_2 + \gamma p_2 & \gamma q_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}$. We have four cases to consider:

(1) $a_1 \neq 0$, $a_2 \neq 0$, and hence, $\delta_{ab} \neq 0$, $P_1^{(2)}(\theta_*) = P_2^{(2)}(\theta_*) = 0$, where $P_i^{(2)}(\theta) = a_i^2\theta^3 - 2a_ib_i\theta^2 + (a_ic_i + b_i^2)\theta + a_id_i - b_ic_i$ $(i = 1, 2)$, $2\theta_* = \delta_{ac}\delta_{ab}^{-1}$, and $\alpha = 1$, $2\beta = \theta_*$, $\gamma = \theta_*^2 - (b_i\theta_* - c_i)a_i^{-1}$, $p_i = a_i$, $q_i = b_i - a_i\theta_*$;

(2) $a_1 \neq 0$, $a_2 = 0$, and, hence,

$$b_2 \neq 0, \quad P_1^{(2)}(\theta_*) = 0, \quad \theta_*^2 - (b_1\theta_* - c_1)a_1^{-1} = d_2b_2^{-1}, \tag{2.15}$$

where $\theta_* = c_2b_2^{-1}$ and $\alpha = 1$, $2\beta = \theta_*$, $\gamma = d_2b_2^{-1}$, $p_1 = a_1$, $q_1 = b_1 - a_1\theta_*$, $p_2 = 0$, $q_2 = b_2$;

(3) $a_1 = 0$, $a_2 \neq 0$, and now everything is similar to (2) with a substitution in the subscripts;

(4) $a_1 = 0$, $a_2 = 0$, and, hence, $d_1 = 0$, $d_2 = 0$, $\delta_{bc} \neq 0$ and $\alpha = 0$, $\beta = 1/2$, $\gamma = 0$, $p_i = b_i$, $q_i = c_i$.

Here in case (1), the value of θ_* was obtained from the equality of the right-hand sides of the formula for γ. Thus, if $\theta_* = \delta_{ac} = 0$ and $\delta_{ab} = 0$, then the rows A_1, A_2 are proportional; that is, $l = 3$; in case (4) $\delta_{bc} = \delta_{pq} \neq 0$.

The eigenvalues of H and the discriminant of the characteristic polynomial are as follows:

$$\lambda_{1,2} = \frac{p_1 + q_2 \pm \sqrt{D}}{2} \neq 0, \quad D = (p_1 + q_2)^2 - 4\delta_{pq} = (p_1 - q_2)^2 + 4p_2q_1. \tag{2.16}$$

Theorem 2.2. *For $l = 2$ substitution* (2.2) $x = Ly$ *transforms system* (2.1) *of form* (2.14) *into system* (2.3) *of the form*

$$\dot{y} = \tilde{P}_0^2(y)\tilde{H}y, \tag{2.17}$$

where the matrix $\tilde{H} = \begin{pmatrix} \tilde{p}_1 & \tilde{q}_1 \\ \tilde{p}_2 & \tilde{q}_2 \end{pmatrix}$ *and the row of coefficients* $\tilde{p}_0^2 = (\tilde{\alpha}, 2\tilde{\beta}, \tilde{\gamma})$ *of the common factor of* $\tilde{P}_0^2 = \tilde{\alpha} y_1^2 + 2\tilde{\beta} y_1 y_2 + \tilde{\gamma} y_2^2 = \tilde{p}_0^2 q^{[2]}(y)$ *are calculated by the following formulas*:

$$\tilde{p}_0^2 = p_0^2 M \quad (M \text{ from } (2.12)), \quad \tilde{H} = L^{-1} H L \quad \text{or}$$

$$\tilde{\alpha} = \alpha r_1^2 + 2\beta r_1 r_2 + \gamma r_2^2, \quad \tilde{\beta} = \alpha r_1 s_1 + \beta \delta_* + \gamma r_2 s_2, \quad \tilde{\gamma} = \alpha s_1^2 + 2\beta s_1 s_2 + \gamma s_2^2, \tag{2.18}$$

$$\tilde{H} = \delta^{-1} \begin{pmatrix} r_1 \delta_{ps} + r_2 \delta_{qs} & s_1 \delta_{ps} + s_2 \delta_{qs} \\ -r_1 \delta_{pr} - r_2 \delta_{qr} & -s_1 \delta_{pr} - s_2 \delta_{qr} \end{pmatrix} \quad (\delta_{\tilde{p}\tilde{q}} = \delta_{pq}).$$

Furthermore, the discriminant $\tilde{D}_0 = \tilde{\beta}^2 - \tilde{\alpha}\tilde{\gamma}$ is related to D_0 as follows:

$$\tilde{D}_0 = \delta^2 D_0, \tag{2.19}$$

and the eigenvalues of $\tilde{H}$ and the discriminant $\tilde{D}$ agree with λ_1, λ_2 and D.

Proof. Formula (2.17) follows from the following chain of equalities:

$$\tilde{P}(y) \overset{(2.4)}{=} L^{-1} P(Ly) \overset{(2.14)}{=} L^{-1}(p_0^2 q^{[2]}(Ly)) H L y \overset{(2.13)}{=} (p_0^2 M q^{[2]}(y)) L^{-1} H L y \overset{(2.18)}{=} \tilde{p}_0^2 q^{[2]}(y) \tilde{H} y.$$

According to (2.18), we have $\tilde{\beta}^2 - \tilde{\alpha}\tilde{\gamma} = (\alpha r_1 s_1 + \beta \delta_* + \gamma r_2 s_2)^2 - (\alpha r_1^2 + 2\beta r_1 r_2 + \gamma r_2^2)(\alpha s_1^2 + 2\beta s_1 s_2 + \gamma s_2^2) = (\beta^2 - \alpha\gamma)(r_1^2 s_2^2 - 2 r_1 s_1 r_2 s_2 + s_1^2 r_2^2) = (\beta^2 - \alpha\gamma)\delta^2$, that is, formula (2.19) holds. □

2.5. Form and Linear equivalence of Systems with $l = 3$

In system (2.1) with $l = 3$ the polynomials P_1, $P_2 \not\equiv 0$ are proportional and, hence, in view of Convention 2.2,

$$\exists k \neq 0: \quad A = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ k a_1 & k b_1 & k c_1 & k d_1 \end{pmatrix} \quad (a_1^2 + b_1^2 \neq 0; \quad P_2 = k P_1, \quad P_0^3 \equiv P_1). \tag{2.20}$$

We claim that system (2.20) can be written in the form

$$\dot{x} = P_0(x) H x, \tag{2.21}$$

where $H = \begin{pmatrix} p_1 & q_1 \\ k p_1 & k q_1 \end{pmatrix}$ $(p_1^2 + q_1^2 \neq 0, k \neq 0, \delta_{pq} = 0)$, the real common factor reads as $P_0 = x_1^2 + 2\beta x_1 x_1 + \gamma x_2^2$ $(D_0 = \beta^2 - \gamma)$.

The eigenvalues of H and the discriminant of the characteristic polynomial are as follows:

$$\lambda_1 = p_1 + k q_1, \quad \lambda_2 = 0, \quad D = \lambda_1^2 \geq 0. \tag{2.22}$$

As proposed in the case $l = 2$, the structure of system (2.21) is fairly convenient for subsequent analysis (we have already seen this); thus, for system (2.20), we will employ an analogous expansion based on the factoring a common factor out of P_0 that is of a nonmaximal degree (degree 2). The differences are as follows: in system (2.14) $\det H = 0$, one may always yield $\alpha \neq 0$ and there is an uncertainty due to different possible ways of factoring P_0 out of the polynomial P_1 of system (2.20).

Let us refine the principles behind the choice of a quadratic common factor of P_0 in (2.20).

Convention 2.3. *We shall factor the following out of* $P_1(x)$ *into system* (2.20) *as follows*:

(1) *a perfect square, if possible*;

(2) *if further possible,* P_0, *for which* $\lambda_1 \neq 0$ *in* (2.22);

(3) *otherwise*, *two linear cofactors with maximal zeros (if they exist)*.

The coefficients β, γ ($\alpha = 1$) and the elements p_1, q_1 in (2.21) are uniquely expressed in terms of elements of (2.20) from the equalities $a_1 = \alpha p_1$, $b_1 = \alpha q_1 + 2\beta p_1$, $c_1 = 2\beta q_1 + \gamma p_1$, $d_1 = \gamma q_1$ as follows:

(1) $a_1 \neq 0 \Rightarrow \alpha = 1$, $2\beta = \theta_*$, $\gamma = (a_1\theta_*^2 - b_1\theta_* + c_1)a_1^{-1}$, $p_1 = a_1$, $q_1 = b_1 - a_1\theta_*$, where $\theta_* \in \mathbb{R}^1$ is a zero of $P_1^{(2)}(\theta)$ from (2.15), as taken with the consideration of Convention 2.3;

(2) $a_1 = 0$, $(b_1 \neq 0) \Rightarrow \alpha = 1$, $2\beta = c_1 b_1^{-1}$, $\gamma = d_1 b_1^{-1}$, $p_1 = 0$, $q_1 = b_1$.

Let us consider the choice of θ_* in more detail in case (1) $a_1 \neq 0$.

The presence of multiple roots for the polynomial $P_1^{(2)}$ is equivalent to saying that $\gamma = \beta^2$ with $a_1 \neq 0$ and, hence, $\theta_*^\pm = 2(b_1 \pm d_*^{1/2})/3$, where $d_* = b_1^2 - 3a_1c_1$. Hence, if $d_* \geqslant 0$ and $d_1 = a_1^{-2}(b_1^3 - 3b_1d_* \mp 2d_*^{3/2})/27$, then $P_1^{(2)}(\theta)$ has the zero $\theta_*^\pm$ and the double zero $(2b_1 \pm d_*^{1/2})/3$. If there are no multiple zeros, then, if possible, we put $\theta_* \neq 1 + a_1^{-1}b_1$; otherwise, we set $p_1 + kq_1 = 0$. Finally, if a choice is still possible, then we put θ_* to be the minimal zero of $P_1^{(2)}(\theta)$.

Regarding system (2.21), we note that, for this system, as well as for system (2.14) with $l = 2$, the conclusion of Theorem 2.2 holds with $\alpha = 1$ and $\det H = 0$.

2.6. Main Linear Invariants

We now extend the obtained results.

Theorem 2.3. *The degree* l ($l = \overline{0,3}$) *of the common factor* P_0^l, *as introduced for system* (2.1) *in Definition* 2.1, *is invariant with respect to linear nonsingular substitutions. Furthermore, if* $l = 1$, *then the sign of the resultant* R_2 ($R_2 \neq 0$) *of the matrix* G *of system* (2.9) *is invariant and, if* $l = 2$ *or* $l = 3$, *then the signs of the discriminant of the quadratic common factor* P_0 *and of the discriminant of the roots of the characteristic polynomial for the matrix* H *of systems* (2.14) *or* (2.21) *are invariant.*

Corollary 2.1. *In the case* $l = 2$ *or* $l = 3$, *the quadratic common factor* P_0, *which is factored out of the polynomials* P_1, P_2 *of system* (2.1), *and the common factor* $\tilde{P}_0$, *as obtained as a result of substitution* (2.2) *and factored out of the polynomials* $\tilde{P}_1$, $\tilde{P}_2$ *of system* (2.17), *simultaneously expand or do not expand into linear factors with real coefficients and, for them, the perfect squares are conserved.*

In conclusion, we note that subsequent studies will ultimately be related to investigations of the compatibility, various simplifications, the particularization, and the solution of system (2.5).

Moreover, (2.5) should be interpreted as a system that involves eight equations with the unknowns r_1, r_2, s_1, s_2 (the coefficients of substitution (2.2)) with the following structure: the left-hand side is a fourth-order homogeneous polynomial in r_1, r_2, s_1, s_2, the coefficients of which are linear combinations of eight coefficients of the initial system (2.1), the right-hand side vector is formed by the eight coefficients of system (2.17) multiplied by the determinant of the linear nonsingular substitution (2.2).

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