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MATHEMATICS

Two-Dimensional Homogeneous Cubic Systems: Classification and Normal Forms II

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Abstract—This work is the second in a series of papers concerning two-dimensional homogeneous cubic systems. In the first paper of the series, structural principles were developed to introduce a total order on the set of structural forms, i.e., vector polynomials with a fixed number of zero coefficients that are right-hand sides of two-dimensional homogeneous cubic systems of ODEs. Among them, structural forms normalized on the basis of normalization principles and canonical forms (CFs) that are linearly nonequivalent to each other and are the simplest in their class were sequentially distinguished. In this paper, for above-mentioned systems with proportional right-hand side components, all *CF*s with their canonical sets of permissible values are distinguished. For each *CF*, (a) conditions on the coefficients of the original system, (b) linear substitutions that reduce the right-hand side of a system under these conditions to the chosen *CF*, and (c) the resulting values of the *CF*'s coefficients are given.

Keywords: homogeneous cubic system, normal form, canonical form.

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INTRODUCTION

This paper is a direct continuation of [1], so it retains the notation introduced in [1]. In view of the numerous references to formulas obtained in [1], for brevity, they are denoted by superscript 1. For example, system (2.1) from [1] is designated as $(2.1)^1$.

1. CANONICAL FORMS AND THEIR DEFINITION PRINCIPLES

1.1. Structural Forms

Let us consider the homogeneous cubic system $(2.1)^1$

$$\dot{x} = P(x) = Aq^{[3]}(x), \tag{1.1}$$

which is identified with a real matrix A any of whose rows $A_i = (a_i, b_i, c_i, d_i)$ $(i = 1, 2)$ is nonzero and $q_{[3]} = \mathrm{colon}(x_1^3, x_1^2x_2, x_1x_2^2, x_2^3)$.

The main task in this section is to formulate principles that make it possible to distinguish the "simplest" linearly nonequivalent systems, which are referred to hereafter as cubic normal forms (NFs), while their right-hand sides are called canonical forms. For every cubic NF, we specify the conditions on the coefficients of the original system (1.1) and a linear nonsingular substitution $(2.2)^1$,

$$x_1 = r_1y_1 + s_1y_2, \quad x_2 = r_2y_1 + s_2y_2 \quad \text{or} \quad x = Ly \quad (\delta = \det L \neq 0), \tag{1.2}$$

that reduces (1.1) to the chosen cubic NF.

Moreover, the principles behind the choice of canonical forms have to be formulated so as to maximally facilitate the reduction of system $(1.4)^1$ $\dot{x} = P(x) + X(x)$, where the unperturbed part $P(x)$ is a canonical form, to generalized normal forms by applying almost identical substitutions.

As the first step toward the definition of a canonical form, we introduce the formal concept of a structural form and an order on the set of structural forms.

Definition 1.1. A real matrix $A = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}$ with nonzero rows is called a united structural m-form ($m = 2, ..., 8$), denoted by USF^m, if some m of its elements are nonzero, while the others are zero. The finite set of all USF^m is designated as $SUSF^m$ (set of USF^m).

Obviously, one united structural m-form differs from another by the positions of the nonzero elements.

In what follows, for brevity, any USF^m is written in rows, indicating only nonzero elements in each of them, for example, $\begin{pmatrix} a_1 & 0 & c_1 & 0 \\ 0 & 0 & 0 & d_2 \end{pmatrix} = (a_1, c_1; d_2)$.

Consider all possible arrangements of the nonzero elements in the set $SUSF^m$ ($m = \overline{2,8}$).

Definition 1.2. The index of an element a_{ij} ($i = 1, 2; j = 1, 2, 3, 4$) of the matrix A is the number in the position (i, j) in the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$. In turn, the index k of A is the sum of the indices of the nonzero elements in A; if necessary, we write $A_{[k]}$. The indices of rows A_1 and A_2 are introduced in a similar manner.

It can be shown by direct verification that the following three structural principles allow one to completely order the finite set $SUSF = \bigcup_{m=2}^{8} SUSF^m$.

Structural principles (SPs) for ordering the SUSF:

(1) *Any USF^m precedes any USF^n if $m < n$.*

(2) *Any USF^m with a smaller index precedes any USF^m with a larger index.*

(3) *Given any two USF^m with identical indices, the preceding m-form is one in which*

(3_1) *the row A_2 has a smaller index*;

(3_2) *in the case of identical indices of A_2, the left nonzero element in A_1 has a smaller index*;

(3_3) *otherwise, the right nonzero element in A_2 has a smaller index.*

Thus, in any subset of $SUSF$, the structurally "simplest" is a matrix A in which, in order of mentioning, (1) the number m of nonzero elements is minimal; (2) the index k is minimal; (3_1) the index of A_2 is minimal; (3_2) the index of the left nonzero element in A_1 is minimal; and (3_3) the index of the right nonzero element in A_2 is minimal.

Let us discuss the basis for choosing the above SPs as applied to system (1.4)[1] with an unperturbed part $P(x)$ generated by a united structural m-form A.

SP1 requires that the number of zero elements in A be maximal, which is undoubtedly a necessary condition of primary importance for the maximum possible simplification of the bonding system (1.7)[1].

Relying on the arguments used in Section 1.4 of [1], SP2 and SP3 optimize the arrangement of the available nonzero coefficients.

For example, SP2 prefers the least weakly related unperturbed system, i.e., a system in which P_1 and P_2 contain variables x_2 and x_1, respectively, in minimal degrees. Naturally, SP2 makes sense only for $l \le 2$, since, for $l = 3$, the coefficients of the polynomials are proportional and a permutation of the columns in A does not change its index.

SP3s always prefer those structural forms that satisfy condition (1.9)[1], which is important for the normalization of perturbed systems.

Moreover, SPs were chosen so as to minimize whenever possible the number of nonzero elements in the row A_2. This is indicated in Remark 1.1 below.

After introducing the SPs, the number of USF^m used in what follows is considerably reduced, since, from the point of view of the subsequent normalization of perturbed systems, it is of no matter which of two matrices is chosen as an unperturbed part if they are obtained from each other by relabeling (2.7)[1] of $L = \{r_1, s_2 = 0, r_2, s_1 = 1\}$.

Definition 1.3. Given two different united structural m-forms obtained from each other by relabeling, the one that is preceding according to SP3 is called basic (or merely a structural m-form) and is denoted by SF^m, while the other is called additional and is denoted by SF_a^m.

Obviously, there are "symmetric" structural m-forms, i.e., SF^m that do not change after relabeling (2.7)[1].

Since any pair consisting of a basic and an additional structural form is linearly equivalent, the “worst” additional form in terms of SP3 is of no interest in itself, but it can sometimes be used for convenience.

Convention 1.1. According to the order introduced, any basic structural m-form is associated with an index i and is denoted by SF_i^m, while the corresponding additional structural form is denoted by $SF_{a,i}^m$.

List 1.1. 120 ordered structural forms from the *SUSF*:

$$SF_1^2 = (a_1; d_2)_{[2]},\ SF_2^2 = (a_1; c_2)_{[3]},\ SF_3^2 = (a_1; b_2)_{[4]},\ SF_4^2 = (b_1; c_2)_{[4]},\ SF_5^2 = (a_1; a_2)_{[5]},$$
$$SF_6^2 = (b_1; b_2)_{[5]},\ SF_7^2 = (b_1; a_2)_{[6]},\ SF_8^2 = (c_1; b_2)_{[6]},\ SF_9^2 = (c_1; a_2)_{[7]},\ SF_{10}^2 = (d_1; a_2)_{[8]};$$

$$SF_1^3 = (a_1, b_1; d_2)_{[4]},\ SF_2^3 = (a_1, c_1; d_2)_{[5]},\ SF_3^3 = (a_1, b_1; c_2)_{[5]},\ SF_4^3 = (a_1, d_1; d_2)_{[6]},$$
$$SF_5^3 = (b_1, c_1; d_2)_{[6]},\ SF_6^3 = (a_1, c_1; c_2)_{[6]},\ SF_7^3 = (a_1, b_1; b_2)_{[6]},\ SF_8^3 = (b_1, d_1; d_2)_{[7]},$$
$$SF_9^3 = (a_1, d_1; c_2)_{[7]},\ SF_{10}^3 = (b_1, c_1; c_2)_{[7]},\ SF_{11}^3 = (a_1, c_1; b_2)_{[7]},\ SF_{12}^3 = (a_1, b_1; a_2)_{[7]},$$
$$SF_{13}^3 = (c_1, d_1; d_2)_{[8]},\ SF_{14}^3 = (b_1, d_1; c_2)_{[8]},\ SF_{15}^3 = (a_1, d_1; b_2)_{[8]},\ SF_{16}^3 = (b_1, c_1; b_2)_{[8]},$$
$$SF_{17}^3 = (a_1, c_1; a_2)_{[8]},\ SF_{18}^3 = (c_1, d_1; c_2)_{[9]},\ SF_{19}^3 = (b_1, d_1; b_2)_{[9]},\ SF_{20}^3 = (a_1, d_1; a_2)_{[9]},$$
$$SF_{21}^3 = (b_1, c_1; a_2)_{[9]},\ SF_{22}^3 = (c_1, d_1; b_2)_{[10]},\ SF_{23}^3 = (b_1, d_1; a_2)_{[10]},\ SF_{24}^3 = (c_1, d_1; a_2)_{[11]};$$

$$SF_1^4 = (a_1, b_1; c_2, d_2)_{[6]},\ SF_2^4 = (a_1, b_1; c_1; d_2)_{[7]},\ SF_3^4 = (a_1, c_1; c_2, d_2)_{[7]},$$
$$SF_4^4 = (a_1, b_1, d_1; d_2)_{[8]},\ SF_5^4 = (a_1, b_1, c_1; c_2)_{[8]},\ SF_6^4 = (a_1, d_1; c_2, d_2)_{[8]},$$
$$SF_7^4 = (b_1, c_1; c_2, d_2)_{[8]},\ SF_8^4 = (a_1, c_1; b_2, d_2)_{[8]},\ SF_9^4 = (a_1, c_1, d_1; d_2)_{[9]},$$
$$SF_{10}^4 = (a_1, b_1, d_1; c_2)_{[9]},\ SF_{11}^4 = (a_1, b_1, c_1; b_2)_{[9]},\ SF_{12}^4 = (b_1, d_1; c_2, d_2)_{[9]},$$
$$SF_{13}^4 = (a_1, d_1; b_1, d_2)_{[9]},\ SF_{14}^4 = (b_1, c_1; \mathrm{b}_2, d_2)_{[9]},\ SF_{15}^4 = (b_1, c_1, d_1; d_2)_{[10]},$$
$$SF_{16}^4 = (a_1, c_1, d_1; c_2)_{[10]},\ SF_{17}^4 = (a_1, b_1, d_1; b_2)_{[10]},\ SF_{18}^4 = (c_1, d_1, c_2, d_2)_{[10]},$$
$$SF_{19}^4 = (a_1, b_1, c_1; a_2)_{[10]},\ SF_{20}^4 = (b_1, d_1; b_2, d_2)_{[10]},\ SF_{21}^4 = (a_1, d_1; a_2, d_2)_{[10]},$$
$$SF_{22}^4 = (a_1, d_1; b_2, c_2)_{[10]},\ SF_{23}^4 = (b_1, c_1; b_2, c_2)_{[10]},\ SF_{24}^4 = (b_1, c_1, d_1; c_2)_{[11]},$$
$$SF_{25}^4 = (a_1, c_1, d_1; b_2)_{[11]},\ SF_{26}^4 = (a_1, b_1, d_1; a_2)_{[11]},\ SF_{27}^4 = (c_1, d_1; b_2, d_2)_{[11]},$$
$$SF_{28}^4 = (b_1, d_1; a_2, d_2)_{[11]},\ SF_{29}^4 = (b_1, d_1; b_2, c_2)_{[11]},\ SF_{30}^4 = (b_1, c_1, d_1; b_2)_{[12]},$$
$$SF_{31}^4 = (a_1, c_1, d_1; a_2)_{[12]},\ SF_{32}^4 = (\mathrm{c}_1, d_1; a_2, d_2)_{[12]},\ SF_{33}^4 = (c_1, d_1; b_2, c_2)_{[12]},$$
$$SF_{34}^4 = (b_1, d_1; a_2, c_2)_{[12]},\ SF_{35}^4 = (b_1, c_1, d_1; a_2)_{[13]},\ SF_{36}^4 = (c_1, d_1; a_2, c_2)_{[13]},$$
$$SF_{37}^4 = (c_1, d_1; a_2, b_2)_{[14]};$$

$$SF_1^5 = (a_1, b_1, c_1; c_2, d_2)_{[9]},\ SF_2^5 = (a_1, b_1, d_1; c_2, d_2)_{[10]},\ SF_3^5 = (a_1, b_1, c_1; b_2, d_2)_{[10]},$$
$$SF_4^5 = (a_1, b_1, c_1, d_1; d_2)_{[11]},\ SF_5^5 = (a_1, c_1, d_1; c_2, d_2)_{[11]},\ SF_6^5 = (a_1, b_1, d_1; b_2, d_2)_{[11]},$$
$$SF_7^5 = (a_1, b_1, c_1; a_2, d_2)_{[11]},\ SF_8^5 = (a_1, b_1, c_1; b_2, c_2)_{[11]},\ SF_9^5 = (a_1, b_1, c_1, d_1; c_2)_{[12]},$$
$$SF_{10}^5 = (b_1, c_1, d_1; c_2, d_2)_{[12]},\ SF_{11}^5 = (a_1, c_1, d_1; b_2, d_2)_{[12]},\ SF_{12}^5 = (a_1, b_1, d_1; a_2, d_2)_{[12]},$$
$$SF_{13}^5 = (a_1, b_1, d_1; b_2, c_2)_{[12]},\ SF_{14}^5 = (a_1, b_1, c_1; a_2, c_2)_{[12]},\ SF_{15}^5 = (a_1, b_1, c_1, d_1; b_2)_{[13]},$$
$$SF_{16}^5 = (b_1, c_1, d_1; b_2, d_2)_{[13]},\ SF_{17}^5 = (a_1, c_1, d_1; a_2, d_2)_{[13]},\ SF_{18}^5 = (a_1, c_1, d_1; b_2, c_2)_{[13]},$$
$$SF_{19}^5 = (a_1, b_1, d_1; a_2, c_2)_{[13]},\ SF_{20}^5 = (c_1, d_1; b_2, c_2, d_2)_{[13]},\ SF_{21}^5 = (a_1, b_1, c_1, d_1; a_2)_{[14]},$$
$$SF_{22}^5 = (b_1, c_1, d_1; a_2, d_2)_{[14]},\ SF_{23}^5 = (b_1, c_1, d_1; b_2, c_2)_{[14]}\ SF_{24}^5 = (a_1, c_1, d_1; a_2, c_2)_{[14]},$$
$$SF_{25}^5 = (a_1, b_1, d_1; a_2, b_2)_{[14]},\ SF_{26}^5 = (b_1, c_1, d_1; a_2, c_2)_{[15]},\ SF_{27}^5 = (a_1, c_1, d_1; a_2, b_2)_{[15]},$$
$$SF_{28}^5 = (b_1, c_1, d_1; a_2, b_2)_{[16]};$$

$$SF_1^6 = (a_1, b_1, c_1; b_2, c_2, d_2)_{[12]},\quad SF_2^6 = (a_1, b_1, c_1, d_1; c_2, d_2)_{[13]},$$
$$SF_3^6 = (a_1, b_1, d_1; b_2, c_2, d_2)_{[13]},\quad SF_4^6 = (a_1, b_1, c_1, d_1; b_2, d_2)_{[14]},$$
$$SF_5^6 = (a_1, c_1, d_1; b_2, c_2, d_2)_{[14]},\quad SF_6^6 = (a_1, b_1, d_1; a_2, c_2, d_2)_{[14]},$$
$$SF_7^6 = (a_1, b_1, c_1, d_1; a_2, d_2)_{[15]},\quad SF_8^6 = (a_1, b_1, c_1, d_1; b_2, c_2)_{[15]},$$
$$SF_9^6 = (b_1, c_1, d_1; b_2, c_2, d_2)_{[15]},\quad SF_{10}^6 = (a_1, c_1, d_1; a_2, c_2, d_2)_{[15]},$$
$$SF_{11}^6 = (a_1, b_1, c_1, d_1; a_2, c_2)_{[16]},\quad SF_{12}^6 = (b_1, c_1, d_1; a_2, c_2, d_2)_{[16]},$$
$$SF_{13}^6 = (a_1, c_1, d_1; a_2, b_2, d_2)_{[16]},\quad SF_{14}^6 = (a_1, b_1, c_1, d_1; a_2, b_2)_{[17]},$$
$$SF_{15}^6 = (b_1, c_1, d_1; a_2, b_2, d_2)_{[17]},\quad SF_{16}^6 = (b_1, c_1, d_1; a_2, b_2, c_2)_{[18]};$$
$$SF_1^7 = (a_1, b_1, c_1, d_1; b_2, c_2, d_2)_{[16]},\quad SF_2^7 = (a_1, b_1, c_1, d_1; a_2, c_2, d_2)_{[17]},$$
$$SF_3^7 = (a_1, b_1, c_1, d_1; a_2, b_2, d_2)_{[18]},\quad SF_4^7 = (a_1, b_1, c_1, d_1; a_2, b_2, c_2)_{[19]};$$
$$SF_1^8 = (a_1, b_1, c_1, d_1; a_2, b_2, c_2, d_2)_{[20]}.$$

This list demonstrates the sufficiency of the SPs for ordering the basic structural forms. However, the SPs also distinguish all additional structural forms.

Indeed, each asymmetric form from List 1.1 with rows A_1 and A_2 having different indices is basic according to SP3_1. The remaining five asymmetric *SF*s, i.e., SF_6^3, SF_{17}^3, SF_{22}^4, SF_{14}^5, and SF_{25}^5, are basic according to SP3_2.

Remark 1.1. In all SF_i^m from List 1.1, except for SF_{20}^5, the number of nonzero elements in A_2 does not exceed that in A_1.

Definition 1.4. A representative of an arbitrary SF_i^m is any numerical matrix whose structure coincides with that of SF_i^m.

As a result, SF_i^m can be treated as the collection of all its representatives.

An important characteristic of SF_i^m is related to finding all possible values of the maximum degree of a common factor P_0^l (see Definition 2.1 in [1]) that can be taken out from the right-hand side of system (1.1) generated by this structural form for various values of nonzero coefficients. Accordingly, for any SF_i^m, the set of real nonzero values of its elements is divided into subsets $s_i^{m,l}$ $(0 \le l \le 3)$ as follows: $s_i^{m,l}$ contains those and only those values of elements of SF_i^m for which a common factor P_0^l can be taken out from the right-hand side of system (1.1) generated by this form.

Definition 1.5. For any SF_i^m specified by a matrix A, the notation $SF_i^{m,l}$ means the same matrix A, but the values of its nonzero elements belong to $s_i^{m,l} \neq \emptyset$.

In other words, $SF_i^{m,l}$ unites those and only those representatives of SF_i^m whose elements belong to the nonempty set $s_i^{m,l}$ or, equivalently, $SF_i^{m,l}$ generates only those systems that have a common factor of maximum degree l.

Definition 1.5 and Theorem 2.3 from [1] imply the following result.

Proposition 1.1. *SF_i^{m,l_1} is linearly not equivalent to SF_i^{m,l_2} for $l_1 \neq l_2$; i.e., any two representatives of SF_i^{m,l_1} and SF_i^{m,l_2} are linearly not equivalent.*

If SF_i^m has only one set $s_i^{m,l_0} \neq \emptyset$, then it has no constraints and is called trivial. Thus, $SF_i^{m,l_0} = SF_i^m$.

For example, the values of the nonzero elements in $SF_8^4 = (a_1, c_1; b_2, d_2)$ are divided into two subsets: $s_8^{4,0} = \{a_1 d_2 \neq b_2 c_1\}$ and $s_8^{4,2} = \{a_1 d_2 = b_2 c_1\}$. System (1.1) generated by $SF_2^4 = (a_1, b_1, c_1; d_2)$ for any values of elements has no common factor, i.e., $l = 0$ and the only nonempty set $s_2^{4,0}$ is trivial.

1.2. Normalized Structural Forms and Permissible Sets

The next step toward the definition of a canonical form is to introduce the concept of a normalized structural form based on the normalization of all representatives of $SF_i^{m,l}$ with the help of the substitution $(2.6)^1$ $L = \{r_1, s_2$ arbitrary $r_2, s_1 = 0\}$ in order to obtain unit (in absolute value) elements in two properly chosen positions.

Let us formulate the principles behind the choice of elements of A to be normalized. The basic idea is to normalize perturbed systems of elements that are the nastiest for the subsequent normalization, i.e., those having maximum indices. Following the logic of the structural principles (see $SP3_1$), whenever possible, it is preferable to normalize elements from the row A_2.

Normalization principles (NPs) for $SF_i^{m,l}$.

(1) *The normalized elements in A are arranged in the following order*:

(1_1) *The first normalized element is in* A_2 *and has the maximum index.*

(1_2) *If not all nonzero elements of A are from the same zigzag* (see Remark 2.1 in [1]), *then the second normalized element*, *after normalization*, *must have a definite sign for any values of elements of A from* $s_i^{m,l}$.

(1_3) *If* $l = 3$, *then* $A_1 = A_2$; *if* $l \le 2$, *then the second normalized element is*, *if possible*, *in the row* A_2 *and has the maximum index in* A_2 *out of the remaining ones*; *otherwise*, *it is in* A_1 *and has the maximum index there*.

(2) *The values of normalized elements are equal to unity in absolute value; moreover, the following conditions hold*:

(2_1) *If they are from an odd zigzag*, *then the first normalized element is equal to* 1.

(2_2) *If they are from different zigzags*, *then the sign of a normalized element from an odd zigzag must coincide with the sign of a normalized element from an even zigzag.*

Direct verification shows that, relying on the NPs introduced, in any $SF_i^{m,l}$, we can uniquely choose the positions of normalized elements and the values to be obtained by the elements in these positions after normalization. Moreover, a normalizing substitution is uniquely determined for all SF, except $SF_3^{2,2}$ and $SF_4^{2,2}$, for which the element s_2 in $(2.6)^1$ is arbitrary and can be set, for example, to unity (see Remark 2.1 in [1]).

Thus, the representatives of any $SF_i^{m,l}$ (numerical matrices of given structure with elements from $s_i^{m,l}$) can be divided into equivalence classes with respect to normalizing substitutions $(2.6)^1$, while normalized representatives are used as generatrices.

Definition 1.6. $SF_i^{m,l}$ is called a normalized structural form, denoted by $NSF_i^{m,l}$, if it unites only its representatives normalized according to the NPs.

Convention 1.2. Any normalized structural form A will be written as σB, where the factor σ taken out from A is equal to the sign of the first normalized element. The nonzero elements of B remaining unnormalized, if any, will be properly expressed in terms of variables known as parameters of NSF, which are denoted by u, v, w, ... If necessary, NSF will be written as a function of its parameters.

For example, we can write $NSF_7^{5,1} = NSF_7^{5,1}(\sigma, u, v) = \sigma\begin{pmatrix} u & v & v-u & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$; here, $v \ne u$; otherwise, $m \ne 5$.

Relying on Convention (1.2), we can obtain the maximum number of units in the matrix B, which is used below for normalizing perturbed systems, while σ, if negative, can always be set to unity by time substitution.

For example, by making substitution $(2.6)^1$, $SF_2^{2,1} = (a_1; c_2)$ can be reduced to $NSF_2^{2,1} = \sigma\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ with $\sigma = \operatorname{sgn} a_1$. Here, the normalized elements are from different zigzags and, according to Remark 2.1 in [1], the sign of an element of an even zigzag cannot be affected, so it is factored out as σ, while the sign of a normalized element from an odd zigzag can always be made equal to σ, as required in $NP2_2$.

Now we discuss the rationale behind the introduction of $NP1_2$.

If the normalized elements in SF are taken from the same zigzag, then, after normalization, their product can be positive or negative.

For example, let us normalize b_2 and d_2 in $SF_{13}^4 = (a_1, d_1; b_2, d_2)$, as required by NP1$_3$. Then, for $l = 1$, we obtain $NSF_{13}^{4,1} = \sigma\begin{pmatrix} u & 0 & 0 & u \\ 0 & 1 & 0 & -1 \end{pmatrix}$ with $\sigma = \operatorname{sgn} b_2$, while, for $l = 0$, we obtain the system $\sigma\begin{pmatrix} u & 0 & 0 & v \\ 0 & 1 & 0 & \kappa \end{pmatrix}$ with $\kappa = \operatorname{sgn}(b_2 d_2)$ and the same σ; moreover, $v \neq u$ if $\kappa = -1$, i.e., depending on the sign of κ, we have one of two different *NSF*s.

In this case, the bifurcation can be avoided, since $SF_{13}^{4,0}$ contains the nonzero element d_1 from another zigzag, and it is this element that is to be normalized according to NP1$_2$. Therefore, for $l = 0$, we obtain the unique $NSF_{13}^{4,0} = \sigma\begin{pmatrix} u & 0 & 0 & 1 \\ 0 & 1 & 0 & v \end{pmatrix}$ with $v \neq -u^{-2}$, which is preferable to normalizing both elements of A_2 at the cost of the bifurcation of *NSF*.

Definition 1.7. If all nonzero elements of $SF_i^{m,l}$ are from the same zigzag, which results in the second normalized element in B (if any) being equal to 1 or -1 (denote it by κ), then the resulting *NSF* is called dual and is denoted by $NSF_{i,\kappa}^{m,l}$.

Thus, the position of the second normalized element in any $SF_i^{m,l}$ can be uniquely determined by NP1$_2$ and NP1$_3$. For $l \le 2$, they place this element in A_2 and, if impossible, in A_1, in position with a maximum index, so that uniqueness is preserved after the normalization. At the same time, for $l = 3$, the second unit element is automatically placed in A_1 above the first one in view of the natural assumption that A_1 and A_2 are equal, which prevents bifurcation.

Note that the conditions fixing the maximum degree l of the common factor are much easier to write for $NSF_i^{m,l}$ than for $SF_i^{m,l}$.

For example, $NSF_7^5 = \sigma\begin{pmatrix} u & v & w & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ is $NSF_7^{5,2}$ for $-v$, $w = u$; $NSF_7^{5,1}$ for $w = v - u$; and $NSF_7^{5,0}$ if the above constraints on the parameters are not satisfied.

Definition 1.8. Parameter values for which an arbitrary $NSF_i^{m,l}$ is defined are called permissible. The union of permissible parameter values for each form is called a permissible set and is denoted by $ps_i^{m,l}$. A permissible set is said to be trivial (denoted by $tps_i^{m,l}$) if the parameters involved have no constraints.

Definition 1.8 and Theorem 2.3 from [1] imply the following result.

Proposition 1.2. *For* $l = 2, 3$, *all representatives constituting* $NSF_i^{m,l}$ *can be partitioned into three disjoint sets, depending on the sign of the discriminant* $D_0 = \beta^2 - \alpha\gamma$ *from* (2.14)[1]. *These sets are denoted by* $NSF_i^{m,l,>}$, $NSF_i^{m,l,=}$, *and* $NSF_i^{m,l,<}$. *For* $l = 2$, *all representatives constituting* $NSF_i^{m,l,*}$ *can be partitioned into three disjoint sets, depending on the sign of the discriminant* $D = (p_1 - q_2)^2 + 4p_2q_1$ *from* (2.16)[1]. *These sets are denoted by* $NSF_i^{m,l,*,>}$, $NSF_i^{m,l,*,=}$, *and* $NSF_i^{m,l,*,<}$. *For* $l = 3$, *there are two such sets, since in* (2.22)[1] *we have* $D \ge 0$. *Similar partitions can be made in* $ps_i^{m,l}$ ($l = 2, 3$).

Corollary 1.1. The systems generated by any two representatives of $NSF_i^{m,l}$ ($l = 2, 3$) with different pairs of third and fourth superscripts cannot be linearly equivalent.

It should be kept in mind that the positions of normalized elements in $NSF_i^{m,l}$ can be different not only for different l (see the normalizations of $SF_{13}^{4,0}$ and $SF_{13}^{4,1}$ above), but also for superscripts appearing when the action of NP1$_2$ is stopped due to fixing the sign of the second normalized element, which is chosen according to NP1$_3$ from the same zigzag as the first.

1.3. Canonical Sets and Canonical Forms

Consider an arbitrary matrix $NSF_i^{m,l}$ with m nonzero elements in given positions that fix its index i in $SUSF^m$ according to the introduced SPs. Let l denote the degree of a common factor P_0^l taken out from the right-hand side of the system generated by any representative of $NSF_i^{m,l}$. By Theorem 2.3 in [1], l is invariant under linear nonsingular substitutions.

Note that obtaining normalized structural forms is a formal task requiring only normalization (2.6)[1], i.e., a substitution that does not affect the structure of the matrix A generating these forms.

Now, we simplify $NSF_i^{m,l}$, reducing them, by applying suitable linear nonsingular substitutions (1.2) with certain parameter values from $ps_i^{m,l}$, to preceding structural forms, i.e., to $SF_i^{n,l}$ with $n < m$ or $j < i$ for $n = m$.

On the one hand, nearly every $NSF_i^{m,l}$ can be reduced to preceding $SF_i^{n,l}$, i.e., it has "redundant" representatives that are linearly equivalent to some representatives of the preceding forms. The parameter values generating such representatives have to be deleted from $ps_i^{m,l}$.

On the other hand, those $NSF_i^{m,l}$ that are linearly equivalent to some preceding forms for all permissible parameter values are of no interest in themselves, since they cannot be the "simplest."

Definition 1.9. A nonempty set containing those and only those parameter values from $ps_i^{m,l}$ for which $NSF_i^{m,l}$ is linearly equivalent to none of the preceding SF is called canonical and is denoted by $cs_i^{m,l}$.

Definition 1.10. Any $NSF_i^{m,l}$ is called a canonical form, denoted by $CF_i^{m,l}$, if its parameters belong to $cs_i^{m,l}$.

Thus, the matrices $CF_i^{m,l}$ and $NSF_i^{m,l}$ look identical, but the parameters of $CF_i^{m,l}$ belong to $cs_i^{m,l}$, i.e., to $ps_i^{m,l}$ with deleted parameter values for which the representatives of $NSF_i^{m,l}$ are reducible to preceding SF by substitutions (1.2).

Proposition 1.3. *No two canonical forms are linearly equivalent.*

This obvious result means that no two representatives of different CFs or, equivalently, no two systems (1.1) generated by corresponding numerical matrices are related by a linear nonsingular substitution.

For $l = 2, 3$, the concepts of a canonical form and a canonical set have to be refined. The fact is that, for certain discriminant values, $CF_i^{m,l}$ ceases to be canonical, i.e., all its representatives whose values are taken from the permissible set with certain third and fourth superscripts are reducible to preceding forms. In these cases, in every $CF_i^{m,l}$, the values of the discriminants for which it remains canonical will be listed in the third and (or) fourth upper positions and all canonical sets for these discriminant values will be described.

In some cases, canonical sets of parameters can be additionally constrained with the help of linear substitutions transforming CF into itself. In what follows, any such constraint undoubtedly facilitates finding generalized normal forms of perturbed systems.

Definition 1.11. The canonical set of any $CF_i^{m,l}$ is called minimal and is denoted by $mcs_i^{m,l}$ if there is a linear nonsingular substitution that transforms $CF_i^{m,l}$ into itself and constrains the values of elements of $cs_i^{m,l}$, namely, if possible, at least one of the nonunit elements becomes bounded from above and (or) below and (or) the sign of the factor σ is fixed.

Thus, if $CF_i^{m,l}$ does not contain parameters or they cannot be constrained, we automatically have $cs_i^{m,l} = mcs_i^{m,l}$, i.e., this canonical set is minimal.

Definition 1.12. The set containing those parameter values from $cs_i^{m,l}$ that can be eliminated by applying linear nonsingular substitutions transforming $CF_i^{m,l}$ into itself is called additional and is denoted by $acs_i^{m,l}$.

Thus, we can write $mcs_i^{m,l} = cs_i^{m,l} \backslash acs_i^{m,l}$.

The concept of *acs* was introduced because it is more convenient to writing it than *mcs* in practice.

1.4. Degenerate Forms for $l = 3$

In addition to the nondegenerate system (1.1) identified with the coefficient matrix $A = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{pmatrix}$, in the case $l = 3$, there are systems such that one row of A is zero.

Definition 1.13. A structural form A ($\mu = \overline{1,4}$) is called degenerate and is denoted by SF_d^{μ} if $A_2 = 0$.

Then, in view of $SP3_1$ and Definition 1.3, the form with a zero first row obtained from SF_d^{μ} with the help of relabeling (2.7)[1] is additional.

Note that the possible use of relabeling in obtaining SF_d^{μ} leads to the refusal of Convention 2.8 from [1] and admits the case $a_1, b_1 = 0, A_2 = 0$.

System (1.1) generated by a degenerate SF is naturally called degenerate. It arises only for $l = 3$ and is system (2.20)[1] with $k = 0$.

For $SF_d^{\mu,3}$, all SPs remain valid, except for $SP3_3$, which is replaced by the following principle: (3_{d3}) otherwise, the subsequent nonzero element of A_1 has a smaller index.

Now, for any $\mu = \overline{1,4}$, according to the introduced ordering, every $SF_d^{\mu,3}$ is assigned its index ι and is denoted by $SF_{d,\iota}^{\mu,3}$. All NPs and the subsequent definitions and notation are naturally extended to $SF_{d,\iota}^{\mu,3}$ with the only difference being that, in NP1, both elements to be normalized are taken from the row A_1.

To conclude, we describe the possibilities provided by degenerate canonical forms for the normalization of perturbed systems in the case $l = 3$.

Supplement 1.1. With the use of $CF_d^{\mu,3}$, there are three different ways of normalizing the system (1.4)[1] $\dot{x} = P(x) + X(x)$, where P corresponds to the case $l = 3$:

(1) $CF_d^{\mu,3}$ itself is used as an unperturbed part.

(2) $CF^{m,3}$ is used as an unperturbed part, and the corresponding linear substitution transforming $CF_d^{\mu,3}$ into $CF^{m,3}$ is made in the perturbed system.

(3) The unperturbed part $CF_d^{\mu,3}$ is made nondegenerate by adding some terms from the perturbation of system (1.4)[1] to $P_2 \equiv 0$ so that the new unperturbed part becomes a quasihomogeneous polynomial due to introducing a corresponding weight.

Convention 1.3. In what follows, (1) the notation "... $\zeta = [\varsigma_1 \vee v_1]$ ... $\eta = [\varsigma_2 \vee v_2]$ ..." means that either $\zeta = \varsigma_1$ and $\eta = \varsigma_2$ or $\zeta = v_1$ and $\eta = v_2$; (2) a condition in round brackets given after another condition is not a requirement, but is presented as a reminder to clarify the subsequent argument; and (3) in the formulations of results, the fact that the denominator is nonzero is not assumed, but is shown in the course of the proof.

2. CANONICAL FORMS OF A HOMOGENEOUS CUBIC SYSTEM WITH A COMMON FACTOR OF THIRD DEGREE

2.1. Six Classes of Linear Equivalence of Systems for $l = 3$

Consider the system (1.1) $\dot{x} = P(\mathrm{x}) = Aq^{[3]}(x)$. For $l = 3$, since the nonzero rows of the matrix A in (2.20)[1] are proportional ($P_2 \equiv kP_1$) and in view of Convention 2.3 from [1], this system can be uniquely written in the form of (2.21)[1] by using formulas (2.23)[1], namely,

$$\dot{x} = P_0(x)Hx, \quad P_0 = x_1^2 + 2\beta x_1 x_2 + \gamma x_2^2, \quad H = \begin{pmatrix} p_1 & q_1 \\ kp_1 & kq_1 \end{pmatrix}, \quad \begin{matrix} p_1^2 + q_1^2 \neq 0, \quad k \neq 0, \\ \delta_{pq} = \det H = 0. \end{matrix} \tag{2.1}$$

By Theorem 2.2 in [1], any substitution (1.2) $x = Ly$ reduces (2.1) to system (2.17)[1] of the form

$$\dot{y} = (\tilde{\alpha}, 2\tilde{\beta}, \tilde{\gamma})q^{[2]}(y)\tilde{H}y, \tag{2.2}$$

where $\tilde{\alpha} = r_1^2 + 2\beta r_1 r_2 + \gamma r_2^2$, $\tilde{\beta} = r_1 s_1 + \beta(r_1 s_2 + r_2 s_1) + \gamma r_2 s_2$, and $\tilde{\gamma} = s_1^2 + 2\beta s_1 s_2 + \gamma s_2^2$ $(\alpha = 1)$ according to $(2.18)^1$ and the matrix $\tilde{H}$ also introduced in $(2.18)^1$ is singular, i.e., $\delta_{\tilde{p}\tilde{q}} = 0$.

Any linear nonsingular substitution transforms the matrix

$$A = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ ka_1 & kb_1 & kc_1 & kd_1 \end{pmatrix} \quad \text{into} \quad \tilde{A} = \begin{pmatrix} (s_2 - ks_1)a^* & (s_2 - ks_1)b^* & (s_2 - ks_1)c^* & (s_2 - ks_1)d^* \\ (kr_1 - r_2)a^* & (kr_1 - r_2)b^* & (kr_1 - r_2)c^* & (kr_1 - r_2)d^* \end{pmatrix}.$$

By using the form of $\tilde{A}$, we can find some relations between the coefficients of the substitution and the structures of the bonded systems.

Proposition 2.1. *Suppose that substitution* (1.2) *reduces* (2.1) *to system* (2.2). *Then the following assertions hold*:

(1) *If* $P_2 \equiv 0 \Leftrightarrow k = 0$, *then* $\tilde{P}_2 \equiv 0 \Leftrightarrow r_2 = 0$.

(2) *If* $P_2 \equiv 0 \Leftrightarrow k = 0$, *then* $\tilde{P}_1 \equiv \tilde{P}_2 \Leftrightarrow k = 1 \Leftrightarrow r_2 = -s_2 \neq 0$.

(3) *If* $P_1 \equiv P_2 \Leftrightarrow k = 1$, *then* $\tilde{P}_1 \equiv \tilde{P}_2 \Leftrightarrow k = 1 \Leftrightarrow r_2 = r_1 + s_1 - s_2$.

Substitution (1.2) is chosen so that $\tilde{H}$ in system (2.2) is a Jordan matrix.

Of course, the form of the substitution depends on the sign of the discriminant $D = \lambda_1^2$ of the characteristic polynomial of H, which is a linear invariant. Here, according to $(2.22)^1$, the eigenvalues satisfy the equalities $\lambda_1 = p_1 + kq_1$ and $\lambda_2 = 0$. Therefore, the set of systems (2.1) is divided into two linearly nonequivalent classes depending on the sign of D.

The substitution $J_1^3 = \begin{pmatrix} 1 & q_1 \\ k & -p_1 \end{pmatrix}$ for $D > 0$ or the substitution $J_2^3 = \begin{pmatrix} 1 & 0 \\ k & q_1^{-1} \end{pmatrix}$ for $D = 0$ transforms system (2.1) into system $(2.17)^1$ of one of the following two forms, respectively:

$$\tilde{A} = \lambda_1 \begin{pmatrix} \tilde{\alpha} & 2\tilde{\beta} & \tilde{\gamma} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$\tilde{\alpha} = 1 + 2\beta k + \gamma k^2, \quad \tilde{\beta} = (1 + \beta k)q_1 - (\beta + \gamma k)p_1, \quad \tilde{\gamma} = \alpha q_1^2 - 2\beta p_1 q_1 + \gamma p_1^2;$$
$$p_1 + kq_1 \neq 0;$$
$$\tilde{A} = \begin{pmatrix} 0 & \tilde{\alpha} & 2\tilde{\beta} & \tilde{\tilde{\gamma}} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{2.3}$$
$$\tilde{\alpha} = 1 + 2\beta k + \gamma k^2, \quad \tilde{\beta} = (\beta + \gamma k)q_1^{-1}, \quad \tilde{\gamma} = \gamma q_1^{-2};$$
$$p_1 + kq_1 = 0 \quad (q_1 \neq 0).$$

Obviously, (2.3_1) and (2.3_2) are degenerate systems of form $(2.20)^1$ with $k = 0$.

By virtue of $(2.19)^1$ $\tilde{D}_0 = \delta^2 D_0$, the set of systems (2.1) is divided into three linearly nonequivalent classes depending on the sign of $D_0 = \beta^2 - \gamma$, which is the common factor P_0.

In what follows, sequentially fixing the various combinations of the signs of D_0 and D, in each of the six equivalence classes, we can simplify system (2.3) as much as possible, while preserving its degeneracy.

2.2. Construction of Degenerate Canonical Forms

Let us prove that the list presented below contains all possible degenerate canonical forms of system (2.1) with their canonical sets from Definitions 1.10 and 1.9.

List 2.1. Ten $CF_{d,\iota}^{\mu,3}$ and their nontrivial $cs_{d,\iota}^{\mu,3}$ $(\sigma, \kappa = \pm 1)$:

$$CF_{d,1}^{1,3,=,>} = \sigma(1,0,0,0), \quad CF_{d,2}^{1,3,=,=} = \sigma(0,1,0,0), \quad CF_{d,3}^{1,3,=,>} = \sigma(0,0,1,0),$$
$$CF_{d,4}^{1,3,=,=} = \sigma(0,0,0,1); \quad CF_{d,1}^{2,3,=,>} = \sigma(1,1,0,0), \quad CF_{d,2,\kappa}^{2,3,\gtrless,>} = \sigma(\kappa,0,1,0),$$

$$CF_{d,3}^{2,3,<,>} = \sigma(1,0,0,1),\quad CF_{d,4}^{2,3,>,>} = \sigma(0,1,1,0),\quad CF_{d,5,+1}^{2,3,<,=} = \sigma(0,+1,0,1);\quad CF_{d,1}^{3,3,\gtrless,>} = \sigma(v,1,1,0);$$
$$cs_{d,2,-1}^{2,3,>,>} = \{\kappa = -1\},\quad cs_{d,2,+1}^{2,3,<,>} = \{\kappa = 1\};\quad cs_{d,1}^{3,3,>,>} = \{v < 1/4,\ v \neq 2/9\},\quad cs_{d,1,}^{3,3,<,>} = \{v > 1/4,\ v \neq 1/3\}.$$

Here, the third and fourth superscripts indicate the signs of D_0 and D for which the forms are canonical, and the right-hand sides contain only the rows A_1, since all A_2 are zero.

Proposition 2.2. *For* $v = 2/9$, $NSF_{d,1}^{3,3}$ *is reduced to* $SF_{d,2,-}^{2,3,>,>}$ *by substitution* (1.2) *with* $s_1 = -3s_2/2$ *and* $r_2 = 0$; *for* $v = 1/4$, *it is reduced to* $SF_{d,1}^{2,3,=,>}$ *by a substitution with* $s_1 = -2s_2$ *and* $r_2 = 0$; *for* $v = 1/3$, *it is reduced to* $SF_{d,3}^{2,3,<,>}$ *by a substitution with* $s_1 = -2s_2$ *and* $r_2 = 0$; *and it is not reduced to preceding forms for the other values of* v.

Family 2.1. The substitutions used in what follows in Section 2:

$$J_1^3 = \{r_1 = 1,\ s_1 = q_1,\ r_2 = k,\ s_2 = -p_1\},\quad J_2^3 = \{r_1 = 1,\ s_1 = 0,\ r_2 = k,\ s_2 = -q_1^{-1}\};$$
$$L_{d,4}^{2,3,>,>} = \{r_1 = (2\beta)^{-1}\tilde{\gamma}s_2,\ s_1,\ r_2 = 0,\ s_2 = |\tilde{\gamma}\lambda_1|^{-1/2}\};$$
$$L_{d,2,-1}^{2,3,>,>} = \{r_1 = |\tilde{\alpha}\lambda_1|^{-1/2},\ s_1 = [0 \vee -\tilde{\alpha}^{-1}\tilde{\beta}|\tilde{\alpha}|^{1/2}|\tilde{\beta}^2\lambda_1|^{-1/2},\ r_2 = 0,\ s_2 = [|\tilde{\gamma}\lambda_1|^{-1/2} \vee \sqrt{2}|\tilde{\gamma}\lambda_1|^{-1/2}]\};$$
$$L_{d,1}^{3,3,>,>} = \{r_1 = (2\tilde{\beta})^{-1}\tilde{\gamma}s_2,\ s_1,\ r_2 = 0,\ s_2 = |\tilde{\gamma}\lambda_1|^{-1/2}\};$$
$$L_{d,1}^{1,3,=,>} = \{r_1 = (\tilde{\alpha}|\lambda_1|)^{-1/2},\ s_1,\ r_2 = 0,\ s_2 = 1\};$$
$$L_{d,3}^{1,3,=,>} = \{r_1 = 1,\ s_1,\ r_2 = 0,\ s_2 = (\tilde{\gamma}|\lambda_1|)^{-1/2}\};$$
$$L_{d,1}^{2,3,=,>} = \{r_1,\ s_1 = (\tilde{\alpha}|\lambda_1|)^{-1/2},\ r_2 = 0,\ s_2 = -\tilde{\alpha}\tilde{\beta}^{-1}s_1\};$$
$$L_{d,2}^{1,3,=,=} = \{r_1 = \tilde{\alpha}^{-1},\ s_1 = -\tilde{\alpha}^{-1}\tilde{\beta},\ r_2 = 0,\ s_2 = 1\};$$
$$L_{d,4}^{1,3,=,=} = \{r_1 = \tilde{\gamma},\ s_1,\ r_2 = 0,\ s_2 = 1\};$$
$$L_{d,2,+1}^{2,3,<,>} = \{r_1 = |\tilde{\alpha}\lambda_1|^{-1/2},\ s_1,\ r_2 = 0,\ s_2 = (\tilde{\gamma}|\lambda_1|)^{-1/2}\};$$
$$L_{d,3}^{2,3,<,>} = \{r_1,\ s_1 = |\tilde{\alpha}\lambda_1|^{-1/2},\ r_2 = 0,\ s_2 = -3\tilde{\alpha}(2\tilde{\beta})^{-1}s_1\};$$
$$L_{d,1}^{3,3,<,>} = \{r_1 = (2\tilde{\beta})^{-1}\gamma^{1/2}|\lambda_1|^{-1/2},\ s_1,\ r_2 = 0,\ s_2 = |\tilde{\gamma}\lambda_1|^{-1/2}\};$$
$$L_{d,5,+1}^{2,3,<,=} = \{r_1 = \tilde{\alpha}^{-1}s_2^{-1},\ s_1 = -\tilde{\alpha}^{-1}\tilde{\beta}s_2,\ r_2 = 0,\ s_2 = \tilde{\tau}^{-1/2}\}.$$

Theorem 2.1. *Any system* (1.1) *with* $l = 3$ *written in the form of* (2.1) *according to* (2.23)[1] *is linearly equivalent to the system generated by a certain representative of the corresponding degenerate canonical form from List* 2.1. *Below, for every* $CF_{d,\iota}^{\mu,3,*,*}$, *we present* (a) *conditions on the coefficients of system* (2.1), (b) *substitutions* (1.2) *that transform the right-hand side of* (2.1) *under the indicated conditions into the chosen form, and* (c) *the resulting values of the factor* σ *and the parameters from* $cs_{d,\iota}^{\mu,3,*,*}$.

$CF_{d,4}^{2,3,>,>}$: (a) $\gamma < \beta^2$, $\lambda_1 \neq 0$, in (2.3$_1$) $\tilde{\alpha} = 0$, $\tilde{\gamma} \neq 0$; (b) J_1^3, $L_{d,4}^{2,3,>,>}$; (c) $\sigma = \mathrm{sgn}(\tilde{\gamma}\lambda_1)$;

$CF_{d,2,-1}^{2,3,>,>}$: (a) $\gamma < \beta^2$, $\lambda_1 \neq 0$, in (2.3$_1$) $\tilde{\alpha} \neq 0$, $\tilde{\gamma} \neq 0$, $\tilde{\beta}^2 = [0 \vee 9\tilde{\alpha}\tilde{\gamma}/8]$; (b) J_1^3, $L_{d,2,-1}^{2,3,>,>}$; (c) $\sigma = [\mathrm{sgn}(\tilde{\gamma}\lambda_1) \vee -\mathrm{sgn}(\tilde{\gamma}\lambda_1)]$;

$CF_{d,1}^{3,3,>,>}$: (a) $\gamma < \beta^2$, $\lambda_1 \neq 0$, in (2.3$_1$) $\tilde{\alpha} \neq 0$, $\tilde{\gamma} \neq 0$, $\tilde{\beta}^2 \neq 0, 9\tilde{\alpha}\tilde{\gamma}/8$; (b) J_1^3, $L_{d,1}^{3,3,>,>}$; (c) $\sigma = \mathrm{sgn}(\tilde{\gamma}\lambda_1)$, $v = \tilde{\alpha}\tilde{\gamma}(2\tilde{\beta})^{-2}$;

$CF_{d,1}^{1,3,=,>}$: (a) $\gamma = \beta^2$, $\lambda_1 \neq 0$, in (2.3$_1$) $\tilde{\gamma} = 0$; (b) J_1^3, $L_{d,1}^{1,3,=,>}$; (c) $\sigma = \mathrm{sgn}\,\lambda_1$;

$CF_{d,3}^{1,3,=,>}$: (a) $\gamma = \beta^2$, $\lambda_1 \neq 0$, in (2.3$_1$) $\tilde{\alpha} = 0$; (b) J_1^3, $L_{d,3}^{1,3,=,>}$; (c) $\sigma = \mathrm{sgn}\,\lambda_1$;

$CF_{d,1}^{2,3,=,>}$: (a) $\gamma = \beta^2$, $\lambda_1 \neq 0$, in (2.3$_1$) $\tilde{\alpha} \neq 0$, $\tilde{\gamma} \neq 0$; (b) J_1^3, $L_{d,1}^{2,3,=,>}$; (c) $\sigma = \mathrm{sgn}\,\lambda_1$;

$CF_{d,2}^{1,3,=,=}$: (a) $\gamma = \beta^2$, $\lambda_1 = 0$, in (2.3_2) $\tilde{\alpha} \neq 0$; (b) J_2^3, $L_{d,2}^{1,3,=,=}$; (c) $\sigma = 1$;

$CF_{d,4}^{1,3,=,=}$: (a) $\gamma = \beta^2$, $\lambda_1 = 0$, in (2.3_2) $\tilde{\alpha} = 0$; (b) J_2^3, $L_{d,4}^{1,3,=,=}$; (c) $\sigma = 1$;

$CF_{d,2,+1}^{2,3,<,>}$: (a) $\gamma > \beta^2$, $\lambda_1 \neq 0$, in (2.3_1) $\tilde{\beta} = 0$; (b) J_1^3, $L_{d,2,+1}^{2,3,<,>}$; (c) $\sigma = \operatorname{sgn}\lambda_1$;

$CF_{d,3}^{2,3,<,>}$: (a) $\gamma > \beta^2$, $\lambda_1 \neq 0$, in (2.3_1) $\tilde{\beta}^2 = 3\tilde{\alpha}\tilde{\gamma}/4$; (b) J_1^3, $L_{d,3}^{2,3,<,>}$; (c) $\sigma = \operatorname{sgn}\lambda_1$;

$CF_{d,1}^{3,3,<,>}$: (a) $\gamma > \beta^2$, $\lambda_1 \neq 0$, in (2.3_1) $\tilde{\beta}^2 \neq 0, 3\tilde{\alpha}\tilde{\gamma}/4$; (b) J_1^3, $L_{d,1}^{3,3,<,>}$; (c) $\sigma = \operatorname{sgn}\lambda_1$, $v = \tilde{\alpha}\tilde{\gamma}(2\tilde{\beta})^{-2}$;

$CF_{d,5,+1}^{2,3,<,=}$: (a) $\gamma > \beta^2$, $\lambda_1 = 0$, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ from (2.3_2); (b) J_2^3, $L_{d,5,+1}^{2,3,<,=}$; (c) $\sigma = 1$.

Proof. Systems (2.3_1) and (2.3_2) are obtained from (2.1) by making substitutions J_1^3 and J_2^3. Let us maximally simplify them, while preserving the condition $\tilde{P}_2 \equiv 0$. For this purpose, according to Proposition 2.1_1, we used an arbitrary substitution (1.2) with $r_2 = 0$, which reduces (2.3_1) and (2.3_2) to systems with a zero second row:

$$\breve{A} = \lambda_1\left(\tilde{\alpha}r_1^2,\, (3\tilde{\alpha}s_1 + 2\tilde{\beta}s_2)r_1,\, 3\tilde{\alpha}s_1^2 + 4\tilde{\beta}s_1s_2 + \tilde{\gamma}s_2^2,\, (\tilde{\alpha}s_1^2 + 2\tilde{\beta}s_1s_2 + \tilde{\gamma}s_2^2)r_1^{-1}s_1\right),$$
$$\breve{A} = \left(0,\, \tilde{\alpha}r_1s_2,\, 2(\tilde{\alpha}s_1 + \tilde{\beta}s_2)s_2,\, (\tilde{\alpha}s_1^2 + 2\tilde{\beta}s_1s_2 + \tilde{\gamma}s_2^2)r_1^{-1}s_2\right), \tag{2.4}$$

respectively. Elements of these systems will be marked by the symbol $\smile$.

(1) Consider $D_0 > 0$, i.e., $P_0(x)$ is factored into two different factors.

(1_1) $\lambda_1 = p_1 + kq_1 \neq 0$ $(D = \lambda_1^2 > 0)$. Substitution J_1^3 reduces (2.1) to system (2.3_1).

(1_1^0) $\tilde{\alpha} = 0$, $\tilde{\gamma} = 0$. Then, in (2.3_1) we obtain $\tilde{P}_1 = (2\tilde{\beta}y_1y_2)(\lambda_1y_1)$, which is not possible, since, according to Convention 2.3_1 and Corollary 2.1 from [1], y_1^2 has to be taken out from system (2.3_1).

(1_1^1) $\tilde{\alpha} \neq 0$, $\tilde{\gamma} = 0$. Then, in (2.3_1), we have $\tilde{P}_1 = (\tilde{\alpha}y_1^2 + 2\tilde{\beta}y_1y_2)(\lambda_1y_1)$, which is the situation from case (1_1^0).

(1_1^2) $\tilde{\alpha} = 0$, $\tilde{\gamma} \neq 0$ $(\tilde{\beta} \neq 0)$. System (2.4_1) is transformed into $\lambda_1s_2(0, 2\tilde{\beta}r_1, 4\tilde{\beta}s_1 + \tilde{\gamma}s_2, (2\tilde{\beta}s_1 + \tilde{\gamma}s_2)s_1r_1^{-1})$ and $\breve{c}_1^2 + \breve{d}_1^2 \neq 0$. In view of SP2, we obtain $\breve{d}_1 = 0 \Leftrightarrow s_1 = 0$. Then system (2.4_1) can be written as $\lambda_1s_2(0, 2\tilde{\beta}r_1, \tilde{\gamma}s_2, 0)$. For $r_1 = \tilde{\gamma}(2\tilde{\beta})^{-1}s_2$ and $s_2 = |\tilde{\gamma}\lambda_1|^{-1/2}$, this is $CF_{d,4}^{2,3,>,>}$ with $\sigma = \operatorname{sgn}(\tilde{\gamma}\lambda_1)$.

(1_1^3) $\tilde{\alpha}$, $\tilde{\gamma} \neq 0$.

(1_1^{3a}) $\tilde{\beta} = 0$ $(\tilde{\alpha}\tilde{\gamma} < 0)$. Then, for $s_1 = 0$, system (2.4_1) is transformed into $\lambda_1(\tilde{\alpha}r_1^2, 0, \tilde{\gamma}s_2^2, 0)$. For $r_1 = |\tilde{\alpha}\lambda_1|^{-1/2}$ and $s_2 = |\tilde{\gamma}\lambda_1|^{-1/2}$, this is $CF_{d,2,-1}^{2,3,>,>}$ with $\sigma = \operatorname{sgn}(\tilde{\gamma}\lambda_1)$.

(1_1^{3b}) $\tilde{\beta} \neq 0$ $(\tilde{\tau} = (\tilde{\beta}^2 - \tilde{\alpha}\tilde{\gamma})^{1/2} \neq |\tilde{\beta}|)$.

The case $\breve{b}_1$, $\breve{c}_1 = 0$ would require that $s_1 = (-2\tilde{\beta} \pm (4\tilde{\beta}^2 - 3\tilde{\alpha}\tilde{\gamma})^{1/2})(3\tilde{\alpha})^{-1}s_2$ and $s_1 = -2\tilde{\beta}(3\tilde{\alpha})^{-1}s_2$, but then we obtain $4\tilde{\beta}^2 - 3\tilde{\alpha}\tilde{\gamma} = 0$, which is not possible, since $\tilde{\beta}^2 > \tilde{\alpha}\tilde{\gamma}$.

Therefore, in view of SP2, we obtain $\breve{d}_1 = 0 \Leftrightarrow s_1 = \tilde{\alpha}^{-1}(-\tilde{\beta} \pm \tilde{\tau})s_2$ or $s_1 = 0$.

Under these constraints, system (2.4_1) has the form

$$\lambda_1\left(\tilde{\alpha}r_1^2,\, \tilde{\beta}(3\tilde{\tau}|\tilde{\beta}|^{-1} - 1)r_1s_2,\, 2\tilde{\alpha}^{-1}\tilde{\tau}(\tilde{\tau} - |\tilde{\beta}|)s_2^2,\, 0\right) \quad or \quad \lambda_1\left(\tilde{\alpha}r_1^2,\, 2\tilde{\beta}r_1s_2,\, \tilde{\gamma}s_2^2,\, 0\right), \tag{2.5}$$

respectively.

(1_1^{3b1}) $|\tilde{\beta}| = 3\tilde{\tau} \Leftrightarrow |\tilde{\beta}| = 3(2\tilde{\alpha}\tilde{\gamma})^{1/2}/4$ $(\tilde{\alpha}\tilde{\gamma} > 0)$. For $r_1 = |\tilde{\alpha}\lambda_1|^{-1/2}$, $s_2 = \sqrt{2}|\tilde{\gamma}\lambda_1|^{-1/2}$, and $s_1 = -\tilde{\alpha}^{-1}\beta|\tilde{\alpha}|^{1/2}|\beta^2\lambda_1|^{-1/2}$, system (2.5_1) is $CF_{d,2,-}^{2,3,>,>}$ with $\sigma = -\operatorname{sgn}(\tilde{\gamma}\lambda_1)$.

(1_1^{3b2}) $3\tilde{\tau} \neq |\tilde{\beta}| \Leftrightarrow \tilde{\beta}^2 \neq 9\tilde{\alpha}\tilde{\gamma}/8$. Then system (2.5_2) with $r_1 = (2\tilde{\beta})^{-1}\tilde{\gamma}s_2$, $s_1 = 0$, and $s_2 = |\tilde{\gamma}\lambda_1|^{-1/2}$ is $CF_{d,1}^{3,3,>,>}$ with $\sigma = -\operatorname{sgn}(\tilde{\gamma}\lambda_1)$ and $v = \tilde{\alpha}\tilde{\gamma}(2\tilde{\beta})^{-2}$ $(v < 1/4,\ v \neq 0, 2/9)$.

(1_2) $\lambda_1 = p_1 + kq_1 = 0$ $(q_1 \neq 0)$. The substitution J_2^3 made in (2.1) yields system (2.3_2).

(1_2^0) $\tilde{\alpha}, \tilde{\gamma} = 0$. Then, in system (2.3_2), $\tilde{P}_1 = (2\tilde{\beta}y_1y_2)(y_2)$, which is the situation from case (1_1^0).

(1_2^1) $\tilde{\alpha} \neq 0, \tilde{\gamma} = 0$. In (2.3_2), we obtain $\tilde{P}_1 = (\tilde{\alpha}y_1^2 + 2\tilde{\beta}y_1y_2)(y_2)$, which is not possible, since, according to Convention 2.3_2 from [1], with an appropriate grouping, $\tilde{P}_1 = (\tilde{\alpha}y_1y_2 + 2\tilde{\beta}y_2^2)(y_1)$ and $\lambda_1 = 1$.

(1_2^2) $\tilde{\alpha} = 0, \tilde{\gamma} \neq 0$. Then, in (2.3_2), $\tilde{P}_1 = (2\tilde{\beta}y_1y_2 + \tilde{\gamma}y_2^2)(y_2)$, which is the situation from case (1_1^0).

(1_2^3) $\tilde{\alpha}, \tilde{\gamma} \neq 0$. In (2.3_2), $\tilde{P}_1 = (\tilde{\alpha}y_1^2 + 2\tilde{\beta}y_1y_2 + \tilde{\gamma}y_2^2)(y_2) = (\tilde{\alpha}(y_1 + \tilde{\zeta}y_2)(y_1 + \tilde{\eta}y_2))(y_2)$ with $\tilde{\zeta} \neq \tilde{\eta}$, which is the situation from case (1_2^1).

(2) Consider $D_0 = 0$, i.e., the common factor $P_0(x)$ is a perfect square.

(2_1) $\lambda_1 = p_1 + kq_1 \neq 0$ $(D = \lambda_1^2 > 0)$. The substitution J_1^3 made in (2.1) yields system (2.3_1).

(2_1^1) $\tilde{\gamma} = 0$ $(\tilde{\beta} = 0, \tilde{\alpha} > 0)$. Then, for $s_1 = 0$, system (2.4_1) is transformed into the form $(\lambda_1\tilde{\alpha}r_1^2, 0, 0, 0)$.

For $r_1 = (\tilde{\alpha}|\lambda_1|)^{-1/2}$ and $s_2 = 1$, this is $CF_{d,1}^{1,3,=,>}$ with $\sigma = \operatorname{sgn}\lambda_1$.

(2_1^2) $\tilde{\alpha} = 0$ $(\tilde{\beta} = 0, \tilde{\gamma} > 0)$. Then, for $s_1 = 0$, system (2.4_1) can be written as $(0, 0, \lambda_1\tilde{\gamma}s_2^2, 0)$.

For $r_1 = 1$ and $s_2 = (\tilde{\gamma}|\lambda_1|)^{-1/2}$, this is $CF_{d,3}^{1,3,=,>}$ with $\sigma = \operatorname{sgn}\lambda_1$.

(2_1^3) $\tilde{\alpha} > 0$, $\tilde{\gamma} > 0$ $(\tilde{\gamma} = \tilde{\alpha}^{-1}\tilde{\beta}^2)$. Then, in (2.4_1), we have $\breve{c}_1 = \lambda_1\tilde{\alpha}^{-1}(3\tilde{\alpha}s_1 + \tilde{\beta}s_2)(\tilde{\alpha}s_1 + \tilde{\beta}s_2)$, $\breve{d}_1 = \lambda_1(\tilde{\alpha}r_1)^{-1}s_1(\tilde{\alpha}s_1 + \tilde{\beta}s_2)^2$, and, hence, for $s_2 = -\tilde{\alpha}\tilde{\beta}^{-1}s_1$, system (2.4_1) can be rewritten as $\lambda_1\tilde{\alpha}r_1(r_1, s_1, 0, 0)$. For $r_1, s_1 = (\tilde{\alpha}|\lambda_1|)^{-1/2}$, this is $CF_{d,1}^{2,3,=,>}$ with $\sigma = \operatorname{sgn}\lambda_1$.

(2_2) $\lambda_1 = p_1 + kq_1 = 0$ $(q_1 \neq 0)$. Making the substitution J_2^3 in (2.1), we obtain system (2.3_2).

(2_2^1) $\tilde{\alpha} > 0$ $(\tilde{\gamma} = \tilde{\alpha}^{-1}\tilde{\beta}^2)$. Then, in (2.4_2), we have $\breve{c}_1 = 2s_2(\tilde{\alpha}s_1 + \tilde{\beta}s_2)$ and $\breve{d}_1 = (\tilde{\alpha}r_1)^{-1}s_2(\tilde{\alpha}s_1 + \tilde{\beta}s_2)^2$; therefore, for $s_1 = -\tilde{\alpha}^{-1}\tilde{\beta}s_2$, system (2.4_2) can be rewritten as $(0, \tilde{\alpha}r_1s_2, 0, 0)$. For $r_1 = \tilde{\alpha}^{-1}$ and $s_2 = 1$, this is $CF_{d,2}^{1,3,=,=}$.

(2_2^2) $\tilde{\alpha} = 0$ $(\tilde{\beta} = 0, \tilde{\gamma} > 0)$. Then system (2.4_2) has the form $(0, 0, 0, \tilde{\gamma}r_1^{-1}s_2^3)$. For $r_1 = \tilde{\gamma}$, $s_1 = 0$, and $s_2 = 1$, this is $CF_{d,4}^{1,3,=,=}$.

(3) Consider $D_0 < 0$, i.e., P_0 has no real zeros.

(3_1) $\lambda_1 = p_1 + kq_1 \neq 0$ $(D = \lambda_1^2 > 0)$. Making the substitution J_1^3 in (2.1) yields system (2.3_1).

(3_1^1) $\tilde{\beta} = 0$ $(\tilde{\alpha}\tilde{\gamma} > 0)$. Then, for $s_1 = 0$, system (2.4_1) is transformed into the form $\lambda_1(\tilde{\alpha}r_1^2, 0, \tilde{\gamma}s_2^2, 0)$. For $r_1 = |\tilde{\alpha}\lambda_1|^{-1/2}$ and $s_2 = |\tilde{\gamma}\lambda_1|^{-1/2}$, this is $CF_{d,2,+1}^{2,3,>,>}$ with $\sigma = \operatorname{sgn}\lambda_1$.

(3_1^2) $\tilde{\beta} \neq 0$.

(3_1^{2a}) $\tilde{\gamma} = 4(3\tilde{\alpha})^{-1}\tilde{\beta}^2$ $(> \tilde{\alpha}^{-1}\tilde{\beta}^2)$. Then, for $s_2 = -3\tilde{\alpha}(2\tilde{\beta})^{-1}s_1$, system (2.4_1) can be written as $\tilde{\alpha}\lambda_1(r_1^2, 0, 0, r_1^{-1}s_1^3)$. For $r_1, s_1 = |\tilde{\alpha}\lambda_1|^{-1/2}$, this is $CF_{d,3}^{2,3,<,>}$ with $\sigma = \operatorname{sgn}\lambda_1$.

(3_1^{2b1}) $\tilde{\gamma} \neq 4(3\tilde{\alpha})^{-1}\tilde{\beta}^2$. In (2.4_1), we obtain $\breve{d}_1 = \lambda_1(\tilde{\alpha}s_1^2 + 2\tilde{\beta}s_1s_2 + \tilde{\gamma}s_2^2)r_1^{-1}s_2$. Then, for $s_1 = 0$, system (2.4_1) is transformed into the form $\lambda_1(\tilde{\alpha}r_1^2, 2\tilde{\beta}r_1s_2, \tilde{\gamma}s_2^2, 0)$. For $r_1 = (2\tilde{\beta})^{-1}\gamma^{1/2}|\lambda_1|^{-1/2}$ and $s_2 = |\tilde{\gamma}\lambda_1|^{-1/2}$, this is $CF_{d,1}^{3,3,<,>}$ with $\sigma = \operatorname{sgn}\lambda_1$ and $v = \tilde{\alpha}\tilde{\gamma}(2\tilde{\beta})^{-2}$ $(v > 1/4, v \neq 1/3)$.

(3_2) $\lambda_1 = p_1 + kq_1 = 0$ $(q_1 \neq 0)$. Making the substitution J_2^3 in (2.1), we obtain system (2.3_2). For $s_1 = -\tilde{\alpha}^{-1}\tilde{\beta}s_2$, system (2.4_2) can be written as $(0, \tilde{\alpha}r_1s_2, 0, \tilde{\alpha}^{-1}\tilde{\tau}^2r_1^{-1}s_2^3)$. For $r_1 = \tilde{\alpha}^{-1}s_2^{-1}$ and $s_2 = \tilde{\tau}^{-1/2}$, this is $CF_{d,5,+}^{2,3,<,=}$ with $\sigma = 1$.

As a result, we have proved the completeness of List 2.1 and the linear nonequivalence of the forms involved. □

Let us single out the minimal canonical sets introduced in Definition 1.11.

Proposition 2.3. The parameter values in *cs* can be constrained only in the following forms from List 2.1: (1) the sign of σ reverses in $CF_{d,2}^{1,3,=,=}$, $CF_{d,4}^{1,3,=,=}$, and $CF_{d,5,+}^{2,3,<,=}$ by using normalization (2.6)[1] with $r_1 = 1$ and $s_2 = -1$ and in $CF_{d,4}^{2,3,>,>}$ by using a substitution with r_1, $s_1 = 1$, $r_2 = 0$, and $s_2 = -1$; (2) in $CF_{d,1}^{3,3,>,>}$ for $\tilde{v} = v \in (2/9,\ 1/4)$, a substitution with $r_1 = (8\tilde{v} - 2 + 2\varrho)^{1/2}(6\tilde{v} - 1 - \varrho)(4\tilde{v})^{-1}(9v - 2)^{-1}$, $s_1 = (8\tilde{v} - 2 + 2\varrho)^{1/2}(4\tilde{v} - 1 - \varrho)(4\tilde{v})^{-1}(4\tilde{v} - 1)^{-1}$, $r_2 = 0$, and $s_2 = -(8\tilde{v} - 2 + 2\varrho)^{1/2}(2\varrho)^{-1}$, where $\varrho = (1 - 4\tilde{v})^{1/2}$, gives $v = (36\tilde{v}^2 - 13\tilde{v} + 1 + \varrho(1 - 3\tilde{v}))(9\tilde{v} - 2)^{-2}/2$; therefore $v \in (0,\ 2/9)$.

Corollary 2.1. According to Definition 1.12, we have the following additional *cs*: $acs_{d,2}^{1,3,=,=}$, $acs_{d,4}^{1,3,=,=}$, $acs_{d,4}^{2,3,>,>}$, $acs_{d,5,+1}^{2,3,<,=} = \{\sigma = -1\}$, $acs_{d,1}^{3,3,>,>} = \{2/9 < v < 1/4\}$. For the other degenerate canonical forms from List 2.1, $mcs_d^{\mu,3,*,*} = cs_d^{\mu,3,*,*}$.

2.3. Reduction of Degenerate Canonical Forms to Canonical Ones

Let us prove that the list presented below contains all possible canonical forms of system (2.1) and their canonical sets introduced in Definitions 1.10 and 1.9.

List 2.2. Seven $CF_{d,i}^{m,3}$ and their nontrivial $cs_{d,i}^{m,3}$ $(\sigma, \kappa = \pm 1)$:

$$CF_5^{2,3,=,>} = \sigma\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad CF_6^{2,3,=,>} = \sigma\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix};$$

$$CF_{18,-1}^{4,3,=,=} = \sigma\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad CF_{20}^{4,3,\gtrless,>} = \sigma\begin{pmatrix} 0 & 1 & 0 & u \\ 0 & 1 & 0 & u \end{pmatrix}, \quad CF_{21,\kappa}^{4,3,<,\geq} = \sigma\begin{pmatrix} 1 & 0 & 0 & \kappa \\ 1 & 0 & 0 & \kappa \end{pmatrix};$$

$$CF_9^{6,3,=,>} = \sigma\begin{pmatrix} 0 & 1 & -2 & 1 \\ 0 & 1 & -2 & 1 \end{pmatrix}; \quad CF_1^{8,3,=,=} = \sigma\begin{pmatrix} 1 & -3 & 3 & -1 \\ 1 & -3 & 3 & -1 \end{pmatrix};$$

$$cs_{20}^{4,3,>,>} = \{u < 0\}, \quad cs_{20}^{4,3,<.>} = \{u > 0\}; \quad cs_{21,+1}^{4,3,<,>} = \{\kappa = 1\}, \quad cs_{21,-1}^{4,3,<,>} = \{\kappa = -1\}.$$

Family 2.2. The substitutions used in what follows in Section 2:

$$L_5^{2,3,=,>} = \{r_1, -r_2, s_2 = 1, s_1 = 0\}; \quad L1_{20}^{4,3,<,>} = \{r_1, s_1, -r_2, s_2 = 3^{-1/2}\};$$

$$L_6^{2,3,=,>} = \{r_1, -r_2, s_2 = 1, s_1 = 0\}; \quad L2_{20}^{4,3,<,>} = \{r_1 = 0, s_1 = -2, -r_2, s_2 = 1\};$$

$$L_{18,-1}^{4,3,=,=} = \{r_1 = 0, s_1, r_2, -s_2 = 1\}; \quad L_{21,+1}^{4,3,<,>} = \{r_1, s_1 = 2^{-1/2}, -r_2, s_2 = (3/2)^{1/2}\};$$

$$L2_{20}^{4,3,>,>} = \{r_1, s_1, s_2 = 1, r_2 = -1\}; \quad L_{21,-1}^{4,3,<,=} = \{r_1, s_1 = 3^{3/4}2^{-1/2}, r_2, -s_2 = 3^{1/4}2^{-1/2}\};$$

$$L1_{20}^{4,3,>,>} = \{r_1, s_2 = 2^{-1/2}, s_1 = s_2/3, r_2 = -s_2\}; \quad L_9^{6,3,=,>} = \{r_1 = 0, s_1, -r_2, s_2 = 1\};$$

$$L3_{20}^{4,3,>,>} = \{r_1 = 0, s_1 = -2, -r_2, s_2 = 1\}; \quad L_1^{8,3,=,=} = \{r_1, r_2, -s_2 = 1, s_1 = 0\}.$$

Let us establish the linear relations between degenerate and nondegenerate canonical forms, thereby proving the linear nonequivalence of all $CF_i^{m,3}$.

Theorem 2.2. *It holds that*

$CF_{d,2,-1}^{2,3,>,>}$ *with* $\tilde{\sigma} = \sigma$ *is reduced by the substitution* $L1_{20}^{4,3,>,>}$ *to* $CF_{20}^{4,3,>,>}$ *with* $\sigma = -\tilde{\sigma}$ *and* $u = -1/9$;

$CF_{d,4}^{2,3,>,>}$ *with* $\tilde{\sigma} = \sigma$ *is reduced by the substitution* $L2_{20}^{4,3,>,>}$ *to* $CF_{20}^{4,3,>,>}$ *with* $\sigma = -\tilde{\sigma}$ *and* $u = -1$;

$CF_{d,1}^{2,3,>,>}$ *is reduced by the substitution* $L3_{20}^{4,3,>,>}$ *to* $CF_{20}^{4,3,>,>}$ *with* $u = 4v - 1$ $(u < 0,\ u \neq -1, -1/9)$;

$CF_{d,1}^{1,3,=,>}$ *is reduced by the substitution* $L_5^{2,3,=,>}$ *to* $CF_5^{2,3,=,>}$;

$CF_{d,3}^{1,3,=,>}$ is reduced by the substitution $L_9^{6,3,=,>}$ to $CF_9^{6,3,=,>}$;

$CF_{d,1}^{2,3,=,>}$ *is reduced by the substitution* $L_5^{2,3,=,>}$ *to* $CF_5^{2,3,=,>}$;

$CF_{d,2}^{1,3,=,=}$ *is reduced by the substitution* $L_{18,-1}^{4,3,=,=}$ to $CF_{18,-1}^{4,3,=,=}$;

$CF_{d,4}^{1,3,=,=}$ *is reduced by the substitution* $L_1^{8,3,=,=}$ *to* $CF_1^{8,3,=,=}$;

$CF_{d,2,+1}^{2,3,<,>}$ *is reduced by the substitution* $L_{21,+1}^{4,3,<,>}$ to $CF_{21,+1}^{4,3,<,>}$;

$CF_{d,3}^{2,3,<,>}$ *is reduced by the substitution* $L1_{20}^{4,3,<,>}$ *to* $CF_{20}^{4,3,<,>}$ *with* $u = 1/3$;

$CF_{d,1}^{3,3,<,>}$ *is reduced by the substitution* $L2_{20}^{4,3,<,>}$ *to* $CF_{20}^{4,3,<,>}$ *with* $u = 4v - 1$ $(u > 0,\ u \neq 1/3)$; *and*

$CF_{d,5,+1}^{2,3,<,=}$ *with* $\tilde{\sigma} = \sigma$ *is reduced by the substitution* $L_{21,-1}^{4,3,<,=}$ *to* $CF_{21,-1}^{4,3,<,=}$ *with* $\sigma = -\tilde{\sigma}$.

Proof. For each $CF_{d,\iota}^{\mu,3}$ from List 2.1, we make substitution (1.2), in which, according to Proposition 2.12, $r_2 = -s_2 \neq 0$. For brevity, we introduce $\delta_1 = (r_1 + s_1)^{-1}$.

$CF_{d,4}^{2,3,>,>}$ is reduced to $\sigma(r_1(r_1 - s_2),\ r_1^2 - 2r_1s_1 - 2r_1s_2 + s_1s_2,\ 2r_1s_1 + r_1s_2 - s_1^2 - 2s_1s_2,\ s_1(s_1 + s_2))\delta_1 s_2$. For r_1, $s_1 = s_2$, we obtain $SF_{20}^{4,3,>,>}$ of the form $\sigma(0,-1,0,1)s_2^2$.

$CF_{d,2,-1}^{2,3,>,>}$ is reduced to $\sigma(r_1(r_1 - s_2)(r_1 + s_2), 3r_1^2s_1 + 2r_1s_2^2 - s_1s_2^2, 3r_1s_1^2 - r_1s_2^2 + 2s_1s_2^2, s_1(s_1 - s_2)(s_1 + s_2))\delta_1$. For r_1, $3s_1 = s_2$, we obtain $SF_{20}^{4,3,>,>}$ of the form $2\sigma(0,-1,0,1/9)\,s_2^2$. Instead of $c_i = 0$, we can use $d_i = 0$, thus obtaining $SF_{23}^{4,3}$, which is preceded by $SF_{20}^{4,3,>,>}$.

$CF_{d,1}^{3,3,>,>}$ $(v < 1/4,\ v \neq 0,\ 2/9)$ is reduced to $\sigma(r_1(vr_1^2 - r_1s_2 + s_2^2),\ 3vr_1^2s_1 - 2r_1s_1s_2 + r_1^2s_2 - 2r_1s_2^2 + s_1s_2^2,$ $3vr_1s_1^2 + 2r_1s_1s_2 - s_1^2s_2 - 2s_1s_2^2 + r_1s_2^2,\ s_1(vs_1^2 + s_1s_2 + s_2^2))\delta_1$. For $r_1 = 0$ and $s_1 = -2s_2$, we obtain $SF_{20}^{4,3}$ of the form $\sigma(0,1,0,4v-1)s_2^2$. It is also possible to obtain $SF_{23}^{4,3}$.

$CF_{d,1}^{1,3,=,>}$ is reduced to $\sigma(r_1^3, 3r_1^2s_1, 3r_1s_1^2, s_1^3)\delta_1$. For $s_1 = 0$, we obtain $SF_5^{2,3,=,>}$ of the form $\sigma(1,0,0,0)r_1^2$.

$CF_{d,3}^{1,3,=,>}$ is reduced to $\sigma(r_1, s_1 - 2r_1, r_1 - 2s_1, s_1)\delta_1 s_2^2$. For $r_1 = 0$, we obtain $SF_9^{6,3,=,>}$ of the form $\sigma(0,1,-2,1)s_2^2$. Instead of $a_i = 0$, we can use $b_i = 0$ to obtain $SF_{10}^{6,3}$.

$CF_{d,1}^{2,3,=,>}$ is reduced to $\sigma(r_1^2(r_1 - s_2),\ r_1(3r_1s_1 - 2s_1s_2 + r_1s_2),\ s_1(3r_1s_1 + 2r_1s_2 - s_1s_2),\ s_1^2(s_1 + s_2))\delta_1$. For $r_1 = s_2$ and $s_1 = 0$, we obtain $SF_6^{2,3,=,>}$ of the form $\sigma(0,1,0,0)s_2^2$.

$CF_{d,2}^{1,3,=,=}$ is reduced to $\sigma(-r_1^2, r_1(r_1 - 2s_1),\ s_1(2r_1 - s_1), s_1^2)\delta_1 s_2$. For $r_1 = 0$, we obtain $SF_{18,-1}^{4,3,=,=}$ of the form $\sigma(0,0,-1,1)s_1s_2$.

$CF_{d,4}^{1,3,=,=}$ is reduced to $\sigma(-1,3,-3,1)\delta_1 s_2^3$.

$CF_{d,2,+1}^{2,3,<,>}$ is reduced to $\sigma(r_1(r_1^2 + s_2^2),\ 3r_1^2s_1 - 2r_1s_2^2 + s_1s_2^2,\ 3r_1s_1^2 - 2s_1s_2^2 + r_1s_2^2,\ s_1(s_1^2 + s_2^2))\delta_1$. For r_1, $s_1 = 3^{-1/2}s_2$, we obtain $SF_{21,+}^{4,3,<,>}$ of the form $2\sigma(1,0,0,1)s_2^2/3$.

$CF_{d,3}^{2,3,<,>}$ is reduced to $\sigma((r_1 - s_2)(r_1^2 + r_1s_2 + s_2^2),\ 3(r_1^2s_1 + s_2^3),\ 3(r_1s_1^2 - s_2^3),\ (s_1 + s_2)(s_1^2 - s_1s_2 + s_2^2))\delta_1$. For r_1, $s_1 = s_2$, we obtain $SF_{20}^{4,3,<,>}$ of the form $3\sigma(0,1,0,1/3)s_2^2$.

$CF_{d,1}^{3,3,<,>}$ $(v > 1/4,\ v \neq 1/3)$. Everything is similar to $CF_{d,1}^{3,3,>,>}$.

$CF_{d,5,+}^{2,3,<,=}$ is reduced to $\sigma(-r_1^2 - s_2^2,\ r_1^2 - 2r_1s_1 + 3s_2^2,\ 2r_1s_1 - s_1^2 - 3s_2^2,\ s_1^2 + s_2^2)\delta_1 s_2$. For r_1, $s_1 = 3^{1/2}s_2$, we obtain $SF_{21,-1}^{4,3,<,=}$ of the form $2\sigma(-1,0,0,1)s_2^2/3^{1/2}$.

Now, following the NPs, it remains to make normalization $(2.6)^1$ in all $SF^{m,3}$ obtained.

Since $CF_d^{\mu,3}$ from List 2.1 are pairwise linearly nonequivalent and any original system (2.1) is reduced to one of them, we conclude that List 2.2 covers all $CF^{m,3}$. □

Theorems 2.1 and 2.2 imply an assertion establishing linear relations between the original system (2.1) and various canonical forms from the list.

Theorem 2.3. *Any system* (1.1) *with* $l = 3$ *written in the form of* (2.1) *according to* $(2.23)^1$ *is linearly equivalent to the system generated by some representative of the corresponding canonical form from List* 2.2. *Below, for every* $CF_i^{m,3,*,*}$, *we present* (a) *conditions on the coefficients of system* (2.1), (b) *substitutions* (1.2) *that transform the right-hand side of* (2.1) *under the indicated conditions into the chosen form, and* (c) *the resulting values of the factor* σ *and the parameters from* $cs_i^{m,3,*,*}$:

$CF_{20}^{4,3,>,>}$: (a) $\gamma < \beta^2$, $\lambda_1 \neq 0$, *in* (2.3$_1$) $\tilde{\gamma} \neq 0$ *and* (a$_1$) $\tilde{\alpha} = 0$; (b$_1$) J_1^3, $L_{d,4}^{2,3,>,>}$, $L1_{20}^{4,3,>,>}$; (c$_1$) $\sigma = -\operatorname{sgn}(\tilde{\gamma}\lambda_1)$, $u = -1$; (a$_2$) $\tilde{\alpha} \neq 0$, $\tilde{\beta}^2 = [0 \vee 9\tilde{\alpha}\tilde{\gamma}/8]$; (b$_2$) J_1^3, $L_{d,2,-1}^{2,3,>,>}$, $L2_{20}^{4,3,>,>}$; (c$_2$) $\sigma = [-1 \vee 1]\operatorname{sgn}(\tilde{\gamma}\lambda_1)$, $u = -1/9$; (a$_3$) $\tilde{\alpha} \neq 0$, $\tilde{\beta}^2 \neq 0, 9\tilde{\alpha}\tilde{\gamma}/8$; (b$_3$) J_1^3, $L_{d,1}^{3,3,>,>}$, $L3_{20}^{4,3,>,>}$; (c$_3$) $\sigma = \operatorname{sgn}(\tilde{\gamma}\lambda_1)$, $u = \tilde{\alpha}\tilde{\gamma}\tilde{\beta}^{-2} - 1$ ($u < 0$, $u \neq -1, -1/9$);

$CF_5^{2,3,=,>}$: (a) $\gamma = \beta^2$, $\lambda_1 \neq 0$, *in* (2.3$_1$) $\tilde{\gamma} = 0$; (b) J_1^3, $L_{d,1}^{1,3,=,>}$, $L_5^{2,3,=,>}$; (c) $\sigma = \operatorname{sgn} \lambda_1$;

$CF_9^{6,3,=,>}$: (a) $\gamma = \beta^2$, $\lambda_1 \neq 0$, *in* (2.3$_1$) $\tilde{\alpha} = 0$; (b) J_1^3, $L_{d,3}^{1,3,=,>}$, $L_9^{6,3,=,>}$; (c) $\sigma = \operatorname{sgn} \lambda_1$;

$CF_6^{2,3,=>}$: (a) $\gamma = \beta^2$, $\lambda_1 \neq 0$, *in* (2.3$_1$) $\tilde{\alpha}, \tilde{\gamma} \neq 0$; (b) J_1^3, $L_{d,1}^{2,3,=>}$, $L_6^{2,3,=>}$; (c) $\sigma = \operatorname{sgn} \lambda_1$;

$CF_{18,-1}^{4,3,=,=}$: (a) $\gamma = \beta^2$, $\lambda_1 = 0$, *in* (2.3$_2$) $\tilde{\alpha} \neq 0$; (b) J_2^3, $L_{d,2}^{1,3,=,=}$, $L_{18,-1}^{4,3,=,=}$; (c) $\sigma = 1$;

$CF_1^{8,3,=,=}$: (a) $\gamma = \beta^2$, $\lambda_1 = 0$, *in* (2.3$_2$) $\tilde{\alpha} = 0$; (b) J_2^3, $L_{d,4}^{1,3,=,=}$, $L_1^{8,3,=,=}$; (c) $\sigma = 1$;

$CF_{21,+}^{4,3,<,>}$: (a) $\gamma > \beta^2$, $\lambda_1 \neq 0$, *in* (2.3$_1$) $\tilde{\beta} = 0$; (b) J_1^3, $L_{d,2,+1}^{2,3,<,>}$, $L_{21,+}^{4,3,<,>}$; (c) $\sigma = \operatorname{sgn} \lambda_1$;

$CF_{20}^{4,3,<,>}$: (a) $\gamma > \beta^2$, $\lambda_1 \neq 0$, *in* (2.3$_1$): (a$_1$) $\tilde{\beta}^2 = 3\tilde{\alpha}\tilde{\gamma}/4$; (b$_1$) J_1^3, $L_{d,3}^{2,3,<,>}$, *and* $L1_{20}^{4,3,<,>}$; (c$_1$) $\sigma = \operatorname{sgn} \lambda_1$, $u = 1/3$; (a$_2$) $\tilde{\beta}^2 \neq 0, 3\tilde{\alpha}\tilde{\gamma}/4$; (b$_2$) J_1^3, $L_{d,1}^{3,3,<,>}$, *and* $L2_{20}^{4,3,<,>}$; (c$_2$) $\sigma = \operatorname{sgn} \lambda_1$, $u = \tilde{\alpha}\tilde{\gamma}\tilde{\beta}^{-2}$ ($u > 0$, $u \neq 1/3$);

$CF_{21,-1}^{4,3,<,=}$: (a) $\gamma > \beta^2$, $\lambda_1 = 0$, $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$ *from* (2.3$_2$); (b) J_2^3, $L_{d,5,+1}^{2,3,<,=}$, *and* $L_{21,-1}^{4,3,<,=}$; (c) $\sigma = -1$.

Here, the substitutions J_1^3, J_2^3, *and* $L_{d,\iota}^{\mu,3,*,*}$ *are given in Family* 2.1, *while* $L_i^{m,3,*,*}$, *in Family* 2.2.

Let us single out the minimal canonical sets introduced in Definition 1.11.

Proposition 2.4. *The parameter values in* $cs^{m,3}$ *can be constrained in view of Proposition* 2.1$_3$ *only in the following* $CF^{m,3}$ *from List* 2.2:

(1) *The sign of* σ *is reversed by a substitution with* $r_1 = 1$, $s_1 = -2$, $r_2 = 0$, *and* $s_2 = -1$ *in* $CF_{18,-1}^{4,3,=,=}$ *and by relabeling* (2.7)[1] *in* $CF_{21,-1}^{4,3,<,=}$ *and* $CF_1^{8,3,=,=}$.

(2) In $CF_{20}^{4,3,>,>}$, *for* $\tilde{u} = u \in (-\infty, -1) \cup (-1, -1/9)$, *a substitution with* $r_1 = \varrho(2\tilde{u} + 2\varrho)^{-1/2}$, $s_1 = -3\varrho(\varrho - 1)(3\varrho + 1)^{-1}(2\tilde{u} + 2\varrho)^{-1/2}$, $r_2 = (2\tilde{u} + 2\varrho)^{-1/2}$, *and* $s_2 = (\varrho - 1)(3\varrho + 1)^{-1}(2\tilde{u} + 2\varrho)^{-1/2}$, *where* $\varrho = \sqrt{-\tilde{u}}$, *yields* $u = -(\varrho - 1)^2(3\varrho + 1)^{-2}$, *so* $u \in (-1/9, 0)$, *while, for* $u = -1$, *a substitution with* $r_1 = 1/2$, $s_1 = -3/2$, *and* $r_2, s_2 = -1/2$ *changes the sign of* σ.

Corollary 2.2. According to Definition 1.12, we have the following additional *cs*: $acs_{18,-1}^{4,3,=,=}$, $acs_{21,-1}^{4,3,<,=}$, $acs_1^{8,3,=,=} = \{\sigma = -1\}$, and $acs_{20}^{4,3,>,>} = \{u \in (-\infty, -1) \cup (-1, -1/9)$, $\sigma = -1$ for $u = -1\}$. For the other canonical forms from List 2.2, $mcs^{m,3,*,*} = cs^{m,3,*,*}$.

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