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ON AN INVERSE DYNAMIC PROBLEM FOR THE WAVE EQUATION WITH A POTENTIAL ON A REAL LINE

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The inverse dynamic problem for the wave equation with a potential on a real line is considered. The forward initial-boundary value problem is set up with the help of boundary triplets. As an inverse data, an analog of the response operator (dynamic Dirichlet-to-Neumann map) is used. Equations of the inverse problem are derived; also, a relationship between the dynamic inverse problem and the spectral inverse problem from a matrix-valued measure is pointed out. Bibliography: 16 *titles.*

1. Introduction

For a potential $q \in C^2(R) \cap L_1(\mathbb{R})$ we consider an operator H in $L_2(\mathbb{R})$ given by

$$(Hf)(x) = -f''(x) + q(x)f(x), \quad x \in \mathbb{R},$$
$$\operatorname{dom} H = \{f \in H^2(\mathbb{R}) \,|\, f(0) = f'(0) = 0\}.$$

Then

$$(H^*f)(x) = -f''(x) + q(x)f(x), \quad x \in \mathbb{R},$$
$$\operatorname{dom} H^* = \{f \in L_2(\mathbb{R}) \,|\, f \in H^2(-\infty, 0),\ f \in H^2(-\infty, 0)\}.$$

For a continuous function g we denote

$$g_\pm := \lim_{\varepsilon \to 0} g(0 \pm \varepsilon).$$

Let $B := \mathbb{R}^2$. The *boundary operators* $\Gamma_{0,1} : \operatorname{dom} H^* \mapsto B$ are introduced by the rules

$$\Gamma_0 w := \begin{pmatrix} w_+ - w_- \\ w'_+ - w'_- \end{pmatrix}, \quad \Gamma_1 w := \frac{1}{2} \begin{pmatrix} w'_+ + w'_- \\ -w_+ - w_- \end{pmatrix}.$$

Integrating by parts for $u, v \in \operatorname{dom} H^*$ shows that the abstract second Green identity holds:

$$(H^*u, v)_{L_2(\mathbb{R})} - (u, H^*v)_{L_2(\mathbb{R})} = (\Gamma_1 u, \Gamma_0 v)_B - (\Gamma_0 u, \Gamma_1 v)_B\,.$$

The mapping

$$\Gamma := \begin{pmatrix} \Gamma_0 \\ \Gamma_1 \end{pmatrix} : \operatorname{dom} H^* \mapsto B \times B$$

is obviously surjective. Then a triplet $\{B, \Gamma_0, \Gamma_1\}$ is a *boundary triplet* for H^* (see [9]). With the help of boundary triplets, one can describe self-adjoint extensions of H, see [10, 12, 16]. In [6] the authors used the concept of boundary triplets to set up and study a boundary value problem for an abstract dynamical system with a boundary control in Hilbert space; they also used it for the purpose of describing a special (wave) model of the one-dimensional Schrödinger operator on an interval [8].

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Let $T > 0$ be fixed. We use the triplet $\{B, \Gamma_0, \Gamma_1\}$ to set up the dynamical system with a special boundary control (acting in the origin) for a wave equation with a potential on a real line:

$$u_{tt} + H^* u = 0, \quad t > 0, \tag{1.1}$$

$$(\Gamma_0 u)(t) = \begin{pmatrix} f_1(t) \\ f_2'(t) \end{pmatrix}, \quad t > 0, \tag{1.2}$$

$$u(\,\cdot\,, 0) = u_t(\,\cdot\,, 0) = 0. \tag{1.3}$$

Here the function $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $f_1\, f_2 \in L_2(0,T)$, is interpreted as a *boundary control.* The solution to (1.1)–(1.3) is denoted by u^F. The *response operator*, an analog of the Dirichlet-to-Neumann map, is introduced by the rule

$$\left(R^T F\right)(t) := \left(\Gamma_1 u^F\right)(t), \quad t > 0.$$

The speed of wave propagation in system (1.1)–(1.3) is equal to one, that is why the natural set up of the dynamic inverse problem is to find a potential $q(x)$, $x \in (-T, T)$, from the knowledge of a response operator R^{2T} (cf. [1,3,7]).

In the second section, we derive the representation formula for the solution u^F and introduce the operators of the Boundary Control method. In the third section, we derive the Krein and Gelfand–Levitan equations of the dynamic inverse problem and point out the the relationship between the dynamic and spectral inverse problems.

2. Forward problem, operators of the Boundary Control method

It is straightforward to check that when $q = 0$, the solution to (1.1)–(1.3) is given by:

$$u^F(x,t) = \begin{cases} \frac{1}{2} f_1(t-x) - \frac{1}{2} f_2(t-x), & x > 0, \\ -\frac{1}{2} f_1(t+x) - \frac{1}{2} f_2(t+x), & x < 0, \\ 0, & 0 < t < |x|. \end{cases}$$

Everywhere we consider operators acting in L_2-spaces; for this reason it is appropriate to introduce the *outer space* of system (1.1)–(1.3), the space of controls as $\mathcal{F}^T := L_2(0, T; \mathbb{R}^2)$, $F \in \mathcal{F}^T$, $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.

Theorem 1. *The solution to* (1.1)–(1.3) *with a control* $F \in \mathcal{F}^T \cap C_0^\infty(\mathbb{R}_+)$ *admits the following representation:*

$$u^F(x,t) = \begin{cases} \frac{1}{2} f_1(t-x) - \frac{1}{2} f_2(t-x) + \int\limits_x^t w_1(x,s) f_1(t-s) \\ \qquad + w_2(x,s) f_2(t-s)\, ds, & 0 < x < t, \\ -\frac{1}{2} f_1(t+x) - \frac{1}{2} f_2(t+x) + \int\limits_{-x}^t w_1(x,s) f_1(t-s) \\ \qquad + w_2(x,s) f_2(t-s)\, ds, & 0 < -x < t, \\ 0, & 0 < t < |x|. \end{cases} \tag{2.1}$$

where the kernels $w_1(x,t)$ *and* $w_2(x,t)$ *satisfy the following Goursat problems:*

$$\begin{cases} w_{1tt}(x,t) - w_{1xx}(x,t) + q(x) w_1(x,t), & 0 < |x| < t, \\ \frac{d}{dx} w_1(x,x) = -\frac{q(x)}{4}, & x > 0, \\ \frac{d}{dx} w_1(x,-x) = -\frac{q(x)}{4}, & x < 0, \end{cases} \tag{2.2}$$

$$\begin{cases} w_{2tt}(x,t) - w_{2xx}(x,t) + q(x)w_2(x,t), & 0 < |x| < t, \\ \frac{d}{dx} w_2(x,x) = \frac{q(x)}{4}, & x > 0, \\ \frac{d}{dx} w_2(x,-x) = -\frac{q(x)}{4}, & x < 0. \end{cases} \tag{2.3}$$

Proof. Take arbitrary $F \in \mathcal{F}^T \cap C_0^\infty(0,T;\mathbb{R}^2)$ and look for u^F in the form (2.1). Then for $x > 0$ we have

$$\begin{aligned} u_{xx}(x,t) = &\frac{1}{2} f_1''(t-x) - \frac{1}{2} f_2''(t-x) - \frac{d}{dx} w_1(x,x) f_1(t-x) \\ &+ w_1(x,x) f_1'(t-x) - \frac{d}{dx} w_2(x,x) f_2(t-x) + w_2(x,x) f_2'(t-x) \\ &- w_{1x}(x,x) f_1(t-x) - w_{2x}(x,x) f_2(t-x) \\ &+ \int_x^t w_{1xx}(x,s) f_1(t-s) + w_{2xx}(x,s) f_2(t-s)\, ds, \end{aligned}$$

$$\begin{aligned} u_{tt}(x,t) = &\frac{1}{2} f_1''(t-x) - \frac{1}{2} f_2''(t-x) + w_1(x,x) f_1'(t-x) \\ &+ w_2(x,x) f_2'(t-x) + w_{1s}(x,x) f_1(t-x) + w_{2s}(x,x) f_2(t-x) \\ &+ \int_x^t \left(w_{1ss}(x,s) f_1(t-s) + w_{2ss}(x,s) f_2(t-s) \right) ds, \end{aligned}$$

Plugging these expressions into (1.1), we derive that for $x > 0$ the following relation holds true:

$$\begin{aligned} 0 = &\int_x^t ((w_{1ss}(x,s) - w_{1xx}(x,s) + q(x) w_1(x,s)) f_1(t-s) \\ &+ (w_{2ss}(x,s) - w_{2xx}(x,s) + q(x) w_2(x,s)) f_2(t-s))\, ds \\ &+ f_1(t-x)\Big[2\frac{d}{dx} w_1(x,x) + \frac{q(x)}{2}\Big] + f_2(t-x)\Big[2\frac{d}{dx} w_2(x,x) - \frac{q(x)}{2}\Big]. \end{aligned} \tag{2.4}$$

Similarly, for $x < 0$:

$$\begin{aligned} u_{xx}(x,t) = &-\frac{1}{2} f_1''(t+x) - \frac{1}{2} f_2''(t+x) \\ &+ \frac{d}{dx} w_1(x,-x) f_1(t+x) + w_1(x,-x) f_1'(t+x) \\ &+ \frac{d}{dx} w_2(x,-x) f_2(t+x) + w_2(x,-x) f_2'(t+x) \\ &+ w_{1x}(x,-x) f_1(t+x) + w_{2x}(x,-x) f_2(t+x) \\ &+ \int_{-x}^t w_{1xx}(x,s) f_1(t-s) + w_{2xx}(x,s) f_2(t-s)\, ds, \end{aligned}$$

$$\begin{aligned} u_{tt}(x,t) = &-\frac{1}{2} f_1''(t+x) - \frac{1}{2} f_2''(t+x) \\ &+ w_1(x,-x) f_1'(t+x) + w_2(x,-x) f_2'(t+x) \\ &+ w_{1s}(x,-x) f_1(t+x) + w_{2s}(x,-x) f_2(t+x) \end{aligned}$$

$$+\int_{-x}^{t}\big(w_{1ss}(x,s)f_1(t-s)+w_{2ss}(x,s)f_2(t-s)\big)\,ds.$$

Then for $x<0$ we have the relation

$$\begin{aligned}0=&\int_{-x}^{t}\big((w_{1ss}(x,s)-w_{1xx}(x,s)+q(x)w_1(x,s))f_1(t-s)\\&+(w_{2ss}(x,s)-w_{2xx}(x,s)+q(x)w_2(x,s))f_2(t-s)\big)\,ds\\&+f_1(t+x)\Big[-2\frac{d}{dx}w_1(x,-x)-\frac{q(x)}{2}\Big]\\&+f_2(t+x)\Big[-2\frac{d}{dx}w_2(x,-x)-\frac{q(x)}{2}\Big].\end{aligned}\tag{2.5}$$

The condition $\Gamma_0 u=F$ at $x=0$ yields that

$$\begin{aligned}u^+(\,\cdot\,,t)-u^-(\,\cdot\,,t)=f_1(t)+&\int_0^t(w_1{}^+(0,s)-w_1{}^-(0,s))f_1(t-s)\\&+(w_2{}^+(0,s)-w_2{}^-(0,s))f_2(t-s)\,ds,\\u_x^+(\,\cdot\,,t)-u_x^-(\,\cdot\,,t)=f_2'(t)+&\int_0^t(w_{1x}^+(0,s)-w_{1x}^-(0,s))f_1(t-s)\\&+(w_{2x}^+(0,s)-w_{2x}^-(0,s))f_2(t-s)\,ds.\end{aligned}$$

The above relations imply the continuity of the kernels w_1, w_2 at $x=0$:

$$w_1{}^+(0,s)=w_1{}^-(0,s),\quad w_2{}^+(0,s)=w_2{}^-(0,s),\tag{2.6}$$

$$w_{1x}^+(0,s)=w_{1x}^-(0,s),\quad w_{2x}^+(0,s)=w_{2x}^-(0,s).\tag{2.7}$$

Using the arbitrariness of $F\in\mathcal{F}^T\cap C_0^\infty(0,T;\mathbb{R}^2)$ in (2.4), (2.5) and continuity conditions (2.6), (2.6), we conclude that w_1, w_2 satisfy (2.2), (2.3). ☐

Remark 1. When $F\in\mathcal{F}^T$, the function u^F defined by (2.1) is a generalized solution to (1.1)–(1.3).

The *response operator* $R^T:\mathcal{F}^T\mapsto\mathcal{F}^T$ with the domain

$$D_R=\big\{\mathcal{F}^T\cap C_0^\infty(0,T;\mathbb{R}^2)\big\}$$

is defined by

$$(R^TF)(t):=\big(\Gamma_1u^F\big)(t),\quad 0<t<T.$$

Representation (2.1) implies that the response operator has the form

$$\begin{aligned}(R^TF)(t)=&\begin{pmatrix}(R_1F)(t)\\(R_2F)(t)\end{pmatrix}=-\frac{1}{2}\begin{pmatrix}f_1'(t)\\-f_2(t)\end{pmatrix}+R*\begin{pmatrix}f_1\\f_2\end{pmatrix}\\=&\begin{pmatrix}-\frac12 f_1'(t)+\int_0^t(w_{1x}(0,s)f_1(t-s)+w_{2x}(0,s)f_2(t-s))\,ds\\\frac12 f_2(t)-\int_0^t(w_1(0,s)f_1(t-s)+w_2(0,s)f_2(t-s))\,ds\end{pmatrix},\end{aligned}\tag{2.8}$$

where

$$R(t) := \begin{pmatrix} r_{11}(t) & r_{12}(t) \\ r_{21}(t) & r_{22}(t) \end{pmatrix} = \begin{pmatrix} w_{1x}(0,t) & w_{2x}(0,t) \\ -w_1(0,t) & -w_2(0,t) \end{pmatrix}$$

is a *response matrix.* We introduce the *inner space*, the space of states of system (1.1)–(1.3) as $\mathcal{H}^T := L_2(-T,T)$. The representation (2.1) implies that $u^F(\,\cdot\,,T) \in \mathcal{H}^T$.

A *control operator* $W^T : \mathcal{F}^T \mapsto \mathcal{H}^T$ is defined by the formula $W^T F := u^F(\,\cdot\,,T)$. The *reachable set* is defined by the rule

$$U^T := W^T \mathcal{F}^T = \left\{ u^F(\,\cdot\,,T) \,\middle|\, F \in \mathcal{F}^T \right\}.$$

We introduce the notation

$$S := \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \quad J^T : \mathcal{F}^T \mapsto \mathcal{F}^T, \quad \left(J^T F\right)(t) = F(T-t),$$

and note that

$$S = S^*, \quad SS = \frac{1}{2} I.$$

It will be convenient to us to associate the outer space $\mathcal{H}^T = L_2(-T,T)$ with a vector space $L_2(0,T;\mathbb{R}^2)$ by setting for $a \in L_2(-T,T)$ (we keep the same notation for a function)

$$a = \begin{pmatrix} a_1(x) \\ a_2(x) \end{pmatrix} \in L_2(0,T;\mathbb{R}^2), \quad a_1(x) := a(x), \quad a_2(x) := a(-x), \quad x \in (0,T).$$

Thus, bearing in mind this association, we consider the control operator W^T, which maps $\mathcal{F}^T$ to $\mathcal{H}^T = L_2(0,T;\mathbb{R}^2)$, acting (cf. (2.1)) by the rule:

$$\left(W^T F\right)(x) = \begin{pmatrix} \frac{1}{2} f_1(T-x) - \frac{1}{2} f_2(T-x) \\ -\frac{1}{2} f_1(T-x) - \frac{1}{2} f_2(T-x) \end{pmatrix} + \begin{pmatrix} \int\limits_x^T w_1(x,s) f_1(T-s) + w_2(x,s) f_2(T-s)\, ds \\ \int\limits_x^T w_1(-x,s) f_1(T-s) + w_2(-x,s) f_2(T-s)\, ds \end{pmatrix}.$$

On introducing the operator $W : \mathcal{F}^T \mapsto \mathcal{H}^T = L_2(0,T;\mathbb{R}^2)$ defined by the formula

$$(WF)(x) = \begin{pmatrix} \int\limits_x^T w_1(x,s) f_1(s) + w_2(x,s) f_2(s)\, ds \\ \int\limits_x^T w_1(-x,s) f_1(s) + w_2(-x,s) f_2(s)\, ds \end{pmatrix}$$

and noting that $\mathcal{F}^T = \mathcal{H}^T$, without abusing the notation we can rewrite W^T in the form

$$W^T F = S\,(I + 2SW)\, J^T F = S\,(I + K)\, J^T F, \tag{2.9}$$

where

$$K = 2SW, \quad (KF)(x) = \begin{pmatrix} \int\limits_x^T k_{11}(x,s) f_1(s) + k_{12}(x,s) f_2(s)\, ds \\ \int\limits_x^T k_{21}(x,s) f_1(s) + k_{22}(x,s) f_2(s)\, ds \end{pmatrix}. \tag{2.10}$$

Theorem 2. *The control operator is a boundedly invertible isomorphism between $\mathcal{F}^T$ and $\mathcal{H}^T$, and $U^T = \mathcal{H}^T$.*

Proof. It is clear that in representation (2.9) each of the operators $S : \mathcal{H}^T \mapsto \mathcal{H}^T$, $I + K : \mathcal{F}^T \mapsto \mathcal{H}^T$, $J^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ is a boundedly invertible isomorphism. □

The *connecting operator* $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ is introduced via the quadratic form

$$\left(C^T F_1, F_2\right)_{\mathcal{F}^T} = \left(u^{F_1}(\,\cdot\,,T), u^{F_2}(\,\cdot\,,T)\right)_{\mathcal{H}^T}.$$

The crucial fact in the Boundary Control method is that the connecting operator is expressed in terms of inverse dynamic data:

Theorem 3. *The connecting operator C^T admits the following representation:*

$$\left(C^T F\right)(t) = \frac{1}{2}\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} + \int_0^T C(t,s)\begin{pmatrix} f_1(s) \\ f_2(s) \end{pmatrix} ds,$$

where

$$C_{1,1}(t,s) = p_1(2T-t-s) - p_1(|t-s|), \quad p_1(s) = \int_0^s r_{11}(\alpha)\,d\alpha,$$

$$C_{1,2}(t,s) = \widetilde{p}_1(2T-t-s) - \widetilde{p}_1(t-s), \quad \widetilde{p}_1(s) = \begin{cases} \int_0^s r_{12}(\alpha)\,d\alpha, & s > 0, \\ -\int_0^{-s} r_{12}(\alpha)\,d\alpha, & s < 0, \end{cases}$$

$$C_{2,1}(t,s) = -\widetilde{r}_{21}(t-s) - \widetilde{r}_{21}(2T-t-s), \quad \widetilde{r}_{21}(s) = \begin{cases} r_{21}(s), & s > 0, \\ -r_{21}(-s), & s < 0, \end{cases}$$

$$C_{2,2}(t,s) = -r_{22}(|t-s|) - r_{22}(2T-t-s).$$

Proof. We take $F, G \in \mathcal{F}^T \cap C_0^\infty(0,T;\mathbb{R}^2)$ and introduce the Blagoveshchenskii function by setting

$$\Psi(t,s) = \left(u^F(\,\cdot\,,t), u^G(\,\cdot\,,s)\right)_{\mathcal{H}^T}, \qquad s,t > 0.$$

Our aim is to show that Ψ satisfies the wave equation. Indeed, using the fact that $u^F_{tt} = -H^* u^F$ and the Green identity, we can evaluate:

$$\begin{aligned}\Psi_{tt}(t,s) - \Psi_{ss}(t,s) &= \left(-H^* u^F(\,\cdot\,,t), u^G(\,\cdot\,,s)\right)_{\mathcal{H}^T} + \left(u^F(\,\cdot\,,t), H^* u^G(\,\cdot\,,s)\right)_{\mathcal{H}^T} \\ &= \left((\Gamma_0 u^F)(t), (\Gamma_1 u^G)(s)\right)_B - \left((\Gamma_1 u^F)(t), (\Gamma_0 u^G)(s)\right)_B \\ &=: P(t,s).\end{aligned}$$

Note that Ψ satisfies $\Psi(0,s) = \Psi_t(0,s) = 0$ and

$$\Psi(T,T) = \left(u^F(\,\cdot\,,T), u^G(\,\cdot\,,T)\right)_{\mathcal{H}^T} = \left(C^T F, G\right)_{\mathcal{F}^T}.$$

So, by the d'Alembert formula,

$$\left(C^T F, G\right)_{\mathcal{F}^T} = \int_0^T \int_\tau^{2T-\tau} P(\tau,\sigma)\,d\sigma\,d\tau. \tag{2.11}$$

We rewrite the right-hand side:

$$P(t,s) = \left(\begin{pmatrix} f_1(t) \\ f_2'(t) \end{pmatrix}, (RG)(s)\right)_B - \left((RF)(t), \begin{pmatrix} g_1(s) \\ g_2'(s) \end{pmatrix}\right)_B, \tag{2.12}$$

and continue the functions g_1, g_2 (we keep the same notation) from $(0,T)$ to the interval $(0,2T)$ by the rule

$$g_1(s) = \begin{cases} g_1(s), & 0 < s < T, \\ -g_1(2T-s), & T < s < 2T, \end{cases}$$
$$g_2(s) = \begin{cases} g_2(s), & 0 < s < T, \\ g_2(2T-s), & T < s < 2T. \end{cases} \tag{2.13}$$

After such a continuation the second term in (2.12) become odd in s with respect to $s = T$ and disappears after integration in (2.11), so we come to the following expression for the quadratic form:

$$\left(C^T F, G\right)_{\mathcal{F}^T} = \int_0^T \int_\tau^{2T-\tau} \left(\begin{pmatrix} f_1(\tau) \\ f_2'(\tau) \end{pmatrix}, (RG)(\sigma) \right)_B d\sigma\, d\tau. \tag{2.14}$$

Integrating by parts in (2.14) and using the fact that $C^T = \left(C^T\right)^*$ and the arbitrariness of F yield

$$\left(C^T G\right)(\tau) = \begin{pmatrix} \int_\tau^{2T-\tau} (R_1 G)(\sigma)\, d\sigma \\ (R_2 G)(\tau) + (R_2 G)(2T-\tau) \end{pmatrix}. \tag{2.15}$$

Evaluating (2.15), making use of (2.8) and the continuation of g_1, g_2 (2.13), we obtain

$$(C^T G)(\tau) = \frac{1}{2} \begin{pmatrix} g_1(\tau) \\ g_2(\tau) \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \int_\tau^{2T-\tau} \int_0^\sigma (r_{11}(s) g_1(\sigma - s) + r_{12}(s) g_2(\sigma - s))\, ds \\ -\int_0^\tau (r_{21}(s) g_1(\tau - s) + r_{22}(s) g_2(\tau - s))\, ds \end{pmatrix}$$
$$+ \begin{pmatrix} 0 \\ \int_0^{2T-\tau} (r_{21}(s) g_1(2T - \tau - s) + r_{22}(s) g_2(2T - \tau - s))\, ds \end{pmatrix}.$$

Consider the term

$$\int_\tau^{2T-\tau} \int_0^\sigma r_{11}(s) g_1(\sigma - s)\, ds\, d\sigma = I(2T-\tau) - I(\tau), \tag{2.16}$$

where

$$I(\tau) = \int_0^\tau \int_\alpha^\tau r_{11}(\sigma - \alpha) g_1(\alpha)\, d\sigma\, d\alpha.$$

We evaluate (2.16), using that g_1 is odd with respect to T:

$$I(\tau) = \int_0^\tau \int_0^{|\tau-\alpha|} r_{11}(\sigma)\, d\sigma g_1(\alpha)\, d\alpha = \int_0^\tau p_1(|\tau - \alpha|) g_1(\alpha)\, d\alpha, \tag{2.17}$$

where $p_1(s) = \int\limits_0^s r_{11}(\alpha)\,d\alpha$. We can rewrite the first term in (2.16) in the form

$$\begin{aligned} I(2T-\tau) &= \left(\int\limits_0^T + \int\limits_\tau^{2T-\tau}\right) \int\limits_0^{2T-\tau-\alpha} r_{11}(\sigma)\,d\sigma g_1(\alpha)\,d\alpha \\ &= \int\limits_0^T p_1(2T-\tau-\alpha)g_1(\alpha)\,d\alpha - \int\limits_\tau^T p_1(\alpha-\tau)g_1(\alpha)\,d\alpha. \end{aligned} \tag{2.18}$$

Then from (2.17) and (2.18) we get

$$\int\limits_\tau^{2T-\tau}\int\limits_0^\sigma r_{11}(s)g_1(\sigma-s)\,ds\,d\sigma = \int\limits_0^T \left(p_1(2T-\tau-\alpha) - p_1(|\alpha-\tau|)g_1(\alpha)\right)\,d\alpha,$$

which proves the formula for C_{11}. Now we consider the term

$$\int\limits_\tau^{2T-\tau}\int\limits_0^\sigma r_{12}(s)g_2(\sigma-s)\,ds\,d\sigma. \tag{2.19}$$

Note that it has the same structure as (2.16), but we should take into account that $g_2(s)$ is odd with respect to $s = T$. Counting this, we have

$$I(2T-\tau) = \int\limits_0^T p_2(2T-\tau-\alpha)g_2(\alpha)\,d\alpha + \int\limits_\tau^T p_2(\alpha-\tau)g_2(\alpha)\,d\alpha,$$

where $p_2(s) = \int\limits_0^s r_{12}(\alpha)\,d\alpha$. Then

$$\begin{aligned} I(2T-\tau) - I(\tau) &= \int\limits_0^T p_2(2T-\tau-\alpha)g_2(\alpha)\,d\alpha \\ &+ \int\limits_\tau^T p_2(\alpha-\tau)g_2(\alpha)\,d\alpha - \int\limits_0^T p_2(|\alpha-\tau|)g_2(\alpha)\,d\alpha. \end{aligned} \tag{2.20}$$

After we introduce the notation

$$\widetilde{p}_1(s) = \begin{cases} \int\limits_0^s r_{12}(\alpha)\,d\alpha, & s > 0, \\ -\int\limits_0^{-s} r_{12}(\alpha)\,d\alpha, & s < 0, \end{cases} = \begin{cases} p_2(s), & s > 0, \\ -p_2(-s), & s < 0, \end{cases}$$

we can rewrite (2.19), taking into account (2.20), as

$$\int\limits_\tau^{2T-\tau}\int\limits_0^\sigma r_{12}(s)g_2(\sigma-s)\,ds\,d\sigma = \int\limits_0^T \left(\widetilde{p}_1(2T-\tau-\alpha) - \widetilde{p}_1(\tau-\alpha)\right) g_2(\alpha)\,d\alpha,$$

which proves the formula for C_{12}. Similarly, one can prove formulas for C_{21} and C_{22}. □

We note that the symmetry of C^T implies the restriction on the entries; specifically, the following relation should hold:

$$C_{2,1}(t,s) = C_{1,2}(t,s).$$

This relation is equivalent to

$$-\widetilde{r}_{21}(t-s) - \widetilde{r}_{21}(2T-t-s) = \widetilde{p}_1(2T-t-s) - \widetilde{p}_1(s-t),$$

which yields

$$-\widetilde{r}_{21}(s) = \widetilde{p}_1(s).$$

Remark 2. The components of the response matrix must be connected by the relation

$$r_{21}'(s) = -r_{12}(s), \quad s > 0.$$

3. Dynamic inverse problem

In this section we derive equations of the inverse dynamic problem; using them, we answer the question on recovering a potential $q(x)$, $x \in (-T,T)$, from the response operator R^{2T}.

3.1. Krein equations. Let $y(x)$ be a solution to the following Cauchy problem:

$$\begin{cases} -y'' + qy = 0, & x \in (-T,T), \\ y(0) = 0, & y'(0) = 1. \end{cases} \tag{3.1}$$

We set up the *special control problem*: to find $F \in \mathcal{F}^T$ such that $W^T F = y$ in $\mathcal{H}^T$. By Theorem 2, such a control F exists, but we can say even more.

Theorem 4. *The solution to a special control problem is a unique solution to the following equation:*

$$\left(C^T F\right)(t) = (T-t)\begin{pmatrix}1\\0\end{pmatrix}, \quad t \in (0,T). \tag{3.2}$$

Proof. We observe that if $G \in \mathcal{F}^T \cap C_0^\infty(0,T;\mathbb{R}^2)$, then integration by parts shows that

$$u^G(x,T) = \int_0^T (T-t) u_{tt}^G(x,t)\,dt.$$

Using this observation, we can evaluate the quadratic form

$$\begin{aligned}
(C^T F, G)_{\mathcal{F}^T} &= (W^T F, W^T G)_{\mathcal{H}^T} = \left(y(\,\cdot\,), u^G(\,\cdot\,,T)\right)_{\mathcal{H}^T} \\
&= \int_{-T}^{T} y(x) \int_0^T (T-t) u_{tt}^G(x,t)\,dt\,dx \\
&= \int_0^T (T-t)\left(y(\,\cdot\,), -H^* u^G(\,\cdot\,,t)\right)_{\mathcal{H}^T}\,dx\,dt \\
&= \int_0^T (t-T)\Big[\left((\Gamma_0 y(\,\cdot\,))(t), (\Gamma_1 u^G)(t)\right)_B - \left((\Gamma_1 y(\,\cdot\,))(t), \left(\Gamma_0 u^G\right)(t)\right)_B\Big]\,dt \\
&= \int_0^T (T-t)\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}g_1(t)\\g_2'(t)\end{pmatrix}\right)dt,
\end{aligned}$$

from where (3.2) follows because of the arbitrariness of G. $\square$

Representation formulas (2.1) imply that that the solution F to a special control problem satisfies the relations

$$y(T) = u^F(T,T) = \frac{1}{2}f_1(0) - \frac{1}{2}f_2(0),$$
$$y(-T) = u^F(-T,T) = -\frac{1}{2}f_1(0) - \frac{1}{2}f_2(0).$$

Thus solving (3.2) for all $T \in (0,T)$, we recover the solution $y(x)$ to (3.1) on the interval $(-T,T)$. Then the potential $q(x)$, $x \in (-T,T)$, can be recovered as $q(x) = \frac{y''(x)}{y(x)}$, $x \in (-T,T)$.

3.2. Gelfand–Levitan equations. We introduce the notation

$$C^T = \frac{1}{2}(I+C), \quad (CF)(t) = 2\int_0^T C(t,s)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} ds. \tag{3.3}$$

For $F, G \in \mathcal{F}^T$ we set $W^T F = a$, $W^T G = b$, where $a, b \in \mathcal{H}^T$; on using the controllability (Theorem 2), we have (see (2.9))

$$F = J^T(I+K)^{-1}S^{-1}a = 2J^T(I+K)^{-1}Sa,$$
$$G = J^T(I+K)^{-1}S^{-1}b = 2J^T(I+K)^{-1}Sb.$$

Using above representations we can rewrite the quadratic form as:

$$\begin{aligned}(C^T F, G)_{\mathcal{H}^T} &= \Big(\frac{1}{2}(I+C)2J^T(I+K)^{-1}Sa, 2J^T(I+K)^{-1}Sb\Big)_{\mathcal{H}^T} \\ &= \Big(2\big((I+K)^{-1}\big)^* J^T(I+C)J^T(I+K)^{-1}Sa, Sb\Big)_{\mathcal{H}^T}.\end{aligned} \tag{3.4}$$

On the other hand,

$$\big(C^T F, G\big)_{\mathcal{H}^T} = \big(W^T F, W^T G\big)_{\mathcal{H}^T} = (a,b)_{\mathcal{H}^T} = (2Sa, Sb)_{\mathcal{H}^T}. \tag{3.5}$$

On comparing (3.4) and (3.5), we obtain the following operator identity:

$$\big((I+K)^{-1}\big)^* J^T(I+C)J^T(I+K)^{-1} = I. \tag{3.6}$$

We introduce the following notation

$$I + M = (I+K)^{-1}, \tag{3.7}$$

$$(MF)(x) = \begin{pmatrix} \int_x^T m_{11}(x,s)f_1(s) + m_{12}(x,s)f_2(s)\,ds \\ \int_x^T m_{21}(x,s)f_1(s) + m_{22}(x,s)f_2(s)\,ds \end{pmatrix}$$

$$(M^*a)(t) = \begin{pmatrix} \int_0^t m_{11}(x,t)a_1(x) + m_{21}(x,t)a_2(s)\,dx \\ \int_0^t m_{12}(x,t)a_1(s) + m_{22}(x,t)a_2(x)\,dx \end{pmatrix}.$$

It is easy to check that on a diagonal the kernels of the operators K and M satisfy the relation

$$m_{ij}(x,x) = -k_{ij}(x,x), \quad i,j = \{1,2\}, \quad x \in (0,T). \tag{3.8}$$

Rewritten in the new notation, the operator relation (3.6) has the form

$$(I+M)^*(I+\widetilde{C})(I+M) = I, \tag{3.9}$$

where

$$\widetilde{C} = J^T C J^T, \quad \left(\widetilde{C}F\right)(t) = \int_0^T \widetilde{C}(t,s)F(s)\,ds. \tag{3.10}$$

The relation (3.9) is equivalent to the equality

$$M^* + (I+M)^*\left(M + \widetilde{C} + \widetilde{C}M\right) = 0. \tag{3.11}$$

On introducing a function

$$\Phi(x,s) = m(x,s) + \widetilde{C}(x,s) + \int_0^T \widetilde{C}(x,\alpha)m(\alpha,s)\,d\alpha, \quad x,s \in (0,T),$$

we can write down a relation for the kernel of the operator on the left-hand side in (3.11) $M^* + \Phi + M^*\Phi = 0$:

$$m(s,x) + \Phi(x,s) + \int_0^t m(\alpha,x)\Phi(\alpha,s)\,d\alpha = 0, \quad x,s \in (0,T).$$

Since $m(s,x) = 0$ when $x < s$, we derive that Φ satisfies the relation

$$\Phi(x,s) + \int_0^t m(\alpha,x)\Phi(\alpha,s)\,d\alpha = 0, \quad x < s.$$

Thus the function Φ satisfies a Volterra equation of the second kind, and thus we obtain $\Phi(x,s) = 0$ for $x < s$, which immediately yields the following equation on the matrix function m:

$$m(x,s) + \widetilde{C}(x,s) + \int_0^T \widetilde{C}(x,\alpha)m(\alpha,s)\,d\alpha = 0, \quad 0 < x < s < T. \tag{3.12}$$

As a result, we can state the following theorem.

Theorem 5. *The matrix kernel of the operator* M (3.7) *satisfy the Gelfand–Levitan equation* (3.12), *where the kernel* $\widetilde{C}$ *is defined by* (3.3), (3.10). *Solving this equation, one can recover the potential using relations between kernels* (2.10), (3.8) *and relations on diagonals* $\{x = t\}$, $\{-x = t\}$ *in* (2.2), (2.3):

$$q(x) = 2\frac{d}{dx}\left(m_{11}(x,x) - m_{12}(x,x)\right), \quad x \in (0,T),$$
$$q(-x) = -2\frac{d}{dx}\left(m_{11}(x,x) + m_{12}(x,x)\right), \quad x \in (0,T).$$

3.3. Relationship between dynamic and spectral inverse data. The problem of finding relationships between different types of inverse data is very important in inverse problems theory. We can mention [2, 4, 5, 14, 15] some recent results in this direction. Below we show a relationship between the dynamic response function and matrix spectral measure.

Consider two solution to the equation

$$-\phi'' + q(x)\phi = \lambda\phi, \quad -\infty < x < \infty, \tag{3.13}$$

satisfying the Cauchy data:

$$\varphi(0,\lambda) = 0, \quad \varphi'(0,\lambda) = 1, \quad \theta(0,\lambda) = -1, \quad \theta'(0,\lambda) = 0.$$

Note that

$$\Gamma_0\varphi = 0, \quad \Gamma_0\theta = 0, \quad \Gamma_1\varphi = \begin{pmatrix}1\\0\end{pmatrix}, \quad \Gamma_1\theta = \begin{pmatrix}0\\1\end{pmatrix}.$$

We fix some $N > 0$ and prescribe self-adjoint boundary conditions at $x = \pm N$:

$$a_1\phi(-N,\lambda) + b_1\phi'(-N,\lambda) = 0, \quad a_1^2 + b_1^2 \neq 0, \tag{3.14}$$

$$a_2\phi(N,\lambda) + b_2\phi'(N,\lambda) = 0, \quad a_2^2 + b_2^2 \neq 0. \tag{3.15}$$

Eigenvalues and normalized eigenfunctions of (3.13), (3.14), (3.15) are denoted by $\{\lambda_n, y_n\}_{n=1}^{\infty}$. Let $\beta_n, \gamma_n \in \mathbb{R}$ be such that

$$y_n(x) = \beta_n\varphi(x,\lambda_n) + \gamma_n\theta(x,\lambda_n), \quad \text{then} \quad \Gamma_1 y_n = \begin{pmatrix}\beta_n\\ \gamma_n\end{pmatrix}.$$

Let $F \in \mathcal{F}^T \cap C_0^\infty(0,T;\mathbb{R}^2)$, and v^F be a solution to (1.1)–(1.3), (3.14), (3.15), i.e., a solution to the initial boundary value problem for a wave equation on the interval $(-N,N)$. Multiplying the wave equation for v^F by y_n and integrating by parts, we get the following relation:

$$\begin{aligned}
0 &= \int_{-T}^{T} v_{tt}^F y_n\,dx - \int_{-N}^{N} v_{xx}^F y_n\,dx + \int_{-N}^{N} q(x)v^F y_n\,dx \\
&= \int_{-N}^{N} v_{tt}^F y_n\,dx + (v^F, Hy_n) + (\Gamma_1 v^F, \Gamma_0 y_n)_B - (\Gamma_0 v^F, \Gamma_1 y_n)_B \\
&= \int_{-T}^{T} v_{tt}^F y_n\,dx + \lambda_n(v^F, y_n) - \left(\begin{pmatrix} f_1(t)\\ f_2'(t)\end{pmatrix}, \begin{pmatrix}\beta_n\\ \gamma_n\end{pmatrix}\right)_B.
\end{aligned}$$

Looking for the solution to (1.1)–(1.3) in the form

$$v^F = \sum_{k=1}^{\infty} c_k(t) y_k(x), \tag{3.16}$$

we plug (3.16) into (1.1) and multiply by y_n to get:

$$\int_{-N}^{N} \sum_{k=1}^{\infty} c_k''(t) y_k(x) y_n(x)\,dx + \int_{-N}^{N} \sum_{k=1}^{\infty} c_k(t) y_k(x) \lambda_n y_n(x)\,dx = \left(\begin{pmatrix} f_1(t)\\ f_2'(t)\end{pmatrix}, \begin{pmatrix}\beta_n\\ \gamma_n\end{pmatrix}\right)_B.$$

Thus we deduce that $c_n(t)$, $n \geqslant 1$, satisfies the following Cauchy problem:

$$\begin{cases} c_n''(t) + \lambda_n c_n(t) &= \left(\begin{pmatrix} f_1(t)\\ f_2'(t)\end{pmatrix}, \begin{pmatrix}\beta_n\\ \gamma_n\end{pmatrix}\right)_B, \\ c_n(0) = 0, & c_n'(0) = 0. \end{cases}$$

the solution of which is given by the formula

$$c_n(t) = \int_0^t \frac{\sin\sqrt{\lambda_n}(t-s)}{\sqrt{\lambda_n}} \left(f_1(s)\beta_n + f_2'(s)\gamma_n\right)\,ds.$$

Then for v^F (3.16) we have the expansion

$$v^F(x,t) = \sum_{k=1}^{\infty} \int_0^t \frac{\sin\sqrt{\lambda_n}(t-s)}{\sqrt{\lambda_n}} \left(f_1(s)\beta_n + f_2'\gamma_n\right) ds \left(\beta_n\varphi(x,\lambda_n) + \gamma_n\theta(x,\lambda_n)\right)$$

$$= \sum_{k=1}^{\infty} \int_0^t \frac{\sin\sqrt{\lambda_n}(t-s)}{\sqrt{\lambda_n}} \left(\begin{pmatrix}\beta_n\\ \gamma_n\end{pmatrix} \otimes \begin{pmatrix}\beta_n\\ \gamma_n\end{pmatrix} \begin{pmatrix}f_1(s)\\ f_2'(s)\end{pmatrix}, \begin{pmatrix}\varphi(x,\lambda_n)\\ \theta(x,\lambda_n)\end{pmatrix}\right)$$

$$= \int_{-\infty}^{\infty} \int_0^t \frac{\sin\sqrt{\lambda}(t-s)}{\sqrt{\lambda}} \left(d\Sigma_N(\lambda) \begin{pmatrix}f_1(s)\\ f_2'(s)\end{pmatrix}, \begin{pmatrix}\varphi(x,\lambda)\\ \theta(x,\lambda)\end{pmatrix}\right), \tag{3.17}$$

where $d\Sigma_N(\lambda)$ is a matrix measure (see [13]). Owing to the finite speed of wave propagation in system (1.1)–(1.3) (equal to one), we have the relation

$$v^F(\,\cdot\,,t) = u^F(\,\cdot\,,t), \quad \text{for } t < N, \tag{3.18}$$

and for $T < N$, $R^{2T}F = \Gamma_1 v^F$ holds. Thus the response operator R^T for $T < 2N$ is given by

$$(RF)(t) = \Gamma_1 v^F = \sum_{k=1}^{\infty} c_k(t)\Gamma_1 y_k = \sum c_k(t) \begin{pmatrix}\beta_k\\ \gamma_k\end{pmatrix}$$

$$= \sum_{k=1}^{\infty} \int_0^t \frac{\sin\sqrt{\lambda_k}(t-s)}{\sqrt{\lambda_k}} \left(f_1(s)\beta_k + f_2'\gamma_k\right) ds \begin{pmatrix}\beta_k\\ \gamma_k\end{pmatrix} \tag{3.19}$$

$$= \int_{-\infty}^{\infty} \int_0^t \frac{\sin\sqrt{\lambda}(t-s)}{\sqrt{\lambda}} d\Sigma_N(\lambda) \begin{pmatrix}f_1(s)\\ f_2'(s)\end{pmatrix} ds, \quad 0 < t < 2N.$$

Taking $F, G \in \mathcal{F}^T \cap C_0^\infty(0,T;\mathbb{R}^2)$, for $T < N$ we evaluate the quadratic form, using (3.17) and (3.18):

$$(C^T F, G)_{\mathcal{F}^T} = (u^F, u^G)_{\mathcal{H}^T} = (v^F, v^G)_{\mathcal{H}^T} \tag{3.20}$$

$$= \sum_{k=1}^{\infty} \int_0^T \int_0^T \frac{\sin\sqrt{\lambda_n}(t-s)}{\sqrt{\lambda_n}} (f_1\beta_n + f_2'\gamma_n)\, ds \frac{\sin\sqrt{\lambda_n}(t-\tau)}{\sqrt{\lambda_n}} (g_1\beta_n + g_2'\gamma_n)\, d\tau$$

$$= \int_0^T \int_0^T \int_{-\infty}^{\infty} \frac{\sin\sqrt{\lambda}(t-s)}{\sqrt{\lambda}} \frac{\sin\sqrt{\lambda}(t-\tau)}{\sqrt{\lambda}} \left(d\Sigma_N(\lambda) \begin{pmatrix}f_1(s)\\ f_2'(s)\end{pmatrix}, \begin{pmatrix}g_1(\tau)\\ g_2'(\tau)\end{pmatrix}\right) ds\, d\tau$$

We observe that in view of the unit speed of wave propagation in system (1.1)–(1.3), in representation formulas for response operator (3.19) and for connecting operator (3.20), we can replace $d\Sigma_N(\lambda)$ by any $d\Sigma_M(\lambda)$, $M > N$, where $d\Sigma_M(\lambda)$ corresponds to some self-adjoint boundary conditions at $\pm M$, or we can let N go to infinity, and replace $d\Sigma_N(\lambda)$ by the limit measure $d\Sigma(\lambda)$ (see [13]).

The inverse problem for a Schrödinger operator on a half-line from a spectral measure is solved in [11], in [13] the inverse spectral problem for a Schrödinger operator on a real line from a matrix measure is discussed, but some questions remain open. At the same time, in the case of a half-line in [1, 2, 14] the authors established the relationships between the dynamic and spectral inverse problems.

Remark 3. The control, response and connecting operators admit representations in terms of spectral inverse data (matrix measure $d\Sigma(\lambda)$), see (3.17), (3.19) and (3.20). This circumstance makes it possible to assume that the progress in studying the inverse spectral problem from a matrix measure will be greatly stimulated by the progress in studying the inverse dynamic problem in the spirit of [1, 2, 14].

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