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THE WAVE FIELD NEAR A NARROW CONVEX IMPEDANCE CONE COMPLETELY ILLUMINATED BY A PLANE INCIDENT WAVE

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An acoustic incident plane wave completely illuminates a narrow convex cone satisfying the impedance boundary condition on its surface. The wave field at far distances from the vertex of the cone and in some close neighborhood of the cone's surface is asymptotically computed. Bibliography: 9 titles.

1. Introduction

An incident plane wave propagates from infinity and completely illuminates the surface of a convex cone with the impedace boundary condition on its surface. If the cone is narrow and smooth (except for the vertex), the reflected rays of the far scattered field are of almost the same directions as those of the incident wave, Fig. 1. The spherical wave scattered by the vertex of the cone, considered in some close neighborhood of the cone (i.e., in the interior of the bigger dashed cone in Fig. 1) is also attributed to the the rays that have their directions close to those for the incident and reflected rays. The corresponding phase functions of these waves are close to each other, which means, from the physical point of view, that the waves interfere. As a result, the wave picture in some close neighborhood of the narrow cone cannot be simple. In comparison with known results [3, 4, Chap. 5], its study requires some addtional work in order to describe the far field asymptotics.

The asymptotic analysis of the Sommerfeld integral representation (see [4, Chap. 5]) for the wave field in this case implies the study of the situation where the singularities of the intergand are close to the saddle points. On the other hand, the description of the sigularities is based on the study of the Fourier transform of the so-called spectral function. The latter is determined by means of the asymptotic solution of a problem for the Laplace-Beltrami-Helmholtz equation on the unit sphere with a small hole cut out by the narrow cone having its vertex placed at the center of the sphere ([4, Chap. 5]). The size of the hole is of $O(\beta)$, where $\beta \ll 1$ is a small parameter of the problem describing the "narrowness" of the cone.

It is worth commenting on the assumptions that are implied when constructing the approximate solution of the problem at hand. The formal asymptotic solution of the problem is sought for $\beta \ll 1$, i.e., for a narrow impedance cone. However, we are interested in the description of the far field ($kr \gg 1$, kr is the wave distance from the vertex) in some close neighborhood (of $O(\beta^{1-\delta})$ for some $\delta > 0$) of the conical surface. In this work we, first, use the asymptotic solution of the diffraction problem for $\beta \ll 1$ then it is studied as $kr \gg 1$. The order of these two consequent "limits" is essential, and the small parameters β and $(kr)^{-1}$ are assumed to be independent. The situation where either these parameters are connected or the order of the asymptotic limits is different, is not discussed in this work. We note that the expressions for the scattered far field obtained in [4, Chap. 5] as $kr \to \infty$ and $\beta \sim O(1)$ become formally unapplicable as $\beta \to 0$.

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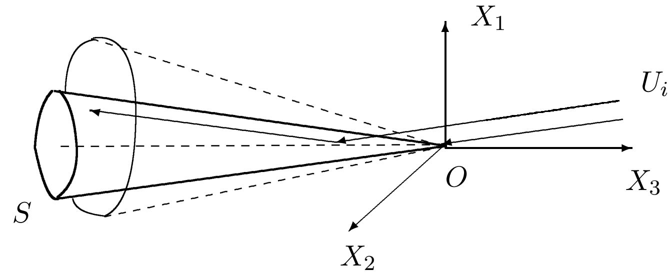

Fig. 1. Diffraction by a narrow completely illuminated cone S.

Let us use the spherical coordinates (r, θ, φ) connected with the Cartesian ones (Fig.1) by the relations

$$X_1 = r\cos\varphi\sin\theta,\ X_2 = r\sin\varphi\sin\theta,\ X_3 = r\cos\theta.$$

We consider a plane wave[1] that is incident from infinity along the direction specified by $\omega_0 = (\theta_0, \varphi_0)$ (Fig. 1),

$$U_i(r, \theta, \varphi) = \exp\{-\mathrm{i}kr\cos\theta_i(\omega, \omega_0)\}, \tag{1}$$

where $\omega = (\theta, \varphi)$ corresponds to the direction of observation and

$$\cos\theta_i(\omega, \omega_0) = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos[\varphi - \varphi_0], \quad \theta_i(\omega, \omega_0),$$

coincides with the geodesic distance between two points ω and ω_0 denoted also by $\theta(\omega, \omega_0) = \theta_i(\omega, \omega_0)$. For a right-circular cone, the equation of its surface S is given by $\theta = \theta_1,\ r \geq 0$, and for the narrow cone we have $\beta = 2(\pi - \theta_1) \ll 1$. Let σ be the curve of intersection of the conical surface S and the unit sphere S^2 centered at the vertex of the cone, $\sigma = S \cap S^2$. The "exterior" of σ is the domain Σ of directions of observation of the wave field. This domain is traditionally divided into an "oasis" Ω_0 and the rest $\Omega_1 = \Sigma \setminus \Omega_0$. The oasis corresponds to all directions of observation, where only the scattered spherical wave is observed in the scattered far field. A more detailed description of the oasis is given in [4, Chap. 5]. In our case of a thin (narrow) cone, the oasis occupies almost the whole Σ except for the asymptotically small part $\Omega_1 = \Sigma \setminus \Omega_0$ of S^2. Indeed, in the case of a circular cone and axially symmetric illumination, the oasis is the domain on Σ described by the inequality $\theta < \theta_1 - \beta/2$. In this case, Ω_1 is described by the inequality $\theta_1 \geq \theta \geq \theta_1 - \beta/2$. In this work, considering an arbitrary thin convex cone, we shall deal with a wider domain than Ω_1, denoting it by Ω_δ. This is an asymptotically thin domain.[2] It is also worth remarking that, provided a thin cone is completely illuminated by the incident plane wave, the point ω_0 belongs to the domain Σ_i ($\Sigma_i \in \Sigma$) being symmetric to the interior of $S^2 \setminus \Sigma$ with respect to the center of the unit sphere.

It is worth mentioning, however, that the far field asymptotics in the oasis for the complete illumination of a narrow convex impedance cone has been computed by Bernard and Lyalinov, [3]. In this case the far field is given by the expression

$$U(r, \theta, \varphi) = D(\omega, \omega_0)\,\frac{\exp\{\mathrm{i}kr\}}{-\mathrm{i}kr}\left(1 + O\left(\frac{1}{kr}\right)\right), \tag{2}$$

[1]The harmonic time-dependence $\mathrm{e}^{-\mathrm{i}\widehat{\omega}t}$ is assumed and omitted throughout the paper.

[2]It is obvious that Ω_δ is a thin ring on S^2 as $\beta \ll 1$ such that the geodesic distance from each point of Ω_δ to the "convex" curve σ, i.e., the boundary of a convex domain, is less than $C\beta^{1-\delta}$ for small $\delta > 0$ and some $C > 0$.

where the diffraction coefficient has been explicitly calculated (as $\beta \ll 1$)

$$D(\omega,\omega_0) \;=\; -\frac{l_\beta}{4\pi}\frac{\eta\,(1+O(\beta\log\beta))}{[\cos\theta(\omega,O)+\theta(O,\omega_0)]^2}\,,\quad \omega\in\Omega_0\,,\quad \theta(\omega,O)+\theta(O,\omega_0)>\pi\,,$$

where η is the surface impedance, l_β is the length of σ and O is a point in the exterior of Σ. The formula for $D(\omega,\omega_0)$ becomes nonapplicable as ω approaches the boundary of Ω_0, i.e. $\theta(\omega,O)+\theta(O,\omega_0)\to\pi$. The analogous results are also known for the perfect narrow cones [5], i.e., with the Dirichlet or Neumann boundary conditions.[3]

In this work we intend to give an expression for the far scattered field as $\omega\in\Omega_\delta$, i.e., in some close vicinity of the conical surface.

2. Statement and reduction to the integral equation for the spectral function

The total wave field $\widehat{U}(r,\theta,\varphi) = U_i(r,\theta,\varphi)+U(r,\theta,\varphi)$ is the sum of the incident wave $U_i(r,\theta,\varphi)$ (see (1) and of the scattered one $U(r,\theta,\varphi)$. The scattered field satisfies the Helmholtz equation

$$(\triangle+k^2)U(r,\theta,\varphi)=0,\quad \omega\in\Sigma,\quad r>0 \tag{3}$$

and the total field is subject to the impedance boundary condition

$$\left(\frac{1}{r}\frac{\partial}{\partial\mathcal{N}}-\mathrm{i}k\eta\right)\widehat{U}(r,\theta,\varphi)=0,\quad \omega\in\sigma,\quad r>0\,, \tag{4}$$

where $\mathcal{N}$ is the normal to σ in the tangent plane to S^2, Meixner's condition at the vertex is implied (see [1, 2, 4, Sec. 5.1]). The wave field at infinity satisfies a radiation condition; in particular, in the oasis (i.e., as $\omega\in\Omega_0$) it has the form (2). The form of the asymptotics in a close neighborhood of the conical surface S, i.e., as $\omega\in\Omega_\delta$, is the main goal of the study in this work.[4]

The details of the derivations in the rest of this section are traditional and can be found by parts in [4, 7] and [3].[5]

The incident plane wave admits the so-called Watson–Bessel integral representation

$$U_i(r,\theta,\varphi)=4\mathrm{i}\sqrt{\frac{\pi}{2}}\int_{C_0}\nu\,\mathrm{e}^{-\mathrm{i}\nu\pi/2}u^i_\nu(\omega,\omega_0)\frac{J_\nu(kr)}{\sqrt{-\mathrm{i}kr}}\,d\nu, \tag{5}$$

where $u^i_\nu(\omega,\omega_0)=-\frac{P_{\nu-1/2}(-\cos\theta_i(\omega,\omega_0))}{4\cos\pi\nu}$, $P_{\nu-1/2}(x)$ is the Legendre function. C_0 is the contour comprising the positive part of the real axis (see [4, 7]). The representation for the scattered wave takes a similar form

$$U(r,\theta,\varphi)=4\mathrm{i}\sqrt{\frac{\pi}{2}}\int_{C_0}\nu\,\mathrm{e}^{-\mathrm{i}\nu\pi/2}u_\nu(\omega,\omega_0)\frac{J_\nu(kr)}{\sqrt{-\mathrm{i}kr}}\,d\nu, \tag{6}$$

where $u_\nu(\omega,\omega_0)$ is an unknown (spectral) function.

Making use of the representations (6), (5) for the problem (3),(4) leads to the equation

$$(\triangle_\omega+(\nu^2-1/4))u_\nu(\omega,\omega_0)=0 \tag{7}$$

[3]It seems that work [6] needs some more detailed justification of the exploited approach, as well as the results.

[4]It is worth noting that we are looking for the asymptotics of the classical solution of the problem at hand.

[5]In this section we actually do not use the fact that the cone is narrow, and the results are known for an arbitrary smooth convex cone.

in Σ and the boundary condition on σ

$$\left.\frac{\partial \widehat{u}_\nu(\omega,\omega_0)}{\partial \mathcal{N}}\right|_\sigma = \frac{\eta}{2\mathrm{i}} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\tau \, \frac{\tau \, \sin \pi\tau \, \widehat{u}_\tau(\omega,\omega_0)|_\sigma}{\cos \pi\tau + \cos \pi\nu}, \quad \nu \in \Pi_\delta, \tag{8}$$

with $\widehat{u}_\nu(\omega,\omega_0) = u_\nu(\omega,\omega_0) + u^i_\nu(\omega,\omega_0)$, $\Pi_\delta = \{\nu \in C : \ |\mathrm{Im}(\nu)| < \delta_0\}$ for some small positive δ_0. The condition (8) is not local with respect to the (spectral) variable ν.

For further derivations it is useful to introduce a formal integral operator

$$\mathcal{A} v_\nu(\omega,\omega_0)|_\sigma = \frac{1}{2\mathrm{i}} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\tau \, \frac{\tau \, \sin \pi\tau \, v_\tau(\omega,\omega_0)|_\sigma}{\cos \pi\tau + \cos \pi\nu}, \tag{9}$$

where ν belongs to the imaginary axis; then from (8) one has

$$\left.\frac{\partial \widehat{u}_\nu(\omega,\omega_0)}{\partial \mathcal{N}}\right|_\sigma = \eta \ \mathcal{A}\widehat{u}_\nu(\omega,\omega_0)|_\sigma \, .$$

The class of functions in which the solution of the problem (7), (8) is looked, is carefully described in [4, Chap. 5] and in more detail in [7].

The problem (7), (8) is followed by the representation (see [3])

$$\alpha_\omega \, u_\nu(\omega,\omega_0) \; = \int\limits_\sigma \mathrm{d}l_s \left(\frac{\partial g_\nu(\omega,s)}{\partial \mathcal{N}} u_\nu(s,\omega_0) \; - \; \eta \, \mathcal{A} u_\nu(s,\omega_0) \, g_\nu(\omega,s) \right) \; + \; \Psi_\nu(\omega,\omega_0) \, , \tag{10}$$

where $g_\nu(s,\omega) = -\frac{P_{\nu-1/2}(-\cos\theta(s,\omega))}{4\cos\pi\nu}$, $\alpha_\omega \ = 1/2$ as $\omega \in \sigma$ and $\alpha_\omega \ = 1$ as $\omega \in \Sigma$,

$$\Psi_\nu(\omega,\omega_0) \; = \int\limits_\sigma \mathrm{d}l_s \left(\frac{\partial u^i_\nu(\omega_0,s)}{\partial \mathcal{N}} g_\nu(s,\omega) \; - \; \eta \, \mathcal{A} u^i_\nu(s,\omega_0) \, g_\nu(\omega,s) \right) \, .$$

The integration in formula (10) is performed along the closed curve σ on S^2.

If $\omega \in \sigma$, the representation (10) converts into an integral equation of the second kind for $u_\nu(\cdot,\omega_0)$. Its asymptotic solution, as $\beta \to 0$, is discussed in [3], and in the leading approximation the spectral function is given by

$$u_\nu(\omega,\omega_0) \; = \; \Psi_\nu(\omega,\omega_0)(1 + O(\beta \log \beta)) \, . \tag{11}$$

It is worth recalling that

$$\omega \in \Omega_\delta \, , \ \ \omega_0 \in \Sigma_0$$

in formula (11), so that $\theta(\omega,O) + \theta(O,\omega_0) = \pi + O(\beta^{1-\delta})$.

For further study of the far field asymptotics of the scattered field based on the approximate solution for the spectral function, we need to give some estimates for (11) as $\nu \to \mathrm{i}\infty$.

3. Some estimates for the approximate spectral function (11)

We write the expression (11) in the form

$$u_\nu(\omega,\omega_0) \; = \int\limits_\sigma \mathrm{d}l_s \, W_\nu(s,\omega,\omega_0)(1 + O(\beta \log \beta)) \tag{12}$$

with

$$W_\nu(s,\omega,\omega_0) \;=\; \frac{P_{\nu-1/2}(-\cos\theta(\omega,s))}{4\cos\pi\nu}\left\{\frac{\eta}{2\mathrm{i}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\tau\,\frac{\tau\,\tan\pi\tau\,P_{\nu-1/2}(-\cos\theta(\omega_0,s))}{\cos\pi\tau+\cos\pi\nu}\right.$$
$$\left. -\,\frac{\partial\theta(s,\omega_0)}{\partial\mathcal{N}}\frac{P^1_{\nu-1/2}(-\cos\theta(\omega_0,s))}{4\cos\pi\nu}\right\},$$

where we made use of

$$\frac{\mathrm{d}}{\mathrm{d}x}P_{\nu-1/2}(x) = -\frac{P^1_{\nu-1/2}(x)}{\sqrt{1-x^2}}\,.$$

We are interested in estimates of $W_\nu(s,\omega,\omega_0)$ that are uniform w.r.t. $s\in\sigma$ for $\omega\in\Omega_\delta$, $\omega_0\in\Sigma_0$ as $\nu\to\mathrm{i}\infty$.

If $\omega\in\Omega_\delta$, $\omega_0\in\Sigma_0$, we can assert that $\theta(\omega_0,s)<\pi-a_0\beta$ for some positive a_0 and $\theta(\omega,s)>A_0\beta$ for some positive A_0. As a result, we have

$$-\cos\theta(\omega_0,s)=\cos[\pi-\theta(\omega_0,s)]<1-a_1\beta^2$$

and

$$-\cos\theta(\omega,s)=\cos[\pi-\theta(\omega,s)]>-1+A_1\beta^2$$

for some positive a_1 and A_1. We assume that β is a small, however, fixed parameter.

We observe that the arguments in $P^1_{\nu-1/2}(-\cos\theta(\omega_0,s))$ and $P_{\nu-1/2}(-\cos\theta(\omega,s))$ are such that the asymptotics ([8], 8.721(3))

$$P^\mu_{\nu-1/2}(\cos\varphi) \;=\; \sqrt{\frac{2}{\pi\sin\varphi}}\,\frac{\Gamma(\nu+\mu+1/2)}{\Gamma(\nu+1)}\,\cos\left(\nu\varphi+\frac{\pi\mu}{2}-\frac{\pi}{4}\right)(1+O(1/\nu)) \tag{13}$$

as $\nu\to\mathrm{i}\infty$ and $0<\epsilon\le\varphi\le\pi-\epsilon$, $|\nu|\gg1/\epsilon$ can be applied. In the estimates that follow, we shall use the asymptotics (13).

Thus we have

$$W_\nu(s,\omega,\omega_0) \;=\; \frac{P_{\nu-1/2}(-\cos\theta(\omega,s))}{4\cos\pi\nu}\left\{I(\nu,s,\omega_0)\;-\;\frac{\partial u^i_\nu(s,\omega_0)}{\partial\mathcal{N}}\right\}$$

with

$$I(\nu,s,\omega_0)=\frac{\eta}{2\mathrm{i}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\tau\,\frac{\tau\,\tan\pi\tau\,P_{\tau-1/2}(-\cos\theta(\omega_0,s))}{\cos\pi\tau+\cos\pi\nu}.$$

First we consider an estimate of the integral $I(\nu,s,\omega_0)$, representing it as

$$I(\nu,s,\omega_0) \;=\; \frac{\eta}{\mathrm{i}}\int\limits_{0}^{\mathrm{i}A} d\tau\,\frac{\tau\,\tan\pi\tau\,P_{\tau-1/2}(-\cos\theta(\omega_0,s))}{\cos\pi\tau+\cos\pi\nu}$$
$$+\;\frac{\eta}{\mathrm{i}}\int\limits_{\mathrm{i}A}^{\mathrm{i}\infty} d\tau\,\frac{\tau\,\tan\pi\tau\,P_{\tau-1/2}(-\cos\theta(\omega_0,s))}{\cos\pi\tau+\cos\pi\nu},$$

where A is large enough. The first summand on the right-hand side of the latter relation is estimated by $O\left(\frac{1}{\cos \pi\nu}\right)$ as $\nu \to \mathrm{i}\infty$. For the second summand, one has

$$\begin{aligned}
&\frac{\eta}{4\mathrm{i}}\int\limits_{\mathrm{i}A}^{\mathrm{i}\infty} d\tau\, \frac{\tau\, \tan\pi\tau\, P_{\tau-1/2}(-\cos\theta(\omega_0,s))}{\cos\pi\tau+\cos\pi\nu}\\
&\quad = C\int\limits_{A}^{\infty} dt \frac{\sqrt{t}\, \tanh\pi t\, \cos(it[\pi-\theta(\omega_0,s)]-\pi/4)}{\cosh\pi t+\cos\pi\nu}(1+O_1(1/t))\\
&\quad = C_1\int\limits_{A}^{\infty} dt \frac{\sqrt{t}\, \tanh\pi t\, \exp[t(\pi-\theta)])}{\cosh\pi t+\cos\pi\nu}(1+O_1(1/t))\\
&\quad = C_2\int\limits_{0}^{\infty} dt \frac{\sqrt{t}\, \sinh\pi t\, [\cosh\pi t]^{-a}}{\cosh\pi t+\cos\pi\nu}(1+O_1(1/t))+O\left(\frac{1}{\cos\pi\nu}\right)\\
&\quad = C_4\int\limits_{1}^{\infty} dp\, \frac{(\log(p+\sqrt{p^2-1}))^{1/2}}{p^a(p+q)}(1+O_1(1/\log p))+O\left(\frac{1}{q}\right),
\end{aligned}$$

where $0 < a = \theta(\omega_0, s)/\pi < 1$ and $p = \cosh\pi t$, $q = \cos\nu$, and the notation $O_1(1/t)$ is used for a function that has the asymptotics C/t when t is large. The latter integral admits the estimate

$$\begin{aligned}
\int\limits_{1}^{\infty} dp\, \frac{(\log(p+\sqrt{p^2-1}))^{1/2}}{p^a(p+q)} &= C_5\int\limits_{1}^{\infty} dp\, \frac{(\log p)^{1/2}}{p^a(p+q)}(1+O_1(1/\log p))\\
&= C_6\frac{\sqrt{\log q}}{q^a}\left(1+O\left(\frac{1}{\log q}\right)\right)
\end{aligned}$$

as $q \to \infty$, $\nu \to \mathrm{i}\infty$. This estimate can be deduced, for instance, in the following way. Consider the identity (see also [2, Appendix])

$$\int\limits_{1}^{\infty} \frac{p^{-a}}{p+q}\, dp \;=\; q^{-a}\left(\frac{\pi}{\sin\pi a}-\frac{1}{1-a}\right)+q^{-a}\left(\frac{1-q^{a-1}}{1-a}\right)+q^{-a}\int\limits_{0}^{1/q}\frac{y^{1-a}}{1+y}\, dy$$

obtained from 3.222(2) in [8].[6] In this identity we compute the derivative $\frac{d^b}{da^b}$ of both sides for natural b, make use of the analytic continuation for real $b \in (0, 1]$ and take $b = 1/2$; therefore, we have

$$\int\limits_{1}^{\infty} dp\, \frac{(\log p)^{1/2}}{p^a(p+q)} \;=\; C_6\, \frac{\sqrt{\log q}}{q^a}\left(1+O\left(\frac{1}{\log q}\right)\right).$$

Make use of the asymptotics (13) and the estimate

$$I(\nu,s,\omega_0) = C_7\sqrt{\nu}\frac{w_I(s,\omega,\omega_0)}{\cos[\nu\theta(\omega_0,s)]}\left(1+O\left(\frac{1}{\nu}\right)\right),$$

[6] The case $a \to 1-0$ can be regarded as a limiting one.

then arrive at

$$
\begin{aligned}
W_\nu(s,\omega,\omega_0) &= \frac{w_0(s,\omega,\omega_0)}{\cos[\nu\theta(\omega_0,s)]\cos[\nu\theta(\omega,s)]}\left(1+O\left(\frac{1}{\nu}\right)\right)\\
&= \frac{w(s,\omega,\omega_0)}{\cos[\nu(\theta(\omega_0,s)+\theta(\omega,s))]}\left(1+O\left(\frac{1}{\nu}\right)\right)
\end{aligned}
\tag{14}
$$

uniformly w.r.t. $s\in\sigma$ and $\omega\in\Omega_\delta$, $\omega_0\in\Sigma_0$ as $\nu\to \mathrm{i}\infty$, w_0 and w, w_I are continuous functions of their arguments. To proceed further, it is convenient to introduce the notation $\theta_*(s,\omega,\omega_0)=\theta(\omega_0,s)+\theta(\omega,s)$.

4. Fourier transform of the spectral function and the Sommerfeld transformant

The Sommerfeld representation for the scattered field (see [4, 7, Chap. 5]) takes the form

$$
U(r,\theta,\varphi)=\frac{1}{2\pi\mathrm{i}}\int\limits_\gamma \frac{\mathrm{e}^{-\mathrm{i}kr\cos\alpha}}{\sqrt{-\mathrm{i}kr}}\,\Phi(\alpha,\omega,\omega_0)\,d\alpha\,,
\tag{15}
$$

where

$$
\Phi(\alpha,\omega,\omega_0)\;=\;\sqrt{2\pi}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\nu\,\mathrm{e}^{\mathrm{i}\nu\alpha}u_\nu(\omega,\omega_0)\,d\nu,
$$

$|\operatorname{Re}(\alpha)|<\theta_*(\omega,\omega_0)$ and $\theta_*(\omega,\omega_0)=\min_{s\in\sigma}(\theta(\omega_0,s)+\theta(\omega,s))$, γ is the double-loop Sommerfeld contour (see [4]). Analogously, integrating by parts, from (15) we have

$$
U(r,\theta,\varphi)=\frac{\sqrt{-\mathrm{i}kr}}{2\pi\mathrm{i}}\int\limits_\gamma \mathrm{e}^{-\mathrm{i}kr\cos\alpha}\,\sin\alpha\,\widetilde{\Phi}(\alpha,\omega,\omega_0)\,d\alpha\,,
$$

where

$$
\widetilde{\Phi}(\alpha,\omega,\omega_0)\;=\;\frac{\sqrt{2\pi}}{\mathrm{i}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{e}^{\mathrm{i}\nu\alpha}u_\nu(\omega,\omega_0)\,d\nu\,,
$$

$\Phi(\alpha,\omega,\omega_0)=\frac{\partial}{\partial\alpha}\widetilde{\Phi}(\alpha,\omega,\omega_0)$.

The Sommerfeld transformants $\Phi(\alpha,\omega,\omega_0)$ and $\widetilde{\Phi}(\alpha,\omega,\omega_0)$ in (15) should be analytically continued from the strip $|\operatorname{Re}(\alpha)|<\theta_*(\omega,\omega_0)$, where they are holomorphic functions, onto a wider strip if it is necessary. In our case, the spectral function in the Fourier integral for the Sommerfeld transformants is given by an explicit expression (12).

We have

$$
\begin{aligned}
\Phi(\alpha,\omega,\omega_0) &= \frac{\sqrt{2\pi}}{\mathrm{i}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\left(\frac{d}{d\alpha}\mathrm{e}^{\mathrm{i}\nu\alpha}\int\limits_\sigma \mathrm{d}l_s\,W_\nu(s,\omega,\omega_0)\right)d\nu\,(1+O(\beta\log\beta))\\
&= \frac{\sqrt{2\pi}}{\mathrm{i}}\int\limits_\sigma \mathrm{d}l_s\,\frac{d}{d\alpha}\left(\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{e}^{\mathrm{i}\nu\alpha}\,W_\nu(s,\omega,\omega_0)\,d\nu\right)(1+O(\beta\log\beta))\,,
\end{aligned}
$$

where the change of the orders of integration and differentiation is justified. Introduce the notation

$$
\widetilde{\phi}(s,\alpha,\omega,\omega_0)=\frac{1}{\mathrm{i}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{e}^{\mathrm{i}\nu\alpha}\,W_\nu(s,\omega,\omega_0)\,d\nu\,.
$$

Remark 1. In explicit and formally equivalent form, the approximation (11) is written as

$$u_\nu(\omega,\omega_0) \;=\; -l_\beta\; g_\nu(\omega,O)\left(\eta\,\mathcal{A}u^i_\nu(O,\omega_0) \;-\; \left.\frac{\partial u^i_\nu(\omega_0,s)}{\partial\mathcal{N}}\right|_{s\to O}\right)(1+O(\beta\log\beta)),$$

where l_β is the length of σ and O is a point in the interior of $S^2\setminus\Sigma$. So that one has (the change $s\to O$, $l_\beta=\int\limits_\sigma dl_s$)

$$\Phi(\alpha,\omega,\omega_0) \;=\; l_\beta\frac{\sqrt{2\pi}}{\mathrm{i}}\,\frac{d}{d\alpha}\left(\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{e}^{\mathrm{i}\nu\alpha}\,W_\nu(O,\omega,\omega_0)\,d\nu\right)(1+O(\beta\log\beta))\,.$$

Transform the integral $\widetilde{\phi}(s,\alpha,\omega,\omega_0)$, which is the Fourier transform of $W_\nu(s,\omega,\omega_0)$, and make use of the asymptotics (14)

$$\widetilde{\phi}(s,\alpha,\omega,\omega_0) \;=\; \frac{1}{\mathrm{i}}\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\nu\,\cos(\nu\alpha)\,\frac{w(s,\omega,\omega_0)}{\cos[\nu\theta_*(s,\omega,\omega_0)]}\left(1+O\left(\frac{1}{\nu}\right)\right).$$

In view of the estimates of the preceeding section, the latter integral absolutely converges and is a holomorphic function as $|\operatorname{Re}(\alpha)|<\theta_*$. The strip $|\operatorname{Re}(\alpha)|<\theta_*$, where $\Phi(\alpha,\omega,\omega_0)$ is also defined and regular, can be extended onto $|\operatorname{Re}(\alpha)|<3\pi/2$ with the aid of the analytic continuation. However, we should specify the type and position of singularities of $\Phi(\alpha,\omega,\omega_0)$ on the boundary of the holomorphicity strip. To this end, we use the formula 3.981(3) from [8]

$$\int\limits_0^\infty\frac{\cos(ax)}{\cosh(bx)}\,dx \;=\; \frac{\pi}{2b}\operatorname{sech}\left(\frac{\pi a}{2b}\right)$$

as $\operatorname{Re}b>a>0$ and its analytic continuation w.r.t. $\alpha=-\mathrm{i}a$ onto the strip $|\operatorname{Re}\alpha|<b$.

Then we have that $\alpha=\theta_*$ is a simple pole of

$$\widetilde{\phi}(s,\alpha,\omega,\omega_0),\quad \widetilde{\phi}(s,\alpha,\omega,\omega_0)=O(1/(\alpha-\theta_*))$$

or

$$\widetilde{\phi}(s,\alpha,\omega,\omega_0) = \;O\left([\cos(\alpha/2)-\cos(\theta_*/2)]^{-1}\right)$$

if α is located in the vicinity of θ_*. From the latter estimate one easily derive that $\widetilde{\phi}(\alpha,s,\omega,\omega_0)$ $[\cos(\alpha/2)-\cos(\theta_*(s,\omega,\omega_0)/2)]$ is holomorphic in some neighborhood of $\alpha=\theta_*$. Since

$$\theta_*(s,\omega,\omega_0)=\theta(\omega,s)+\theta(s,\omega_0)=\pi+O(\beta^{1-\delta}),$$

we have that

$$\widetilde{\phi}(\alpha,s,\omega,\omega_0)[\cos(\alpha/2)-\cos(\theta_*(s,\omega,\omega_0)/2)]$$

is also holomorphic in some small vicinity of the point $\alpha=\pi$.

As a result, from

$$\Phi(\alpha,\omega,\omega_0)=\frac{\partial}{\partial\alpha}\widetilde{\Phi}(\alpha,\omega,\omega_0) \;=\; \frac{\sqrt{2\pi}}{\mathrm{i}}\,\frac{\partial}{\partial\alpha}\int\limits_\sigma dl_s\;\widetilde{\phi}(\alpha,\omega,\omega_0)$$

we obtain

$$\begin{aligned}\Phi(\alpha,\omega,\omega_0)=\frac{\sqrt{2\pi}}{\mathrm{i}}\int\limits_\sigma dl_s\Bigg(&\frac{\Psi_1(\alpha,s,\omega,\omega_0)}{[\cos(\alpha/2)-\cos(\theta_*(s,\omega,\omega_0)/2)]}\\ &+\Psi_2(\alpha,s,\omega,\omega_0)\frac{\frac{1}{2}\sin(\alpha/2)}{[\cos(\alpha/2)-\cos(\theta_*(s,\omega,\omega_0)/2)]^2}\Bigg),\end{aligned}\tag{16}$$

where

$$\Psi_2(\alpha,s,\omega,\omega_0)=\widetilde{\phi}(\alpha,s,\omega,\omega_0)[\cos(\alpha/2)-\cos(\theta_*(s,\omega,\omega_0)/2)]$$

and

$$\Psi_1(\alpha,s,\omega,\omega_0)=\frac{\partial}{\partial\alpha}\Psi_2(\alpha,s,\omega,\omega_0)$$

are holomorphic in the vicinity of $\alpha=\pi$ and $\theta_*\sim\pi$.

If $\theta_*>\pi+\delta_1$, $\delta_1>0$, then $\Phi(\alpha,\omega,\omega_0)$ is holomorphic in the vicinity of $\alpha=\pi$ and

$$\Phi(\pi,\omega,\omega_0)=\frac{\sqrt{2\pi}}{\mathrm{i}}\int\limits_{\sigma}dl_s\left(\frac{\Psi_1(\pi,s,\omega,\omega_0)}{(-\cos(\theta_*(s,\omega,\omega_0)/2))}+\frac{\frac{1}{2}\Psi_2(\pi,s,\omega,\omega_0)}{\cos^2(\theta_*(s,\omega,\omega_0)/2)}\right). \tag{17}$$

5. The far field in the close vicinity of a narrow impedance cone

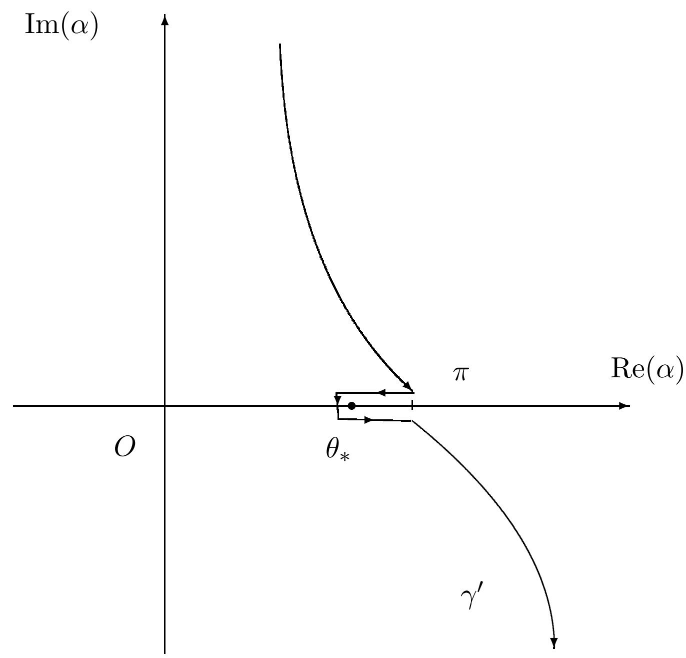

Fig. 2. The contour of integration γ'.

We make use of the integral representation (15) and deform the Sommerfeld contour into the contours of the steepest descent; then[7]

$$\begin{aligned}U(r,\theta,\varphi)=&\frac{\sqrt{2\pi}}{\mathrm{i}}\int\limits_{\sigma}dl_s\frac{1}{\pi\mathrm{i}}\\&\times\int\limits_{\gamma'}\frac{\mathrm{e}^{-\mathrm{i}kr(2\cos^2\alpha/2-1)}}{\sqrt{-\mathrm{i}kr}}\left(\frac{\Psi_1(\alpha,s,\omega,\omega_0)\frac{1}{2}\sin(\alpha/2)}{\frac{1}{2}\sin(\alpha/2)[\cos(\alpha/2)-\cos(\theta_*(s,\omega,\omega_0)/2)]}\right.\\&\left.+\Psi_2(\alpha,s,\omega,\omega_0)\frac{\frac{1}{2}\sin(\alpha/2)}{[\cos(\alpha/2)-\cos(\theta_*(s,\omega,\omega_0)/2)]^2}\right)d\alpha(1+O(\beta\log\beta)),\end{aligned}\tag{18}$$

[7]The function $\Phi(\alpha,\omega,\omega_0)$ is odd w.r.t. α, so that it is sufficient to consider the contribution of only one stationary point $+\pi$ of the Sommerfeld integral; further the result is to be multiplied by two.

where γ' is the contour of the steepest descent comprising the pole of $\Phi(\alpha, \omega, \omega_0)$ in (16) (see Fig. 2). Outside the real axis, the contour γ' coincides with the curve $\operatorname{Re}\alpha = \pi - \operatorname{gd}(\operatorname{Im}\alpha)$ of the steepest descent. We made use of the identity $kr\cos\alpha = -kr + 2kr\cos^2(\alpha/2)$ in the exponent and then change the orders of integration.

It is sufficient to perform the integration w.r.t. α over a neighborhood of the point π of the size $O([kr]^{-1/2+\delta})$ as $kr \to \infty$ in order to compute the leading term of the Sommerfeld integral. Let $B_\rho(\pi)$ be a circle with center at $\alpha = \pi$ and radius $\rho = O([kr]^{-1/2+\delta})$. Replace $2\Psi_1(\alpha, s, \omega, \omega_0)/\sin(\alpha/2)$, $\Psi_2(\alpha, s, \omega, \omega_0)$ by their values at the point $\alpha = \pi$, which contributes the error $O([kr]^{-1/2+\delta})$. Assume that $0 < \beta < \mathrm{Const}[kr]^{-1/2+\delta}$, so that $\theta_* \in B_\rho(\pi)$. Notice that the infinite segments of the integration in the exterior of the circle $B_\rho(\pi)$ give an exponentially small contribution to the leading term of the asymptotics.

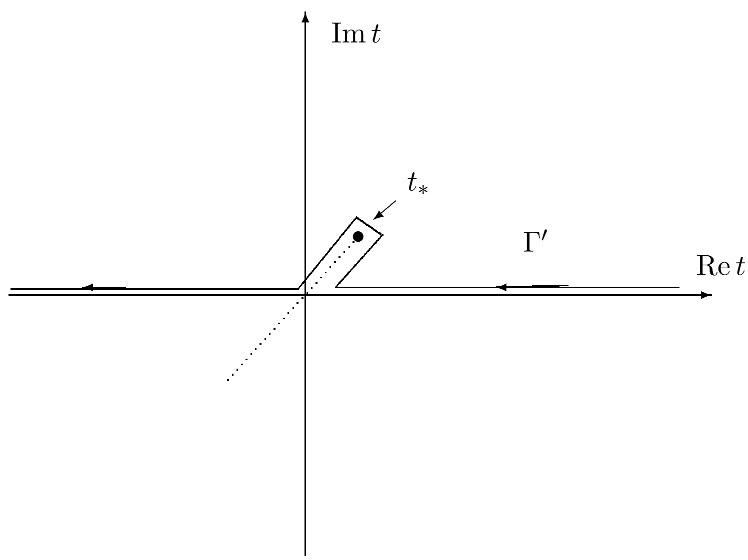

Fig. 3. The contour of integration Γ' and the position of the pole t_*.

Thus we obtain

$$
\begin{aligned}
U(r,\theta,\varphi) \sim\ & -\sqrt{\frac{2}{\pi}}\, \mathrm{e}^{\mathrm{i}kr} \int_\sigma dl_s \int_{\gamma' \cap B_\rho(\pi)} \frac{\mathrm{e}^{-2\mathrm{i}kr\cos^2\alpha/2}}{\sqrt{-\mathrm{i}kr}} \Bigg(\frac{2\Psi_1(\pi, s, \omega, \omega_0)\ \frac{1}{2}\sin(\alpha/2)}{[\cos(\alpha/2) - \cos(\theta_*(s,\omega,\omega_0)/2)]} \\
& + \Psi_2(\pi, s, \omega, \omega_0) \frac{\frac{1}{2}\sin(\alpha/2)}{[\cos(\alpha/2) - \cos(\theta_*(s,\omega,\omega_0)/2)]^2} \Bigg) d\alpha\, (1 + O(\beta\log\beta)),
\end{aligned} \tag{19}
$$

$\theta_* = \pi + O(\beta^{1-\delta}) > 0$ as $\omega \in \Omega_\delta\,,\ \ \omega_0 \in \Sigma_0\,.$

Upon the change of the integration variable α, $t = \sqrt{2\mathrm{i}kr}\cos\alpha/2$, we have

$$
\begin{aligned}
U(r,\theta,\varphi) \sim -\sqrt{\frac{2}{\pi}}\, \mathrm{e}^{\mathrm{i}kr} \int_\sigma dl_s \Bigg(& \frac{(-2)\Psi_1(\pi, s, \omega, \omega_0)}{\sqrt{-\mathrm{i}kr}}\ F_1(t_*(s)) \\
& + \sqrt{2}\mathrm{i}\Psi_2(\pi, s, \omega, \omega_0)\, F_2(t_*(s)) \Bigg) (1 + O(\beta\log\beta)),
\end{aligned} \tag{20}
$$

$t_*(s) = \sqrt{2\mathrm{i}kr}\cos[\theta_*(s,\omega,\omega_0)/2]$, $\omega = (\theta,\varphi)$, the contour Γ' is shown in Fig. 3 (see also [9, Sec. 7],

$$F_1(t_*) = \int\limits_{\Gamma'} \frac{\mathrm{e}^{-t^2}}{(t-t_*)}\,dt, \quad F_2(t_*) = \int\limits_{\Gamma'} \frac{\mathrm{e}^{-t^2}}{[t-t_*]^2}\,dt.$$

If $\theta_* > \pi + \delta_1$, then $t_* \to -\mathrm{e}^{\mathrm{i}\pi/4}\infty$ (Fig. 3); we make use of the asymptotics

$$F_1(t_*) \;=\; \frac{\sqrt{\pi}}{t_*}\left(1 + O\left(\frac{1}{t_*^2}\right)\right),$$
$$F_2(t_*) \;=\; -\frac{\sqrt{\pi}}{t_*^2}\left(1 + O\left(\frac{1}{t_*^2}\right)\right)$$

and formula (17) and arrive at an expression for the far field in the oasis (see (2)):

$$U(r,\omega,\omega_0) = -\sqrt{\frac{2}{\pi}}\,\Phi(\pi,\omega,\omega_0)\,\frac{\exp\{\mathrm{i}kr\}}{-\mathrm{i}kr}\left(1 + O\left(\frac{1}{kr}\right)\right),$$

$D(\omega,\omega_0) = -\sqrt{\frac{2}{\pi}}\,\Phi(\pi,\omega,\omega_0)$ is the diffraction coefficient of the spherical wave from the vertex of the cone.

Remark 2. It seems that a formal simplification of formula (20) is possible by the substitution of the argument of the function $s \to O$, then changing the integral by $l_\beta = \int\limits_\sigma dl_s$ as $\beta \ll 1$.

Asymptotic formula (20) represents the main result of the work. It describes the wave behavior of the far field scattered in some small vicinity of a narrow cone with impedance boundary condition. The applicability domain of the asymptotics was discussed above. In particular, for any $kr \gg 1$ there exists a small β from the interval $0 < \beta < \mathrm{C}[kr]^{-1/2+\delta}$ such that the asymptotic formula (20) in the leading approximation correctly describes the wave field for $\omega \in \Omega_\delta$, $\omega_0 \in \Sigma_0$. Note that as $\theta_* \sim \pi + O([kr]^{-1/2})$, the scattered field $U(r,\theta,\varphi)$ in (20) is formally not vanishing as $kr \to \infty$. However, $U(r,\theta,\varphi) = O(\beta)$ as $\beta \ll 1$; moreover, as has been already remarked, $0 < \beta < \mathrm{C}[kr]^{-1/2+\delta}$. We shall consider computational aspects connected with the application of formula (20) for the numerical calculation of the wave field in the continuation of this study.

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REFERENCES

1. V. M. Babich, M. A. Lyalinov, and V. E. Grikurov, *Diffraction Theory. The Sommerfeld-Malyuzhinets Technique*, Alpha Science Ser. Wave Phenom., Alpha Science, Oxford, United Kingdom (2008).
2. J-M. L. Bernard, "Méthode analytique et transformées fonctionnelles pour la diffraction d'ondes par une singularité conique: équation intégrale de noyau non oscillant pour le cas d'impédance constante," rapport CEA-R-5764, Editions Dist-Saclay (1997), (erratum in J Phys A, 32 L45); an extended version in *Advanced Theory of Diffraction by a Semi-Infinite Impedance Cone*, Alpha Science, Ser. Wave Phenom, Alpha Science, Oxford, United Kingdom (2014).
3. J-M. L. Bernard and M. A. Lyalinov, "Diffraction of acoustic waves by an impedance cone of an arbitrary cross-section," *Wave Motion*, **33**, 155–181 (erratum : p.177 replace $O(1/\cos(\pi(\nu-b)))$ by $O(\nu^d\sin(\pi\nu)/\cos(\pi(\nu-b)))$) (2001).

4. M. A. Lyalinov and N. Y. Zhu, "Scattering of Waves by Wedges and Cones with Impedance Boundary Conditions," *Mario Boella Series on Electromagnetism in Information & Communication* Edison, NJ SciTech-IET, USA (2013).
5. V. M. Babich, "Diffraction of a plane wave by a narrow angle cone in the case of the Dirichlet boundary conditions," *J. Math. Sci.*, **86(3)**, 2657–2663 (1997).
6. I. V. Andronov, "Diffraction by a narrow cone at skew incidence," *Zap. Nauchn. Semin. POMI*, **451**, 5–13 (2017).
7. M. A. Lyalinov, "Acoustic scattering by a semi-infinite angular sector with impedance boundary conditions," submitted to IMA Journ. (2017).
8. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 4th ed., Academic Press, Orlando (1980).
9. M. A. Lyalinov, "Scattering of acoustic waves by a sector," *Wave Motion*, **50**, 739–762 (2013).