

Journal of Algebra and Applied Mathematics

Vol. 17 (2018), No.1, pp.145-156

ISSN: 2319-7234

©SAS International Publications

URL : www.sasip.net

Optimality and Existence of the Solution of Stochastic Multi–Objective Optimization Problem

V.V. Kolbin and D.S. Perestoronin

Abstract: This article focuses on the issue of existence of the solution and its optimality for stochastic multi–objective optimization problem. The presented results make possible to assert the existence and optimality of solutions of stochastic multi–objective optimization problem with expected value from convolution function.

AMS Mathematics Subject Classification (2010): 90C15, 90C29.

Keywords: stochastic multi–objective optimization, existence of the solution of stochastic multi–objective optimization, optimality of the solution of stochastic multi–objective optimization.

1 Introduction

The problem of optimality and existence of the solution of stochastic multi–objective optimization problem is quite important and has been studied by many authors. The first articles on this issue were published by Hanson [11] and Stoddart [20]. Further, their work was continued by the authors of the works [14, 23]. The authors of the articles [4, 18, 19, 16] studied particular cases.

However, the problem of the existence of solutions and their optimality of the stochastic multi-objective optimization problem (SMOOP) [10] has not yet been fully studied, while the use of SMOOP at application fields such as finance [1, 17], manufacturing, production planning [13, 21], supply chain management [2, 3], energy [7], transportation logistics [12], facility location [8], humanitarian logistics [22], telecommunication and health care management [5, 6] or project management [9] is widespread.

At the present time, there is no single approach to the problem of studying the conditions of existence and optimality of SMOOP solutions. Thus, based on the results given in the works [11], that study the problem of scalar stochastic optimization, the authors generalize the results to the multi-objective case, using the approach presented in [15].

2 Stochastic Multi-Objective Optimization Problem

A stochastic multi-objective optimization problem (SMOOP) is a combination of a stochastic optimization problem (SOP) and a multi-objective optimization problem (MOOP). Consider each of this problem separately.

In the simple case, the stochastic optimization problems (SOP) contain the random parameter in the objective function. The math-

emathical formalization of this problem can be represented as follows:

$$\min_x f(x, \omega) \quad (1)$$

subject to.

$$x \in X, \omega \in \Omega \quad (2)$$

where $f(x, \omega)$ is a objective function, X is a set of feasible solutions, and $\omega \in \Omega$ – random parameter in the probability space (Ω, A, Ψ) . Note that in stochastic optimization, the probability model given by (Ω, A, Ψ) is always assumed as known, therefore, to simplify further, we will simply indicate that $\omega \in \Omega$.

To solve this problem, it is necessary to determine how to overcome uncertainty. Most often, to overcome an undefined service, the expectation of a random variable is used:

$$\min_x \mathbb{E}[f(x, \omega)] \quad (3)$$

subject to.

$$x \in X, \omega \in \Omega \quad (4)$$

where $f(x, \omega)$ is a objective function, $\mathbb{E}(\cdot)$ is a expected value of a random variable, X is the set of feasible solutions. Note that there are more complex cases, including uncertainty also in the solution

space:

$$\min_x \mathbb{E}[f(x, \omega)] \quad (5)$$

subject to.

$$x \in X(\omega), \omega \in \Omega \quad (6)$$

where $f(x, \omega)$ is a objective function, $X(\omega)$ is a set of feasible solutions, and $\omega \in \Omega$ - random parameter.

The general form of a multi-objective optimization problem (MOOP) is usually represented as:

$$\min_x (f(x)) \quad (7)$$

subject to.

$$x \in X \quad (8)$$

where X is again a solution space, $f(x) = (f_1(x), \dots, f_m(x))$ is vector functions, $f_j : X \rightarrow R(j = 1, \dots, m)$ are objective functions ($m \geq 2$). From 7 – 8 it is not yet clear what is meant by minimization vector functions $f = (f_1, \dots, f_m)$. There are several approaches to determine what exactly is meant by the minimum vector of a vector function, we list some of them. These are scalarizing methods, lexicographical order, Pareto methods, etc. Scalarization is the most common approach. This approach uses the scalarization or convolution function, which compares the values of multi-objective indicators to the scalar one, for example, by a weighted average or weighted Chebyshev distance. We will write such functions in the

form $F : R^m \rightarrow R$. After scalarization of multi-objective optimization problem is reduced to a scalar optimization problem.

We now turn to the stochastic multi-objective optimization problem (SMOOP), that is, to optimization problems, i.e. an optimization problem that is both a SOP and a MOOP. In particular, consider the tasks of the form:

$$\min_x f(x, \omega), \quad (9)$$

subject to

$$x \in X(\omega), \forall \omega \in \Omega. \quad (10)$$

where $X(\omega)$ is a set of feasible solutions, $f(x, \omega) = (f_1(x, \omega), \dots, f_m(x, \omega))$ is vector function of objective functions, $\omega \in \Omega$ is random parameter.

It is clear that the problem 9–10 is not completely defined. It is necessary to indicate in what sense the vector minimization should be interpreted, and what is being done to eliminate the ambiguity of target values caused by the accidental impact of ω .

One of the methods that answers these two questions at once is the method called in the literature stochastic method. The stochastic method starts with the reduction of SMOOP to SOP, to which the methods of stochastic optimization are applied. Thus, applying first the convolution of the criteria, and then minimizing the mathematical expectation of it, we can rewrite the problem in the

following form:

$$\min_x \mathbb{E}[F(x, \omega)], \quad (11)$$

subject to

$$x \in X(\omega), \forall \omega \in \Omega. \quad (12)$$

Note, since $F(f(x, \omega))$ is completely determined by the x and ω , we simply wrote $F(x, \omega)$ instead of $F(f(x, \omega))$. In the next section, we present a theorem that determines the existence and optimality of the solution for this type of SMOOP methods.

3 Optimality and Existence of the Solution of Stochastic Multi-Objective Optimization

In this section, we consider the SMOO problem as follows:

$$\min_x \mathbb{E}[F(x, \omega)], \quad (13)$$

subject to

$$\mathbb{E}[g(x, \omega)] \leq 0, \quad (14)$$

$$h(x, \omega) \leq 0, \forall \omega \in \Omega. \quad (15)$$

Where $F(x, \omega)$ — convolution of objective functions stochastic multi-objective optimization problem, which is a scalar function of $x \in E^n$, $g(x, \omega)$ and $h(x, \omega)$ — function vector, $F(x, \omega)$, $g(x, \omega)$, $h(x, \omega)$ — are twice continuously differentiable on $S = \{x\} \times \{\omega\} \subset E^n \times \Omega$ for all variables, and distribution density $\Psi(\omega)$ is twice con-

tinuously differentiable on $\omega \in \Omega$.

For this problem the following theorem is correct:

Theorem .1. *x^* is a minimizing vector for problem 13 if the following conditions are satisfied:*

1. $F(x, \omega)$ — is convex in x
2. $g(x, \omega)$ and $h(x, \omega)$ — concave on x
3. there exists $\lambda \leq 0$
4. there exists $\mu(\omega) \leq 0$
5. $F_x(x, \omega)\Psi(\omega) + \lambda g_x(x, \omega)\Psi(\omega) + \mu(\omega)h_x(x, \omega) = 0$
6. $\lambda g(x, \omega) = 0$
7. $\mu(\omega)h(x, \omega) = 0$

where $F_x = (\frac{\delta F(x_1)}{\delta x_1}, \dots, \frac{\delta F(x_n)}{\delta x_n})$, g_x , h_x — matrices of corresponding dimensions from partial derivatives, λ , $\mu(\omega)$ — r dimensional function, respectively.

Proof. Write the problem 13 in an equivalent feed:

$$\min_x \int_{\Omega} F(x, \omega)\Psi(\omega)d\omega, \quad (16)$$

subject to

$$\int_{\Omega} g(x, \omega)\Psi(\omega)d\omega \leq 0, \quad (17)$$

$$h(x, \omega) \leq 0, \forall \omega \in \Omega. \quad (18)$$

Let x^* x — some feasible solutions. As $F(x, \omega)$ is convex in x is continuous and differentiable, from the conditions of the theorem

we have:

$$\begin{aligned} \int_{\Omega} F(x^*, \omega) \Psi(\omega) d\omega &\leq \int_{\Omega} \left((F(x^*, \omega) + (x^* - x) F_x(x, \omega)) \right) \Psi(\omega) d\omega = \\ &= \int_{\Omega} \left(F(x, \omega) \Psi(\omega) - (x^* - x) [\lambda g_x(x, \omega) \Psi(\omega) + \mu(\omega) h_x(x, \omega)] \right) \leq \end{aligned}$$

that, by virtue of the convexity λ_g and μ_h

$$\begin{aligned} &\leq \int_{\Omega} F(x, \omega) \Psi(\omega) d\omega - \int_{\Omega} \lambda (g(x^*, \omega) - g(x, \omega)) \Psi(\omega) d\omega - \\ &\quad - \int_{\Omega} \mu(\omega) (h(x^*, \omega) - h(x, \omega)) d\omega = \int_{\Omega} F(x, \omega) \Psi(\omega) d\omega - \\ &\quad - \int_{\Omega} (x^* - x) \Psi(\omega) d\omega - \int_{\Omega} \mu(\omega) (h(x^*, \omega) - h(x, \omega)) d\omega \leq \\ &\quad \leq \int_{\Omega} (F(x, \omega) \Psi(\omega) d\omega \end{aligned}$$

The latter means that x^* is the optimal solution. \square

Thus, the proved theorem gives the conditions that allow to state the existence and optimality of the solution for the problem 13–15.

4 Conclusion

In the present article we consider problems of optimality and existence of the solution of stochastic multi-optimization with normalizations introduced for them and convolutions.

The main results of this article are presented in Section 3, that demonstrates the theorem of the existence and optimality of solutions of stochastic multi-objective optimization problem with expected value from convolution function.

This theorem allows us to state optimality and existence of the solution of SMOOP depending on the fulfillment of the conditions imposed on the kind of convolution and the set of feasible solutions.

For subsequent studies, it is of interest to study them with no convolution, in the case when the optimality principle is introduced on the set of the objective function.

Acknowledgements. The authors thank the managing editor and anonymous referee for valuable comments which helped us significantly to consolidate the paper.

References

- [1] Abdelaziz, F., Aouni, B., and Fayedh, R. E. (2007). Multiobjective stochastic programming for portfolio selection. *European Journal of Operations Research*, 177:1811–1823.
- [2] Ahmed, S., Schutz, P., and Tomasgard, A. (2009). Supply chain design under uncertainty using sample average approximation and dual decomposition. *European Journal of Operational Research*, 199:409–419.
- [3] Amodeo, L., Prins, C., and Sanchez., D. (2009). Comparison of metaheuristic approaches for multi-objective simulation-based optimization in supply chain inventory management. in applications of evolutionary computing. *Lecture Notes in Computer Science.*, 5484:798–807.
- [4] Andrieu, L., Cohen, G., and Vázquez-Abad, F. J. (2007).

- Stochastic programming with probability constraints. pages 1–27.
- [5] Armbruster, B. and Luedtke., J. (2011). Models and formulations for multivariate dominance constrained stochastic programs. *Technical report.*, 47(1):1–14.
- [6] Baesler, F. and Sepulveda, J. (2001). Multi-objective simulation optimization for a cancer treatment center. *In Proc. Winter Simulation Conference 2001*, 2:1405–1411.
- [7] Bath, S., Dhillon, J., and Kothari, D. (2004). Stochastic multiobjective generation dispatch. *Electric Power Components and Systems*, 32:1083–1103.
- [8] Fonseca, M., Gracia-Sanchez, A., Ortega-Mier, M., and da Gama, F. S. (2010). A stochastic bi-objective location model for strategic reverse logistics. *TOP*, 18:18–158.
- [9] Gutjahr, W. and Reiter., P. (2010). Bi-objective project portfolio selection and staff assignment under uncertainty. *Optimization Methods and Software*, 59:417–445.
- [10] Gutjahr, W. J. and Pichler, A. (2016). Stochastic multi-objective optimization: a survey on non-scalarizing methods. *Annals of Operations Research*, 236:475–499.
- [11] Hanson, M. A. (1964). Stochastic nonlinear programming. *J. Austral. Math. Sec.*, 4:253–263.
- [12] Hassan-Pour, H., Mosadegh-Khah, M., and Tavakkoli-Moghaddam, R. (2009). Solving a multi-objective multi-depot

- stochastic location-routing problem by a hybrid simulated annealing algorithm. in proceedings of the institution of mechanical engineers. *Journal of Engineering Manufacture*, pages 1045–1054.
- [13] Hnaien, F., Delorme, X., and Dolgui., A. (2010). Multi-objective optimization for inventory control in two-level assembly systems under uncertainty of lead times. *Computers and Operations Research*, 20:1835–1843.
- [14] Kolbin, V. (1971). *Stochastic Programming*. Progress in Mathematics.
- [15] Kolbin, V. and Perestoronin, D. (2018). Some approaches to studying the stability of solutions of stochastic multi-objective optimization. *Journal of Algebra and Applied Mathematics*, 16:131–144.
- [16] Penannen, T., Perkkiö, A.-P., and Rásonyi, M. (2015). Non-convex dynamic programming and optimal investment. *Mathematics and Financial Economics*, pages 1–15.
- [17] Rath, S., Gendreau, M., and Gutjahr, W. (2016). Bi-objective stochastic programming models for determining depot locations in disaster relief operations planning. *Technical report*, 23:997–1023.
- [18] Shapiro, A. (2000). *Statistical Inference of Stochastic Optimization Problems*.
- [19] Shapiro, A. and Xu, H. (2008). Stochastic mathematical programs with equilibrium constraints, modelling and sample average approximation. *Optimization*, 57:395–418.

- [20] Stoddart, A. (1968). An existence theorem for optimal stochastic programming. *J. Austral. Math. Soc.*, 8:114–118.
- [21] Syberfeldt, A., A. Ng, R. J., and Moore, P. (2009). Multi-objective evolutionary simulation-optimisation of a real-world manufacturing problem. *Robotics and Computer-Integrated Manufacturing*, 25:926–931.
- [22] Tricoire, F., Graf, A., and Gutjahr., W. (2012). The bi-objective stochastic covering tour problem. *Computers an Operations Research.*, 39:1582–1592.
- [23] Vogel, S. (1985). *Necessary optimality conditions for two-stage stochastic programming problems*. Progress in Mathematics.

Faculty of Applied Mathematics and Control Processes
Saint-Petersburg State University
13B Universitetskaya Emb.
199034, St. Petersburg
Russia
E-mail: vivakolbin@gmail.com
v.kolbin@spbu.ru

Faculty of Applied Mathematics and Control Processes
Saint-Petersburg State University
13B Universitetskaya Emb.
199034 St. Petersburg
Russia
E-mail: perestoronin.daniil@gmail.com

(Received: September, 2018; Revised: December, 2018)