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Surface waves in a polygonal domain with Robin boundary conditions

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A surface wave, propagating from infinity along a semi-infinite part, interacts with the impedance boundary of a 2D polygonal domain and gives rise to the reflected surface wave on this side and to the transmitted surface wave outgoing to infinity along the second semi-infinite side of the domain. The circular outgoing wave also propagates at infinity. It is shown that the classical solution of the problem is unique. By use of some known extension of the Sommerfeld–Malyuzhinets technique the problem at hand is reduced to functional Malyuzhinets equations and then to a system of integral equations of the second kind with the integral operator depending on a characteristic parameter. The integral equations are carefully studied. From the Sommerfeld integral representation of the solution the far field asymptotics is developed.

1 Introduction

The surface wave U^i is incident along l^1 (Fig. 1) from infinity then the scattered wave field consists of the reflected surface wave propagating along l^1 and that transmitted, outgoing to infinity along l^4. At the same time, the circular (or cylindrical in 3D) wave goes to infinity. The excitation coefficients of the surface waves as well as the diffraction coefficient of the circular wave are the most important characteristics of the wave scattering. Their study is one of the main goals in this work. These coefficients cannot be found from any local calculations, for instance, like in the case of calculation of the reflection coefficients for a plane impedance boundary,[1] and require complete solution of the scattering problem at hand. We solve the problem, compute the far field asymptotics and give representations for these characteristics of scattering in terms of the solution of an integral equation. We study Fredholm property, uniqueness and solvability of the latter. We also aim at specifying the range of applicability of the approach [1] considering the problem of scattering of a surface wave in its rigorous formulation. In particular, this requires a new form of the radiation condition enabling possibility to extract the outgoing surface waves. On the other hand, we expect to find some limitations on the geometry of the polygonal domain and on the surface impedances.

We formulate the problem, postulating appropriate radiation and Meixner's conditions, and study uniqueness of the classical solution. From the physical point of view the uniqueness is based on the assumption that one of the finite segments of the boundary absorbs energy. The latter circumstance requires positiveness of the real part of the surface impedance on this segment, see Sect. 2.5 in [2]. The other segments of the boundary are reactive (neither active nor absorbing), i.e. the real parts of their impedances are zero. Note that the impedances are independent of the wave number. In the third section we derive the problem for the functional (Malyuzhinets) equations in terms of the Sommerfeld transformants (i.e. the functions transformed by the Sommerfeld integral) in the framework of the mentioned extension of the Sommerfeld–Malyuzhinets (SM) technique. We reduce the problem for the functional equations on the complex plane to a matrix integral equation of the second kind and give integral representations of the meromorphic transformants in terms of solution of the integral equation. Then we study Fredholm property of the integral equation at hand making use of the so called

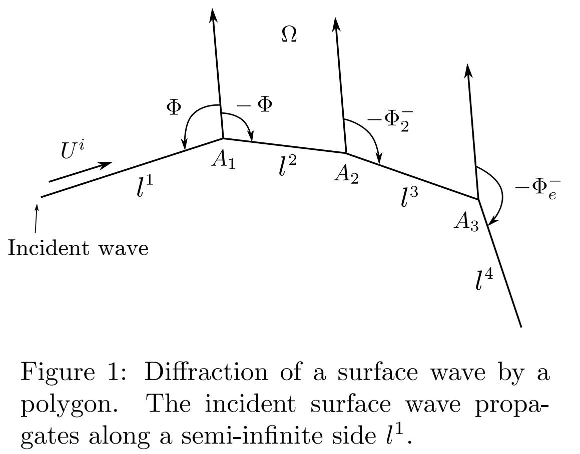

Figure 1: Diffraction of a surface wave by a polygon. The incident surface wave propagates along a semi-infinite side l^1.

[1] In wave physics the boundary, on which the Robin condition is postulated, is also called impedance boundary.

analytic Fredholm alternative. Finally, appropriate singularities of the Sommerfeld transformant are found and the far field asymptotics is developed. The expressions for the excitation or diffraction coefficients are obtained.

1.1 Formulation

In the domain Ω the total wave field

$$u = U^i + u^{sc}$$

is the sum of the incident surface wave U^i propagating along l^1 and the unknown scattered $u^{sc}(r,\varphi)$ waves written in the polar coordinates (r,φ) attributed to the vertex A_1 of the boundary.

The total field satisfies the Helmholtz equation in Ω

$$(\triangle + k^2)u = 0$$

and the impedance boundary conditions on the segments of the boundary

$$\left(\frac{\partial}{\partial n} - \mathrm{i}k\eta_j\right)u\bigg|_{l^j} = 0, \quad j = 1,2,3,4,$$

where n is the external to Ω normal on the boundary $\partial\Omega$, η_j, $j = 1,2,3,4$, are the surface impedances. The finite segment l^2 has the ends A_1, A_2 and the length d_1, the length of the segment l^3 with the end-points A_2, A_3 is d_2. The semi-infinite side l^4 of the polygon with the end-point A_3 supports the transmitted surface wave.

It is convenient to introduce the parameters ϑ_+, ϑ_-, ϑ_1, and ϑ_2 as follows:

$$\eta_1 = \sin\vartheta_+, \;\; \eta_4 = \sin\vartheta_-, \;\; \eta_2 = \sin\vartheta_1, \;\; \eta_3 = \sin\vartheta_2,$$

which are independent of the wave number k. The following limitations for the impedances are assumed

$$\pi/2 \geq \mathrm{Re}(\vartheta_1) > 0, \quad \mathrm{Re}(\vartheta_\pm) = 0, \quad \mathrm{Im}(\vartheta_\pm) < 0.$$

For some technical reason we shall additionally assume that $\mathrm{Re}(\vartheta_2) > 0$, although this assumption is not necessary for uniqueness. The restriction for the real part of ϑ_1 means absorption of the wave field energy on the segment l^2 of the boundary.

It is also implied that in Fig. 1

$$\Phi > \pi/2, \quad \Phi_2^- > \Phi, \quad \Phi_e^- > \Phi_2^-. \qquad (1)$$

These angles and d_1, d_2 completely define the polygonal boundary. Note that the coordinate axes attributed to A_1, A_2, A_3 are assumed to be parallel. Such a choice of the coordinate systems is motivated by the exploited technique (see also [1]). The limitations on the geometry of the polygonal domain Ω in (1) are of the technical nature. They actually mean that the polygonal scatterer $\mathbb{R}^2 \setminus \Omega$ is convex.

The Meixner's conditions at the angular points are satisfied

$$\mathrm{Im}\left(\int_{S_\epsilon^j} \frac{\partial u}{\partial n}\,\overline{u}\,ds\right) \to 0, \quad j = 1,2,3,$$

as $\epsilon \to 0$, where S_ϵ^j is a part of a circumference in Ω centered at j-th vertex, the energy flux through S_ϵ^j vanishes in the limit. Another equivalent form of the Meixner's conditions (see discussion in [2], ch. 1) is also of usage herein:

$$u = C^j + O\big(\rho_j^{\delta_j}\big), \quad \text{as} \quad \rho_j \to 0,$$

where $\delta_j > 0$, $j = 1,2,3$, depend on the openings of the angles with the vertices A_j correspondingly, ρ_j is the distance from A_j in Ω.

Finally, the radiation condition at infinity reads

$$\int_{S_R^i} \left|\frac{\partial u^{sc}}{\partial\rho} - \mathrm{i}ku^{sc}\right|^2 ds + \sum_{\pm}\int_{S_{R,b}^\pm}\left|\frac{\partial u^{sc}}{\partial\rho} - \mathrm{i}k\cos\vartheta_\pm u^{sc}\right|^2 ds \to 0, \qquad (2)$$

as $R \to \infty$ and $S_R^i = S_R \setminus (S_{R,b}^+ \cup S_{R,b}^-)$, where S_R is the part of the circumference in Ω having the radius $\rho = R$ and centered at the origin O,

$$S_{R,b}^\pm = \big\{(\rho,\psi) : \; \rho = R,\; 0 < \Psi \mp \psi < R^{-1+\varkappa}\big\},$$

$\varkappa > 0$ is small. In the definition of the arcs $S_{R,b}^\pm$ we make use of the polar coordinates (ρ,ψ) in the angle contained in Ω with the sides $\psi = \pm\Psi$ and with the vertex O at the point of intersection of the continuations of l^1 $(\psi = \Psi)$ and l^4 $(\psi = -\Psi)$. These arcs correspond to some small angular vicinities of the semi-infinite segments l^1 and l^4 as $R \to \infty$.

The radiation condition in the integral form (2) is a generalization of the standard Sommerfeld radiation condition taking into account the outgoing surface wave propagation at infinity along l^1 and l^4. It implies that for S_R^i, i.e outside close vicinities of l^1 and l^4, in the leading approximation the scattered field has the following far field asymptotic behaviour

$$u^{sc} \sim \frac{e^{\mathrm{i}k\rho + \mathrm{i}\pi/4}}{\sqrt{2\pi k\rho}} D_s(\psi)\left(1 + O\left(\frac{1}{k\rho}\right)\right),$$

$D_s(\cdot)$ is the unknown diffraction coefficient of the circular (cylindrical in 3D) wave, $k\rho = kR \to \infty$, whereas the reflected u_s^+ and transmitted u_s^- surface waves are exponentially small for these directions corresponding to S_R^i. On the other hand, for the directions corresponding to $S_{R,b}^{\pm}$, i.e. in close vicinities of l^1 and l^4, one has

$$u^{sc} \sim u_s^{\pm}(\rho,\psi) + \frac{e^{\mathrm{i}k\rho+\mathrm{i}\pi/4}}{\sqrt{2\pi k\rho}} D_s(\psi)\left(1 + O\left(\frac{1}{k\rho}\right)\right) \qquad (3)$$

with "+" for l^1 and "−" for l^4 in the right-hand side of (3), where

$$u_s^{\pm}(\rho,\psi) = c_s^{\pm} \exp\bigl\{\mathrm{i}k\rho\cos\bigl[\psi \mp \psi_s^{\pm}\bigr]\bigr\}$$

are the outgoing surface waves propagating along l^1 and l^4 correspondingly with yet unknown excitation coefficients $c_s^{\pm}$, $\psi_s^{\pm} = \Psi + \vartheta^{\pm}$.

2 An extension of the Sommerfeld–Malyuzhinets technique and functional equations

We make use of the known extension of the SM technique [1] and apply it in order to get Sommerfeld integral representations of the wave field in Ω and formulate a problem for the corresponding system of Malyuzhinets functional equations.

We introduce the polar axis at A_1 as shown in Fig. 1 and polar coordinates (r,φ), and assume that the polygonal boundary is outside the angle $r > 0$, $|\varphi| < \Phi$. The other two auxiliary polar coordinate systems (ρ_2,φ_2) and (ρ_e,φ_e) are attributed to A_2 and A_3 correspondingly. In the coordinates (r,φ) the wave field is represented by the Sommerfeld integral

$$u(r,\varphi) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \mathrm{d}\alpha\ \mathrm{e}^{-\mathrm{i}kr\cos\alpha} f(\alpha+\varphi), \qquad (4)$$

where γ is the known Sommerfeld double-loop contour [2], $f(\cdot)$ is the so called Sommerfeld transformant which is a meromorphic function depending on the wave number k. The representation (4) is definitely valid as $|\varphi| \le \Phi$ and also satisfies the Helmholtz equation in the whole exterior Ω of the polygon. In the same manner we introduce the representation of the solution in the polar coordinates attributed to A_3

$$u(\rho_e,\varphi_e) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \mathrm{d}\alpha\ \mathrm{e}^{-\mathrm{i}k\rho_e\cos\alpha} h(\alpha+\varphi_e), \qquad (5)$$

where $h(\cdot)$ is the meromorphic Sommerfeld transformant corresponding to the solution of the Helmholtz equation in these coordinates, $-\Phi_e^- < \varphi_e < \pi - \Phi_2^-$. The representations (4) and (5) specify the same wave field in the overlapping domain of the angles $|\varphi| \le \Phi$ and $-\Phi_e^- \le \varphi_e \le \pi - \Phi_2^-$. For $-\Phi_2^- \le \varphi_e \le \pi - \Phi$ we could also introduce the Sommerfeld integral representation of the wave field in the attributed to A_2 polar coordinates (ρ_2,φ_2) with the meromorphic Sommerfeld transformant $g(\cdot)$.

Following the known procedure (see [1]), from the boundary conditions we arrive at the system of Malyuzhinets functional equations for the Sommerfeld transformants

$$\begin{aligned}
&(\sin\alpha + \sin\vartheta_+) f(\alpha+\Phi) \\
&\qquad + (\sin\alpha - \sin\vartheta_+) f(-\alpha+\Phi) = 0, \\
&(\sin\alpha - \sin\vartheta_1) f(\alpha-\Phi) \\
&\qquad + (\sin\alpha + \sin\vartheta_1) f(-\alpha-\Phi) \\
&\quad = \mathrm{e}^{\mathrm{i}kd_1\cos\alpha}\left[(\sin\alpha - \sin\vartheta_1) g(\alpha-\Phi)\right. \\
&\qquad \left. + (\sin\alpha + \sin\vartheta_1) g(-\alpha-\Phi)\right],
\end{aligned} \qquad (6)$$

$$\begin{aligned}
&(\sin\alpha + \sin\vartheta_1) g(\alpha-[\Phi-\pi]) \\
&\qquad + (\sin\alpha - \sin\vartheta_1) g(-\alpha-[\Phi-\pi]) \\
&\quad = \mathrm{e}^{\mathrm{i}kd_1\cos\alpha}\left[(\sin\alpha + \sin\vartheta_1) f(\alpha-[\Phi-\pi])\right. \\
&\qquad \left. + (\sin\alpha - \sin\vartheta_1) f(-\alpha-[\Phi-\pi])\right], \\
&(\sin\alpha - \sin\vartheta_2) g(\alpha-\Phi_2^-) \\
&\qquad + (\sin\alpha + \sin\vartheta_2) g(-\alpha-\Phi_2^-) \\
&\quad = \mathrm{e}^{\mathrm{i}kd_2\cos\alpha}\left[(\sin\alpha - \sin\vartheta_2) h(\alpha-\Phi_2^-)\right. \\
&\qquad \left. + (\sin\alpha + \sin\vartheta_2) h(-\alpha-\Phi_2^-)\right],
\end{aligned} \qquad (7)$$

$$\begin{aligned}
&(\sin\alpha + \sin\vartheta_2) h(\alpha-[\Phi_2^- -\pi]) \\
&\qquad + (\sin\alpha - \sin\vartheta_2) h(-\alpha-[\Phi_2^- -\pi]) \\
&\quad = \mathrm{e}^{\mathrm{i}kd_2\cos\alpha}\left[(\sin\alpha + \sin\vartheta_2) g(\alpha-[\Phi_2^- -\pi])\right. \\
&\qquad \left. + (\sin\alpha - \sin\vartheta_2) g(-\alpha-[\Phi_2^- -\pi])\right], \\
&(\sin\alpha - \sin\vartheta_-) h(\alpha-\Phi_e^-) \\
&\qquad + (\sin\alpha + \sin\vartheta_-) h(-\alpha-\Phi_e^-) = 0.
\end{aligned} \qquad (8)$$

Three connected pairs of the functional equations (6), (7), (8) should be supplemented by additional conditions specifying a class of meromorphic functions which also ensure the Meixner's and radiation condition for the total field represented by the Sommerfeld integrals.

We assume that $f(\cdot)$ is regular in the strip $\Pi(-\Phi,\Phi) = \{\alpha \in \mathrm{C} : -\Phi < \alpha < \Phi\}$, i.e. it is holomorphic in this strip, having a simple pole at $\alpha = \varphi_s$, $(\varphi_s := \Phi - \vartheta_+)$ on its boundary so that

$$f(\alpha) - \frac{1}{\alpha - \varphi_s}$$

is holomorphic in the strip $\Pi(-\Phi - \epsilon, \Phi + \epsilon)$ for any small $\epsilon > 0$. Recall that such a condition is necessary in the framework of the Malyuzhinets technique and enables one to reproduce the incident wave in the far field asymptotics when applying the steepest descent method to the Sommerfeld integral and crossing the pole in the process of deformation of the contour γ into the steepest descent paths. In order to ensure the Meixner's condition at A_1 it is also assumed that $f(\mathrm{i}\infty) = -f(-\mathrm{i}\infty)$ is finite and $f(\alpha) - f(\pm\mathrm{i}\infty)$ is of order $O(\exp(\pm\mathrm{i}\alpha\delta_1))$, i.e. exponentially vanish as $\alpha \to \pm\mathrm{i}\infty$ and $\alpha \in \Pi(-\Phi, \Phi)$.

Similar conditions are formulated for g and h.

3 Reduction to integral equation and its Fredholm property

Instead of the three unknown Sommerfeld transformants f, g, h we introduce the new unknown functions V_f, $V_g^{\pm}$, V_h. The Sommerfeld transformants f, g, h are expressed by means of V_f, $V_g^{\pm}$, V_h in terms of the so called S-integrals, not written in this paper but discussed in [2]. It is possible to determine these new unknowns from the integral equations

$$V_f(\alpha) = V_f^i(\alpha) + \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\tau \left(\mathcal{K}_{1,2}(\alpha,\tau;k)V_g^+(\tau) + \mathcal{K}_{1,3}(\alpha,\tau;k)V_g^-(\tau)\right),$$

$$V_g^+(\alpha) = V_g^{+,i}(\alpha) + \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\tau\, \mathcal{K}_{2,1}(\alpha,\tau;k)V_f(\tau),$$

$$V_g^-(\alpha) = V_g^{-,i}(\alpha) + \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\tau\, \mathcal{K}_{3,4}(\alpha,\tau;k)V_h(\tau),$$

$$V_h(\alpha) = V_h^i(\alpha) + \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\tau \left(\mathcal{K}_{4,2}(\alpha,\tau;k)V_g^+(\tau) + \mathcal{K}_{4,3}(\alpha,\tau;k)V_g^-(\tau)\right),$$

where the expressions of the kernel entries $\mathcal{K}_{ij}(\alpha,\tau;k)$ are some known meromorphic functions of α and τ and V_f^i, $V_g^{\pm,i}$, V_h^i are specified by the incident wave,

$$V = K\,V + V_i, \tag{9}$$

where $K = \{K_{ij}\}_{i,j=1}^4$ is the matrix integral operator with the entries $\mathcal{K}_{ij}(\alpha,\tau;k)$ of the kernel,

$$V = (V_f(\alpha), V_g^+(\alpha), V_g^-(\alpha), V_h(\alpha))^t,$$
$$V_i = (V_f^i(\alpha), V_g^{+,i}(\alpha), V_g^{-,i}(\alpha), V_h^i(\alpha))^t.$$

The equation (9) is studied in

$$\mathcal{L}_2(\mathrm{iR}) := L_2(\mathrm{iR}) \oplus L_2(\mathrm{iR}) \oplus L_2(\mathrm{iR}) \oplus L_2(\mathrm{iR}).$$

The integral operator

$$K : \mathcal{L}_2(\mathrm{iR}) \to \mathcal{L}_2(\mathrm{iR}),$$

holomorphically depends on the wave number k in some domain of the characteristic parameter k. It can be shown that analytic Fredholm alternative can be applied to (9). Thus we have

Theorem. Let k take a non-characteristic value and $\mathrm{Im}(k) \geq 0$, $\mathrm{Re}(k) > 0$, then $(I-K)^{-1}$ is bounded and the equation (9) is uniquely solvable in $\mathcal{L}_2(\mathrm{iR})$.

4 The excitation and diffraction coefficients

Consider the Sommerfeld representation (4) of the total field implying that the Sommerfeld transformant f has been determined via the procedure described above. Herein we are interested in the expressions for the excitation coefficients of the reflected and transmitted surface waves as well as in the determination of the diffraction coefficient of the circular wave in the far field asymptotics. To this end, we consider deformation of the double-loop Sommerfeld contour γ in (4) into the steepest descent paths $\gamma_\pm$ shown in [2], p. 88. In the process of such deformation the poles of the transformant f can be captured. The pole at $\alpha = \varphi_s := \Phi - \vartheta_+$ with unit residue gives rise for the incident surface wave U^i, whereas the pole at $\alpha_*^+ = \pi + \Phi + \vartheta_+$ belongs to the strip $\Pi(\Phi, 3\Phi)$. The latter is responsible for the reflected surface wave. Thus we have the asymptotics

$$u(r,\varphi) = U^i(r,\varphi) + u_s^+(r,\varphi) + \frac{e^{\mathrm{i}kr+\mathrm{i}\pi/4}}{\sqrt{2\pi kr}} D_f(\varphi)\left(1 + O\left(\frac{1}{kr}\right)\right) \tag{10}$$

as $0 \leq \Phi - \varphi \leq -\mathrm{gd}(\mathrm{Im}(\vartheta_+))$ (gd is the Gudermannian function), otherwise, the first two summands in (10) are omitted. Note that the excitation coefficient

$$C_s^+ = \mathrm{res}_{z=\alpha_*^+} f(z) = (-2)\tan\vartheta_+ f(\Phi - \pi - \vartheta_+) \quad (11)$$

with $\Phi - \pi - \vartheta_+ \in \Pi(-\Phi, \Phi)$. The excitation coefficient C_s^+ in (11) is specified by the value $f(\Phi-\pi-\vartheta_+)$, i.e. it requires solution of the integral equations in order to determine f. The reflected surface wave propagating along l^1 takes the form

$$u_s^+(r,\varphi) = C_s^+ \mathrm{e}^{\mathrm{i}kr\cos[\varphi-\Phi-\vartheta_+]}$$

as $0 \le \Phi - \varphi \le -\mathrm{gd}(\mathrm{Im}(\vartheta_+))$, otherwise, the pole is not captured.

In the same manner, making use of the Sommerfeld representation for u in (5) and the second equation in (6), written as

$$h(\alpha) = -h(-2\Phi_e^- - \alpha)\,\frac{\sin(\alpha+\Phi_e^-) + \sin\vartheta_-}{\sin(\alpha+\Phi_e^-) - \sin\vartheta_-},$$

we compute the pole $\alpha_*^- = -(\pi + \Phi_e^- + \vartheta_-)$ and the transmitted surface wave propagating along l^4

$$u_s^-(\rho_e,\varphi_e) = C_s^- \mathrm{e}^{\mathrm{i}k\rho_e\cos[\varphi_e+\Phi_e^-+\vartheta_-]}$$

as $0 \le \Phi_e^- + \varphi_s \le -\mathrm{gd}(\mathrm{Im}(\vartheta_-))$, otherwise, $u_s^- = 0$, with the excitation coefficient

$$C_s^- = 2\tan\vartheta_-\, h(\pi - \Phi_e^- + \vartheta_-)\,.$$

Provided that $\epsilon - \Phi_e^- \ge \varphi_e \ge -\Phi_e^-$, $\epsilon > 0$, i.e. in some angular vicinity of l^4, we obtain the asymptotics

$$u(\rho_e,\varphi_e) = u_s^-(\rho_e,\varphi_e) + \frac{e^{\mathrm{i}k\rho_e+\mathrm{i}\pi/4}}{\sqrt{2\pi k\rho_e}} D_h(\varphi_e) \times \left(1 + O\left(\frac{1}{k\rho_e}\right)\right) \quad (12)$$

with $D_h(\varphi_e) = h(-\pi+\varphi_e) - h(\pi+\varphi_e)$.

In a domain, where both asymptotics (10), (12) are valid, they give asymptotically equivalent results.

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