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ON THE CONCEPT OF ABSOLUTELY CONTINUOUS SUBSPACE FOR NONSELFADJOINT OPERATORS

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ABSTRACT. We give an example of an operator with different weak and strong absolutely continuous subspaces, and a counterexample to the duality problem for the spectral components. Both examples are optimal in the scale of compact operators.

KEYWORDS: *Absolutely continuous subspace, non-selfadjoint operators, weighted shift.*

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INTRODUCTION

The known definitions of the absolutely continuous (a.c.) subspace for nonselfadjoint operators fall into two groups: the weak ones and the strong one. The strong definition was first suggested by L. Sakhnovich [7] in the case of dissipative operators.

DEFINITION 0.1. Let L be a completely non-selfadjoint dissipative operator in a Hilbert space H with a bounded imaginary part V. Any of the following coinciding subspaces is called the *strong absolutely continuous subspace* $H_{\rm ac}$ of L:

(i) the invariant subspace of L corresponding to the canonical factorization of its characteristic function;

(ii) the minimal subspace containing all the invariant subspaces X of L such that $L|_X = WAW^{-1}$ for an a.c. selfadjoint operator A and a bounded and boundedly invertible operator $W\colon X \to X$;

(ii) $\operatorname{Clos}\{u \in H\colon\ V^{1/2}(L-z)^{-1}u|_{\mathbb{C}_+} \in \mathbf{H}^2_+\}$.

The notation we use is given at the end of Introduction. The correspondence in definition (i) is the one between invariant subspaces of an operator and regular factorizations of its characteristic function in the framework of the Szökefalvi–Nagy–Foiaş functional model. The equivalence of (ii) and (iii) is a corollary of this correspondence. The equivalent definition (iii) in model-free terms was found in [4]. In this latter form, the definition was generalized to non-dissipative perturbations of selfadjoint operators [4] and non-contractive perturbations of unitary ones [3].

The weak definitions of the a.c. subspace are obtained if we try to generalize directly the "selfadjoint" definition using the property expressed by the Riesz brothers theorem as a substitute for the absolute continuity of (non-existent, in general) spectral measure. This leads to spaces defined by the requirement that the matrix element of the resolvent be of the Hardy class H^p. They were first introduced and studied in [9], [10] and are called weak a.c. subspaces. A weak a.c. subspace contains the strong one because the restriction of an operator to the a.c. subspace is quasi-similar to an a.c. selfadjoint operator. A natural question is whether these subspaces coincide. So far, it was answered in affirmative in two situations ($p = 2$):

(i) When L is dissipative [5].

(ii) When the characteristic function of L has weak boundary values a.e. on the real axis [6]. This holds true, for instance, for trace class perturbations of a selfadjoint operator, and, more generally, if the function admits scalar multiple.

The first result of the present paper is an example of an operator with different weak and strong a.c. subspaces (Theorems 1.6 and 1.8). The example in Theorem 1.8 is a (non-dissipative) perturbation of a selfadjoint operator. Theorem 1.6 provides an analogous result for perturbations of unitary operators. The operators we construct are in fact similar to the standard bilateral shift [8] in the unit circle case and to the generator of bilateral shift on $\mathbb{R}$ in the real line case.

Whichever definition of the a.c. subspace is used, it is natural to ask whether the orthogonal complement of it coincides with the singular subspace of the adjoint operator. The latter is defined to be the closure of the set of vectors such that the matrix element of the resolvent on such a vector has zero jump a.e. while crossing the essential spectrum. This question is sometimes referred to as the duality problem for spectral components and is known to have affirmative answer in the case of trace class perturbations [3], [9].

Our second result is Theorem 2.3: an example of an operator with trivial singular subspace of the adjoint operator and nontrivial orthogonal complement of the weak a.c. subspace.

The examples in Theorems 1.6 and 2.3 are optimal in the sense that they are additive perturbations of unitary operators by operators whose s-numbers can be chosen to be estimated above by an arbitrary given monotone non-summable sequence.

Theorems 1.6 and 1.8 are proved in Section 1, Theorem 2.3 in Section 2. In Section 3 we analyze the weak definition of the a.c. subspace for $p \neq 2$, and show that in the dissipative case it gives the same subspace as for $p = 2$.

NOTATION. (i) $\mathbb{C}_\pm = \{z : \pm \operatorname{Im} z > 0\}$, $\mathbb{T} = \{z\colon |z| = 1\}$; $\mathbb{D} = \{z\colon |z| < 1\}$;

(ii) $H^p_\pm$, $0 < p \leqslant 2$: Hardy classes of analytic functions in $\mathbb{C}_\pm$, respectively;

(ii) H^2: the Hardy space for the unit disk.

For functions with values in a Hilbert space H:

(iv) $\mathbf{H}^2_\pm$: the Hardy space of H-valued functions in $\mathbb{C}_\pm$, respectively; the norm of an $f \in \mathbf{H}^2_\pm$ is given by $\sup\limits_{\varepsilon>0} \int\limits_{\mathbb{R}} \|f(k \pm \mathrm{i}\varepsilon)\|^2_H \mathrm{d}k$;

(v) $\mathbf{H}^2$: the Hardy space of H-valued functions in the unit disk;

(vi) $\{e_n\}$: the standard basis in $l^2(\mathbb{Z})$.

For an operator L in a Hilbert space:

(vii) $\mathcal{D}(L)$: the domain of L;

(vii) $f_{u,v}(z) = \langle (L-z)^{-1}u, v\rangle$;

(ix) $\mathbf{S}^p$, $p \geqslant 1$: the Shatten–von Neumann classes of compact operators with summable p-th power of their singular numbers.

The subscripts $_\pm$ with functions in the complex plane stand for their respective restrictions to $\mathbb{C}_\pm$.

Various subspaces corresponding to abstract operators are defined in the paper. We will often suppress the explicit indication of the operator in the notation for the subspaces when it is clear which operator it refers to.

1. SECTION

Let L be a closed operator in a Hilbert space H. Throughout the paper, it is assumed that $\sigma(L) \cap \mathbb{C}_\pm$ are discrete sets. For any $p \leqslant 2$ one can define the following subspaces in H:

$$
\begin{aligned}
& H^{w,p}_{\mathrm{ac}}(L) \stackrel{\mathrm{def}}{=} \mathrm{Clos}\widetilde{H^{w,p}_{\mathrm{ac}}}(L), \\
& \widetilde{H^{w,p}_{\mathrm{ac}}}(L) \stackrel{\mathrm{def}}{=} \left\{ \begin{array}{ll} u \in H: & (L-z)^{-1}u \text{ is analytic in } \mathbb{C}\setminus\mathbb{R}, \\ & \langle (L-z)^{-1}u, v\rangle_\pm \in H^p_\pm \text{ for all } v \in H. \end{array} \right\}.
\end{aligned} \tag{1.1}
$$

In the case $p = 2$ such a subspace is called the weak a.c. subspace of the operator L. If L is self-adjoint, then for all $p, 1 < p \leqslant 2$, the subspaces $H^{w,p}_{\mathrm{ac}}(L)$ coincide with the a.c. subspace of the operator L defined in the standard way. We include a proof of this folklore-type assertion in the Appendix. An important property of the weak a.c. subspace is that similarity of operators, obviously, respects it. Notice that by the uniform boundedness principle the H^p-norms of the functions $f_{u,v}(z)$ with $u \in \widetilde{H^{w,p}_{\mathrm{ac}}}$ are bounded above when v ranges over the unit ball in H.

We shall omit throughout the index p in our notation in the case $p = 2$ writing $H^{\mathrm{w}}_{\mathrm{ac}}$ for $H^{w,2}_{\mathrm{ac}}$ etc.

For clarity, we restrict our consideration to the situation of the perturbation theory. From now on, it is assumed additionally that:

(A) *L is a completely nonself-adjoint operator of the form $L = A + \mathrm{i}V$, $A = A^*$, $V = V^*$, $\mathcal{D}(L) := \mathcal{D}(A) \subset \mathcal{D}(V)$, and V is A-bounded with a relative bound less than* 1, *that is, $\|Vu\|^2 \leqslant a\|Au\|^2 + b\|u\|^2$, $a < 1$, for all $u \in \mathcal{D}(A)$.*

This assumption implies, in particular, that

$$\mathrm{i}\tau(L + \mathrm{i}\tau)^{-1} \xrightarrow{s} I, \ \tau \to \pm\infty. \tag{1.2}$$

DEFINITION 1.1 ([4]). The subspace

$$\begin{aligned} &H_{\mathrm{ac}}(L) \stackrel{\mathrm{def}}{=} \operatorname{Clos} \widetilde{H_{\mathrm{ac}}}(L), \\ &\widetilde{H_{\mathrm{ac}}}(L) \stackrel{\mathrm{def}}{=} \left\{ \begin{array}{ll} u \in H: & (L-z)^{-1}u \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}, \\ & (|V|^{1/2}(L-z)^{-1}u)_{\pm} \in \mathbf{H}^2_{\pm} \end{array} \right\}, \end{aligned}$$

is called the *strong absolutely continuous subspace of the operator L*. Elements of the set $\widetilde{H_{\mathrm{ac}}}(L)$ are called *strong smooth vectors*.

Notice that there exists a natural generalization of this definition applicable to operators which do not satisfy the assumption (A) [6].

The main property of the strong smooth vectors is expressed by the following

PROPOSITION 1.2 ([4], Theorem 4). *There exists a Hilbert space $\mathcal{N}$, an a.c. self-adjoint operator A_0 in $\mathcal{N}$, and a bounded operator $P : \mathcal{N} \to H$ such that $P\mathcal{N} = \widetilde{H_{\mathrm{ac}}}(L)$ and the equality*

$$(L - z)^{-1}Pg = P(A_0 - z)^{-1}g$$

holds for all $g \in \mathcal{N}$ and $z \notin \mathbb{R}$, $z \in \rho(L)$.

COROLLARY 1.3. *$H^{\mathrm{w}}_{\mathrm{ac}}(L) \supset H_{\mathrm{ac}}(L)$.*

A similar theory is available for perturbations of unitary operators [3]. Let T be a bounded completely non-unitary operator such that $\sigma(T)$ has no accumulation points off $\mathbb{T}$.

DEFINITION 1.4. The *weak absolutely continuous subspace of the operator T* is the set

$$\begin{aligned} &H^{\mathrm{w}}_{\mathrm{ac}}(T) \stackrel{\mathrm{def}}{=} \operatorname{Clos} \widetilde{H^{\mathrm{w}}_{\mathrm{ac}}}(T), \\ &\widetilde{H^{\mathrm{w}}_{\mathrm{ac}}}(T) \stackrel{\mathrm{def}}{=} \widetilde{H^{\mathrm{w}}_{+}}(T) \cap \widetilde{H^{\mathrm{w}}_{-}}(T), \\ &\widetilde{H^{\mathrm{w}}_{+}}(T) \stackrel{\mathrm{def}}{=} \left\{ \begin{array}{ll} u \in H: & (T-z)^{-1}u \text{ is analytic in } \mathbb{D}, \\ & \langle (T-z)^{-1}u, v\rangle|_{\mathbb{D}} \in H^2 \text{ for all } v \in H \end{array} \right\}, \\ &\widetilde{H^{\mathrm{w}}_{-}}(T) \stackrel{\mathrm{def}}{=} \left\{ \begin{array}{ll} u \in H: & (T-z)^{-1}u \text{ is analytic in } \mathbb{C} \setminus \overline{\mathbb{D}}, \\ & \langle (I-zT)^{-1}u, v\rangle|_{\mathbb{D}} \in H^2 \text{ for all } v \in H \end{array} \right\}. \end{aligned}$$

Let $D_T \stackrel{\text{def}}{=} |I - T^*T|^{1/2}$. The subspace

$$H_{\text{ac}}(T) \stackrel{\text{def}}{=} \operatorname{Clos} \widetilde{H_{\text{ac}}}(T),$$
$$\widetilde{H_{\text{ac}}}(T) \stackrel{\text{def}}{=} \left\{ \begin{array}{lll} u \in H: & \text{(i)} & (T-z)^{-1}u \text{ is analytic in } \mathbb{C} \setminus \mathbb{T}, \\ & \text{(ii)} & D_T(T-z)^{-1}u|_{\mathbb{D}} \in \mathbf{H}^2, \\ & \text{(iii)} & D_T(I-zT)^{-1}u|_{\mathbb{D}} \in \mathbf{H}^2, \end{array} \right\}$$

is called the *strong absolutely continuous subspace of the operator T*. Elements of the linear set $\widetilde{H_{\text{ac}}}(T)$ are called *(strong) smooth vectors*.

An analog of the Proposition 1.2 holds for the strong smooth vectors of T [3].

COROLLARY 1.5. *$H^{\text{w}}_{\text{ac}}(T) \supset H_{\text{ac}}(T)$.*

We now proceed to our results, first for perturbations of unitary operators. Given a selfadjoint operator D, define $\lambda_j(D)$ to be the eigenvalues of D enumerated in the modulus decreasing order. Let $\{\pi_n\}$, $\pi_n > 0$, be a monotone decreasing sequence.

THEOREM 1.6. *There exists a bounded completely non-unitary operator T obeying the following conditions,*
(i) *T is similar to an a.c. unitary operator (and thus $H^{\text{w}}_{\text{ac}}(T) = H$);*
(ii) *$H_{\text{ac}}(T) = \{0\}$;*
(iii) *$I - T^*T \in \mathbf{S}^p$ for all $p > 1$.*
Moreover, for any sequence $\{\pi_n\} \notin l^1$ there exists an operator T satisfying the conditions above with (iii) *replaced by*
(iii′) *$|\lambda_n(I - T^*T)| \leqslant \pi_n$.*

This theorem is optimal in the sense that the subspaces H^{w}_{ac} and H_{ac} are known ([10], Proposition 4.10 and Theorem C) to coincide if $I - T^*T \in \mathbf{S}^1$, provided that $\mathbb{D} \not\subset \sigma_{\text{ess}}(T)$. In the terminology of [2], the theorem says that no condition of the form $I - T^*T \in \mathbf{S}_\pi$ where $\mathbf{S}_\pi$ is a symmetrically-normed ideal of compact operators containing $\mathbf{S}^1$ properly, guarantees the coincidence of H^{w}_{ac} and H_{ac}.

Proof. Let $H = \ell^2(\mathbb{Z})$. We shall construct a sequence $\{\rho_n\}_{n=-\infty}^{+\infty}$ of positive numbers such that the weighted bilateral shift operator T defined by

$$Te_j = \rho_{j-1}e_{j-1}, \quad j \in \mathbb{Z},$$

has the required properties. Assume that

$$\sum_j |\rho_j - 1|^p < \infty \tag{1.3}$$

for any $p > 1$. We are going to need the explicit formula for the resolvent of T,

$$((T-\lambda)^{-1}f)_m = \begin{cases} \sum_{k<m} f_k \frac{\lambda^{m-k-1}}{\prod_k^{m-1}\rho_j} & |\lambda| < 1, \\ -\sum_{k\geqslant m} f_k \frac{\prod_m^{k-1}\rho_j}{\lambda^{k-m+1}} & |\lambda| > 1. \end{cases},$$

the product in the second sum being treated as 1 when $k = m$.

Proceeding, let us check that the following implication holds:

$$D_T(T-z)^{-1}u|_{\mathbb{D}} \in \mathbf{H}^2 \Longrightarrow \sum_{n>0} \frac{|1-\rho_n^2|}{\prod_0^{n-1}\rho_j^2} < \infty, \tag{1.4}$$

provided that $u \neq 0$. This is done by a direct computation. In the situation under consideration

$$D_T = \operatorname{diag}\{|1-\rho_n^2|^{1/2}\}.$$

We have ($z = re^{i\theta}$):

$$\int_{-\pi}^{\pi} \|D_T(T-z)^{-1}f\|^2 d\theta = \sum_n |1-\rho_n^2| \int_{-\pi}^{\pi} \Big| \sum_{k<n} f_k \frac{z^{n-k-1}}{\prod_k^{n-1}\rho_j} \Big|^2 d\theta$$
$$= 2\pi \sum_n |1-\rho_n^2| \sum_{k<n} r^{2(n-k-1)} \frac{|f_k|^2}{\prod_k^{n-1}\rho_j^2}.$$

Thus, the function $D_T(T-z)^{-1}f$ is in $\mathbf{H}^2$ if and only if the quantity

$$\sum_n |1-\rho_n^2| \sum_{k<n} \frac{|f_k|^2}{\prod_k^{n-1}\rho_j^2} = \sum_k \Big(\sum_{n>k} \frac{|1-\rho_n^2|}{\prod_k^{n-1}\rho_j^2} \Big) |f_k|^2$$

is finite. This means that the sum in parentheses in the right hand side must be finite for some k. Since this sum, obviously, converges or diverges for all k simultaneously, the implication (1.4) is established.

The existence of an operator T enjoying the properties (i)–(iii) will be proved if we construct a sequence $\{\rho_j\}$ such that T is similar to an a.c. unitary operator, the sum in (1.4) diverges, and condition (1.3) is satisfied. Let $a_j = 1 + 1/j$ and let

$$\rho_j = 1, \quad j \leqslant 1; \quad \rho_{2j} = a_j, \quad j \geqslant 1; \quad \rho_{2j+1} = a_j^{-1}, \quad j \geqslant 1.$$

With this choice, (1.3) and the divergence of the sum in (1.4) are straightforward. Then, define

$$w_{2j+1} = a_j^{-1} \quad j \geqslant 1; \quad w_j = 1 \quad \text{otherwise}.$$

The diagonal operator $W = \operatorname{diag}\{w_j\}$, defined by the sequence $\{w_j\}$ in H, is obviously bounded, boundedly invertible, and it is easy to check that $W^{-1}TW$ is the unitary operator of (non-weighted) shift in H.

To verify the second assertion of the theorem one can assume without loss of generality that $\pi_{2j+1} = \pi_{2j}$. It is then enough to take $\{a_j\}$ to be any sequence of positive numbers such that $a_j \to 1$, $|1-a_j| \leqslant \pi_j/3$, but $\sum |1-a_j| = \infty$, in the construction above. ∎

REMARK 1.7. Theorem 1.6 shows that the linear resolvent growth condition

$$\sup_{z \notin \mathbb{T}} (|1-|z|\,|(T-z)^{-1}) < \infty \tag{1.5}$$

does not imply the coincidence of $H_{\rm ac}$ and $H^{\rm w}_{\rm ac}$. Also, it shows that in general similarity of operators does not respect the strong a.c. subspace.

We now turn to perturbations of selfadjoint operators. Let $H = L^2(\mathbb{R})$, and $q(x)$ be a bounded real function on $\mathbb{R}$ satisfying

$$\sum_n \Big(\int_n^{n+1} |q|^2 \Big)^{p/2} < \infty \quad \text{for all } p > 1;$$
$$q \notin L^1; \quad q \text{ is conditionally integrable;}$$

($q(x) = \sin x/x$ is the simplest example). Let L be the operator in H defined by the differential expression

$$L = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + \mathrm{i}q(x)$$

on its natural domain. Notice that the operator L is similar to the operator $A = \mathrm{i}(\mathrm{d}/\mathrm{d}x)$:

$$L = WAW^{-1},$$

where W is the operator of multiplication by the function $\exp(-\int_{-\infty}^x q)$.

THEOREM 1.8. *The operator L obeys the following conditions:*

(i) *L is similar to an a.c. selfadjoint operator (and thus $H^{\rm w}_{\rm ac}(L) = H$);*

(ii) *$H_{\rm ac}(L) = \{0\}$;*

(iii) *$(L-z)^{-1} - (A-z)^{-1} \in \mathbf{S}^p$ for all $p > 1$, $\operatorname{Im} z \neq 0$.*

Proof. The operator A is absolutely continuous, hence (i) is immediate. Let $V = \operatorname{Im} L$. Suppose that u is a strong smooth vector of L. Since W commutes with the multiplication by a function, this is equivalent to saying that the restrictions of the function

$$\varphi(z) = |V|^{1/2}(A-z)^{-1}g, \; g = Wu,$$

belong to $\mathbf{H}^2_\pm$ in the respective halfplanes. In turn, the latter is equivalent to the condition

$$\int_{\mathbb{R}} \||V|^{1/2}\mathrm{e}^{\mathrm{i}tA}g\|^2 \mathrm{d}t < \infty$$

by the Parseval equality for the vector Fourier transform. We have

$$\int_{\mathbb{R}} \||V|^{1/2}\mathrm{e}^{\mathrm{i}tA}g\|^2 \mathrm{d}t = \int |q(x)|\,|g(x-t)|^2 \mathrm{d}x\,\mathrm{d}t = \|g\|^2 \int |q(x)|\mathrm{d}x = \infty$$

if $g \neq 0$. This proves (ii). The assertion (iii) is a corollary of the following result from [1]:

For any δ, $1 < \delta < 2$, and any functions f, g satisfying

$$\sum_n \Big(\int_n^{n+1} |f|^2 \Big)^{\delta/2} < \infty, \; \sum_n \Big(\int_n^{n+1} |g|^2 \Big)^{\delta/2} < \infty,$$

the operator T in H defined by

$$(Tu)(x) = \int_{\mathbb{R}} f(x) \mathrm{e}^{\mathrm{i}xy} g(y) u(y) \mathrm{d}y$$

belongs to $\mathbf{S}^\delta$.

Applying it to $f = q$ and $g(y) = (y - z)^{-1}$, $\operatorname{Im} z \neq 0$, we find that the operator $V(A - z)^{-1} \in \mathbf{S}^p$, $p > 1$, which implies (iii). ∎

We have preferred to construct directly the example for the real line case, rather than use the Cayley transform. The reason is that the Cayley transform of the operator in Theorem 1.6 does not belong to the class for which we have defined the strong a.c. subspace. It would give the example required if we used the general definition of $H_{\mathrm{ac}}(L)$ from [6] mentioned above.

2. SECTION

Let T be a bounded operator such that $\sigma(T) \subset \mathbb{T}$.

DEFINITION 2.1. The closure of the linear set of vectors $u \in H$ such that for all $v \in H$ the nontangential limits

$$f^{\pm}_{u,v}(z) = \lim_{w \to z,\, |w|^{\pm 1} \in \mathbb{D}} \langle (T - w)^{-1} u, v \rangle$$

exist and coincide for a.e. $z \in \mathbb{T}$, is called the *singular subspace of the operator T*. It is denoted by $H_{\mathrm{s}}(T)$.

As discussed in the Introduction, the duality problem [10] is the question whether the equality

$$(H^{\mathrm{w}}_{\mathrm{ac}}(T))^{\perp} = H_{\mathrm{s}}(T^*) \tag{2.1}$$

holds.

PROPOSITION 2.2 ([3], Proposition 6.7). *If $I - T^*T \in \mathbf{S}^1$, then* (2.1) *is satisfied.*

In fact, the quoted proposition in [3] establishes (2.1) for completely non-unitary operators with the strong a.c. subspace in the place of $H^{\mathrm{w}}_{\mathrm{ac}}(T)$. These subspaces coincide when $I - T^*T \in \mathbf{S}^1$.

More generally, (2.1) is known to hold if the characteristic function has weak boundary values a.e. [6]. In the following example of a non-contractive conjunction of two bilateral shifts (2.1) fails. In the notation of Definition 1.4 let

$$CN(T) \stackrel{\text{def}}{=} \operatorname{Clos}(\widetilde{H_+^{\mathrm{w}}}(T) \vee \widetilde{H_-^{\mathrm{w}}}(T)).$$

Let $\{\rho_n\}$, $n \geqslant 0$, be a sequence of positive numbers monotone decreasing to 0, and R be an operator in $l^2(\mathbb{Z})$ defined by $Re_n = \rho_{|n|}e_n$, $n \in \mathbb{Z}$. Let $H = l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z})$, and let U be the operator of right shift in $l^2(\mathbb{Z})$, $Ue_n = e_{n+1}$. Define an operator T in H by

$$T = \begin{pmatrix} U & R \\ 0 & U \end{pmatrix}.$$

Obviously, $\sigma(T) = \mathbb{T}$.

THEOREM 2.3. *Let $\{\rho_j\} \notin l^1$. Then the operator T obeys the following conditions:*
(i) *$CN(T) \neq H$;*
(ii) *$H_{\mathrm{s}}(T^*) = \{0\}$;*
(iii) *$T = T_0 + S$, where T_0 is unitary and S is an operator whose singular numbers, $\mu_n(S)$, satisfy $\mu_n(S) \leqslant \rho_{[n/2]}$.*

Proof. The assertion (iii) is obvious (in fact, $\mu_{2n}(S) = \mu_{2n+1}(S) = \rho_n$ for $n \geqslant 1$). Then, for any $\lambda \notin \mathbb{T}$ we have

$$(T^* - \lambda)^{-1} = \begin{pmatrix} (U^* - \lambda)^{-1} & 0 \\ -(U^* - \lambda)^{-1}R(U^* - \lambda)^{-1} & (U^* - \lambda)^{-1} \end{pmatrix}.$$

Let

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

be from the dense set in the definition of $H_{\mathrm{s}}(T^*)$. Considering the matrix element $\langle (T^* - \lambda)^{-1} f, g \rangle$ with g of the form $g = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$, we conclude that $f_1 = 0$, since U is an absolutely continuous unitary operator. Then, by the same reason, taking $g = \begin{pmatrix} 0 \\ f_2 \end{pmatrix}$, we obtain that $f_2 = 0$, hence $H_{\mathrm{s}}(T^*)$ is trivial. Also, the absolute continuity of U implies that $CN(T) \supset \begin{pmatrix} l^2(\mathbb{Z}) \\ 0 \end{pmatrix}$. Let us show that in fact

$$CN(T) = \begin{pmatrix} l^2(\mathbb{Z}) \\ 0 \end{pmatrix}.$$

Actually, we shall show that if for a $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H$ the function

$$\left\langle (T - \lambda)^{-1} u, \begin{pmatrix} e_j \\ 0 \end{pmatrix} \right\rangle = \langle (U - \lambda)^{-1} u_1, e_j \rangle - \langle (U - \lambda)^{-1} R (U - \lambda)^{-1} u_2, e_j \rangle \tag{2.2}$$

is in H^2 for all j, and the H^2-norm of it is bounded above in j, then $u_2 = 0$. By the uniform boundedness principle, this is going to imply that $\widetilde{H_+^{\mathrm{w}}} \subset \begin{pmatrix} l^2(\mathbb{Z}) \\ 0 \end{pmatrix}$.

Taking into account that

$$\|\langle (U-\cdot)^{-1}u_1, e_j\rangle\|_{H^2}^2 = \sum_1^\infty |u_{1,k+j}|^2 \leqslant \|u\|^2,$$

we only have to check that the H^2-norm of the second term in the right hand side in (2.2) is unbounded as $j \to \infty$ if $u_2 \neq 0$. Indeed, a straightforward calculation gives that for $\lambda \in \mathbb{D}$

$$\langle (U-\lambda)^{-1}R(U-\lambda)^{-1}u_2, e_j\rangle = \sum_{s=0}^\infty \lambda^s u_{2,s+j+2} \sum_{m=1}^{s+1} \rho_{|m+j|}.$$

Thus, the H^2-norm of the left hand side is

$$\sum_{s=0}^\infty |u_{2,s+j+2}|^2 \left| \sum_{m=j+1}^{j+s+1} \rho_{|m|} \right|^2.$$

Suppose that $u_{2,r} \neq 0$ for some r. Then for j negative enough this norm is bounded below by $|u_{2,r}|^2 \left| \sum_{m=j+1}^{r-1} \rho_{|m|} \right|^2$ which goes to infinity when $j \to -\infty$ by the assumption about ρ_j. The inclusion $\widetilde{H_-^{\mathrm{w}}} \subset \begin{pmatrix} l^2(\mathbb{Z}) \\ 0 \end{pmatrix}$ is checked similarly. ∎

As is clear from the proof, in this example the subspaces $CN(T)$ and $H_{\mathrm{ac}}^{\mathrm{w}}(T)$ coincide. One should mention that the linear resolvent growth condition (1.5) is violated for the operator T.

3. SECTION

We now turn to the case $p \neq 2$.

LEMMA 3.1. *$\overline{(L-z_0)^{-1}\widetilde{H_{\mathrm{ac}}^{w,p_1}}} = H_{\mathrm{ac}}^{w,p_1} \subset H_{\mathrm{ac}}^{w,p_2}$ for all $z_0 \in \rho(L)$ and $1 \leqslant p_2 \leqslant p_1 \leqslant 2$ save for $p_1 = 1$.*

Proof. An application of the Hölder inequality to the resolvent identity

$$f_{(L-z_0)^{-1}u,v}(z) = \frac{1}{z-z_0}(f_{u,v}(z) - f_{u,v}(z_0))$$

shows that

$$(L-z_0)^{-1}\widetilde{H_{\mathrm{ac}}^{w,p_1}} \subset \widetilde{H_{\mathrm{ac}}^{w,p_2}}$$

for all $z_0 \in \rho(L)$. Since the linear set in the left hand side is independent of the choice of $z_0 \in \rho(L)$, the asymptotics (1.2) implies that the set is dense in H_{ac}^{w,p_1}. ∎

It is also easy to check that

$$\|(L-z)^{-1}u\| \leqslant C_u |\operatorname{Im} z|^{-1/p} \tag{3.1}$$

for any $u \in \widetilde{H_{\mathrm{ac}}^{w,p}}$, $1 < p \leqslant 2$.

THEOREM 3.2. *If L is dissipative ($V \geqslant 0$) and $1 < p \leqslant 2$, then $H_{\mathrm{ac}}^{w,p}(L) = H_{\mathrm{ac}}^{\mathrm{w}}(L)$.*

Proof. Since the inclusion $H_{\mathrm{ac}}^{\mathrm{w}} \subset H_{\mathrm{ac}}^{w,p}$ is contained in Lemma 3.1, it remains to check that $H_{\mathrm{ac}}^{w,p} \subset H_{\mathrm{ac}}^{\mathrm{w}}$. We do so by proving that any vector u from a dense subset in $H_{\mathrm{ac}}^{w,p}$ is a strong smooth vector. Recall [4] that for a dissipative operator L the restriction of the function $V^{1/2}(L-z)^{-1}w$ to $\mathbb{C}_-$ belongs to $\mathbf{H}^2_-$ for all $w \in H$. Hence, we only have to verify that

$$(V^{1/2}(L-z)^{-1}u)_+ \in \mathbf{H}^2_+ \tag{3.2}$$

for all u from the dense subset.

Let $\mathcal{D} = (L+\mathrm{i})^{-2}H_{\mathrm{ac}}^{w,p}$. This is a dense subset in $H_{\mathrm{ac}}^{w,p}$. Let $u \in \mathcal{D}$. Taking into account (3.1) one easily checks that the function $(L - \cdot - \mathrm{i}\varepsilon)^{-1}u \in L^2(\mathbb{R}, H)$ for any $\varepsilon > 0$. We are first going to show that

$$\sup_{\varepsilon>0}\Big(\varepsilon\int_{\mathbb{R}} \|(L-k-\mathrm{i}\varepsilon)^{-1}u\|^2 \mathrm{d}k\Big) < \infty. \tag{3.3}$$

For $\varepsilon > 0$ and $t < 0$ we define

$$u(t) = -\frac{1}{2\pi\mathrm{i}}\lim_{N\to\infty}\int_{-N}^{N} \mathrm{e}^{\mathrm{i}(k+\mathrm{i}\varepsilon)t}(L-k-\mathrm{i}\varepsilon)^{-1}u\,\mathrm{d}k. \tag{3.4}$$

Then:

1° The limit in (3.4) exists for all $t < 0$, and is independent of $\varepsilon > 0$.

2° $\sup\limits_{t<0}\|u(t)\| < \infty$.

The assertion 1° follows immediately from the possibility to rewrite (3.4) in the form ($\lambda = k + \mathrm{i}\varepsilon$)

$$u(t) = -\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}\lambda t}}{(\lambda+\mathrm{i})^2}(L-\lambda)^{-1}u_2\,\mathrm{d}k - \mathrm{e}^t(\mathrm{i}t\,u_1 + u)$$

where $u_1 = (L+\mathrm{i})u$, $u_2 = (L+\mathrm{i})^2u$.

Let us establish 2°. For any $v \in H$ the scalar product $\langle u(t), v\rangle$ equals to

$$-\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}\lambda t}}{(\lambda+\mathrm{i})^2} f_{u_2,v}(\lambda)\,\mathrm{d}k + r_t$$

where $|r_t| \leqslant C\|v\|$, the bound being uniform in $t < 0$. Since $f_{u_2,v} \in H^p$, one can pass to the limit $\varepsilon \to 0$ in the integral obtained, and use the Hölder inequality to estimate the modulus of it by

$$C\|f_{u_2,v}\|_{H^p} \leqslant C\|v\|$$

with a constant C independent of t and v. This implies 2°.

Let $\mathcal{F}$ stand for the conjugate Fourier transform in $L^2(\mathbb{R}, H)$. Then (3.4) means that the restriction of the function

$$\Psi(t) = -\frac{1}{\mathrm{i}\sqrt{2\pi}}\,\mathcal{F}[(L - \cdot - \mathrm{i}\varepsilon)^{-1}u]$$

to $t < 0$ coincides with $\mathrm{e}^{\varepsilon t}u(t)$. On the other hand, $\Psi(t) = 0$ for $t > 0$ by the Paley–Wiener theorem. Applying the Parseval equality and taking into account the property 2°, we find

$$\int_{\mathbb{R}} \|(L - k - \mathrm{i}\varepsilon)^{-1}u\|^2 \mathrm{d}k = C \int_{-\infty}^{0} \mathrm{e}^{2\varepsilon t}\|u(t)\|^2\,\mathrm{d}t \leqslant C\varepsilon^{-1}.$$

The estimate (3.3) is proved.

We are now going to use the following easily verified identity valid for all $\lambda \in \rho(L)$ ($\varepsilon = \operatorname{Im}\lambda$),

$$\|V^{1/2}(L-\lambda)^{-1}u\|^2 = \varepsilon\|(L-\lambda)^{-1}u\|^2 - \operatorname{Im} f_{u,u}(\lambda).$$

The second term in the right hand side can be rewritten in the form

$$\operatorname{Im}\Big[\frac{1}{\lambda + \mathrm{i}}(f_{u_1,u}(\lambda) - f_{u_1,u}(-\mathrm{i}))\Big],$$

which implies that the term is conditionally integrable in $\operatorname{Re}\lambda$ over the real line, and the integrals are uniformly bounded in ε. Together with (3.3), this shows that (3.2) is satisfied for all $u \in \mathcal{D}$. ∎

REMARK 3.3. In a similar way, the subspaces $H_{\mathrm{ac}}^{w,p}(T)$ can be defined for perturbations of unitary operators. These subspaces can be shown to coincide with $H_{\mathrm{ac}}(T)$ for all p, $1 \leqslant p \leqslant 2$. The subspaces $H_{\mathrm{ac}}^{w,p}(T)$ can also be defined for $0 < p < 1$ but they coincide with H for any unitary operator T, rendering the definition meaningless. The reason is that the Cauchy transform of any finite measure is in $H^p(\mathbb{D})$ for $0 < p < 1$. In the real line context, the definition of the subspace for $p = 1$ requires a regularization at infinity.

4. APPENDIX

PROPOSITION 4.1. *Let L be a selfadjoint operator and let $\mathcal{H}_{\mathrm{ac}}(L)$ be its a.c. subspace defined via the spectral theorem. Then $H_{\mathrm{ac}}^{w,p}(L) = \mathcal{H}_{\mathrm{ac}}(L)$ for all $p \in (1,2]$.*

Proof. Let $\mathrm{d}\mu_{u,v}(t)$ be the matrix element of the spectral measure of L on vectors $u, v \in H$. Then

$$f_{u,v}(z) = \int_{\mathbb{R}} \frac{1}{t - z}\,\mathrm{d}\mu_{u,v}(t)$$

for all $u, v \in H$. Let $u = (L - z_0)^{-1}w$ with a $w \in \widetilde{H^{w,p}_{\mathrm{ac}}}$ and $z_0 \in \rho(L)$, then $f_{u,v}$ is represented as the Cauchy transform of its boundary values:

$$f_{u,v}(z) = \int_{\mathbb{R}} \frac{1}{(t-z)(t-z_0)} f_{w,v}(t)\mathrm{d}t.$$

Notice that $(t - z_0)^{-1} f_{w,v}(t)\,\mathrm{d}t$ is a finite Borel measure. Comparing the two representations and using the Riesz brothers theorem, one concludes that the measure

$$\mathrm{d}\mu_{u,v}(t) - (t - z_0)^{-1} f_{w,v}(t)\mathrm{d}t$$

is a.c. for all v. Hence $\mathrm{d}\mu_{u,v}$ is a.c. as well. Since the set of such u's is dense in $H^{w,p}_{\mathrm{ac}}$, the inclusion $H^{w,p}_{\mathrm{ac}} \subset \mathcal{H}_{\mathrm{ac}}$ follows. The inclusion $u \in H^{w,p}_{\mathrm{ac}}$ is obvious for any $u \in \mathcal{H}_{\mathrm{ac}}$ satisfying $\mathrm{d}\mu_{u,u}/\mathrm{d}t \in L^\infty(\mathbb{R})$. Since the set of such u's is dense in $u \in \mathcal{H}_{\mathrm{ac}}$, this implies that $\mathcal{H}_{\mathrm{ac}} \subset H^{w,p}_{\mathrm{ac}}$. ∎

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