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Absence of the absolutely continuous spectrum of a first-order non-selfadjoint Dirac-like system for slowly decaying perturbations

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Abstract. We prove that the absolutely continuous subspace of the completely non-selfadjoint part of a first-order dissipative Dirac-like system is trivial when the imaginary part of the potential is non-integrable.

Introduction

In this article we analyze the structure of the essential spectrum of dissipative operator realizations of the ordinary differential expression

$$l_Q := J\frac{d}{dx}+Q(x), \quad x\in[0,\infty), \tag{1}$$

where

$$J=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} \quad \text{and} \quad Q(x)=\begin{pmatrix}q_1(x) & 0\\ 0 & q_2(x)\end{pmatrix}, \tag{2}$$

in which $Q(x)$ is a bounded function such that $\operatorname{Im} q_1(x)$ and $\operatorname{Im} q_2(x)$ are non-negative a.e. We shall prove the following result.

Theorem 1. *Let L be the operator in $L^2(\mathbf{R}_+,\mathbf{C}^2)$ given by the expression* (1) *and with domain determined by a selfadjoint boundary condition at* 0*. If the absolutely continuous subspace of L is non-trivial, then $\operatorname{Im} q_1$ and $\operatorname{Im} q_2$ both lie in $L^1(\mathbf{R}_+)$.*

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This assertion is an analogue of the following theorem in the paper of Romanov [12] for Schrödinger operators on the half-line: if a bounded potential q on the half-line $x\geq 0$ such that $\operatorname{Im} q(x)\geq 0$, $\operatorname{Im} q(x)\to 0$ as $x\to\infty$, satisfies $\operatorname{Im} q\notin L^1$, then the dissipative operator l, $lu=-u''+qu$, corresponding to a selfadjoint boundary condition at zero has trivial absolutely continuous subspace.

Together with the latter theorem, the result of the present paper suggests that the existence of an absolutely continuous spectrum for dissipative ordinary differential operators is equivalent to having a scattering theory. Let us mention here that the abstract local nuclear scattering theory developed in [6], [7] shows that for any pair of operators B and B_0 such that B_0 is selfadjoint and $B-B_0$ is of (relative) trace class, the absolutely continuous parts of B and B_0 are quasi-similar by means of wave operators and their spectra coincide. In the situation under consideration this implies that the absolutely continuous spectrum of L coincides with that of the selfadjoint Dirac operator $\operatorname{Re} L$ when $\operatorname{Im} Q\in L^1$. This is to be compared with the situation of the selfadjoint theory, where the existence of modified wave operators for the Schrödinger operator in dimension 1 has only been established for specific classes of slowly decaying potentials q (see, e.g. [1]), while the absolutely continuous spectrum is known to coincide with the positive real line for any real $q\in L^2$ [2].

The strategy of the proof of the main result follows that in the paper [12] with one notable exception. We start with an abstract re-formulation of triviality of the absolutely continuous subspace of a dissipative operator, given in Proposition 1.3, and then obtain in Corollary 3.2 a convenient sufficient condition in the situation under consideration which is expressed in terms of the asymptotics of solutions of the equation $l_{Q^*}y=ky$ with real k. Specifically, it says that the absolutely continuous subspace of the completely non-selfadjoint part of L is trivial if $\int^\infty(\operatorname{Im} q_1|y_1|^2+\operatorname{Im} q_2|y_2|^2)\,dx$ is infinite for a.e. real k. Here $y=(y_1,y_2)^T$ is the solution of the Cauchy problem for $l_{Q^*}y=ky$ with the initial data $y(0)=(1,0)^T$ (or any other k-independent non-zero initial data satisfying a selfadjoint boundary condition). In the case of a Schrödinger operator [12], the next step in the argument used spectral averaging [4] for the selfadjoint problem with the potential $\operatorname{Re} q$. The spectral averaging method, as developed by Last–Simon, seems to be only applicable to operators semi-bounded below, while a selfadjoint operator corresponding to a differential expression of the form (1) is never semi-bounded. To circumvent this difficulty, we first establish the result for potentials with "spread" $\operatorname{Im} Q$ (see Proposition 3.4 and the comments after it), and then show that an arbitrary potential can be represented as a sum of a potential with spread $\operatorname{Im} Q$ and an L^1-potential. An appropriate variant of the trace class method, which we develop in the abstract part of the paper in Lemma 1.4, allows us to handle the L^1-perturbation. This argument is similar to that used in [12] for proving a discrete variant of the main result.

Throughout the paper $\mathbf{S}^1$ stands for the trace class of operators. Given an operator B we write B' for the completely non-selfadjoint part of B. For any vector v let v^T stand for its transpose.

1. Preliminaries: Absolutely continuous subspace

Let H be a Hilbert space and L be a maximal dissipative operator in H of the form $L=A+iV$ where $A=A^*$ and $V\geq 0$ is bounded. We shall assume that([1]) $\sigma_{\mathrm{ess}}(L)\subset\mathbf{R}$. Define H_0 to be the maximal reducing subspace of L on which it induces a selfadjoint operator, which will be denoted by L_0. Let L' be the completely non-selfadjoint part of L, $L'=L|_{H'}$, $H'=H\ominus H_0$ [6]. Define the Hardy classes of vector functions $\mathbf{H}^2_\pm$ to be the collections of H-valued analytic functions f on $\mathbf{C}_\pm=\{z\in\mathbf{C}\mid \pm\operatorname{Im} z>0\}$, which satisfy $\sup_{\varepsilon>0}\int_{\mathbf{R}}\|f(k\pm i\varepsilon)\|^2\,dk<\infty$, respectively.

Definition 1.1. The *absolutely continuous* (a.c.) *subspace*, $H_{\mathrm{ac}}(L')$, of the completely non-selfadjoint operator L' is defined as follows [6], [13]

(3)
$$H_{\mathrm{ac}}(L')=\operatorname{clos}\{u\in H':(L-z)^{-1}u \text{ is analytic in } \mathbf{C}_+ \text{ and } V^{1/2}(L-z)^{-1}u\in\mathbf{H}^2_+\}.$$

By the a.c. subspace of the operator L we mean the subspace

$$H_{\mathrm{ac}}(L)=H_{\mathrm{ac}}(L')\oplus H_{\mathrm{ac}}(L_0),$$

where $H_{\mathrm{ac}}(L_0)$ is the a.c. subspace of the selfadjoint operator L_0 defined in the standard way.

The a.c. subspace of L is known [6] to be regular invariant, that is, $\overline{(L-z)^{-1}H_{\mathrm{ac}}}=H_{\mathrm{ac}}$ for all $z\in\rho(L)$.

Various motivations of this definition and its relation to scattering theory are given in [6], [7], [8], [9] and [13]. We only mention here([2]) the following "weak" characterization of the subspace $H_{\mathrm{ac}}(L)$.

Theorem 1.2. ([11])

$$H_{\mathrm{ac}}(L)=\operatorname{clos}\left\{u\in H:\ \begin{array}{l}(L-z)^{-1}u \textit{ is analytic in } \mathbf{C}_+,\\ \langle (L-z)^{-1}u,v\rangle|_{\mathbf{C}_\pm}\in\mathbf{H}^2_\pm \textit{ for all } v\in H\end{array}\right\}.$$

([1]) By $\sigma_{\mathrm{ess}}(L)$ we mean the union of the set of all non-isolated points of $\sigma(L)$ and the set of all isolated points of $\sigma(L)$ of infinite multiplicity.

([2]) Theorem 1.2 is not used in this paper.

This theorem shows, in particular, that (3) can be considered as a generalization of the definition of the a.c. subspace in the selfadjoint theory.

The triviality of the subspace $H_{\mathrm{ac}}(L')$ in the example studied in this paper will be established on the basis of the following criterion which is implicit in [9].

Proposition 1.3. *$H_{\mathrm{ac}}(L')=\{0\}$ if and only if for a.e. $k\in\mathbf{R}$ we have ($z=k+i\varepsilon$)*

$$D(z)\equiv\sqrt{\varepsilon}(L^*-z)^{-1}\sqrt{V}\xrightarrow{s}0, \tag{4}$$

as $\varepsilon\searrow 0$.

Different elementary proofs of the “if” part of the criterion can be found in [12] and [11]. The “only if” part is not used in this paper.

Remark. ([9]) The function $D(z)$ satisfies $\|D(z)\|\leq\frac{1}{2}$ for all $z\in\mathbf{C}_+$.

The proof of this fact is by direct calculation. A similar calculation is contained in (6) in Lemma 1.4 below.

As is well known, if two selfadjoint operators $\mathcal{L}_{1,2}$ satisfy $\mathcal{L}_1-\mathcal{L}_2\in\mathbf{S}^1$, then their a.c. spectra coincide. In the non-selfadjoint case, the question whether $L_1-L_2\in\mathbf{S}^1$ for a pair of dissipative operators L_1 and L_2 implies $\sigma(L_1|_{H_{\mathrm{ac}}(L_1)})=\sigma(L_2|_{H_{\mathrm{ac}}(L_2)})$ appears to be open if $\operatorname{Im} L_{1,2}$ are not of $\mathbf{S}^1$ separately. We shall need a result of this type in a special situation.

Lemma 1.4. *Let L and $\tilde{L}$ be two dissipative operators such that*

$$(L-z)^{-1}-(\tilde{L}-z)^{-1}\in\mathbf{S}^1,\quad z\in\rho(L)\cap\rho(\tilde{L}). \tag{5}$$

Write $\tilde{L}-L=i\Gamma$, and assume that $\Gamma\geq 0$ and that there exists a bounded operator $\Pi\geq 0$ such that $\Gamma=\widetilde{V}\Pi$, $\widetilde{V}=\operatorname{Im}\tilde{L}$. If $H_{\mathrm{ac}}(\tilde{L}')=\{0\}$, then $H_{\mathrm{ac}}(L')=\{0\}$.

This lemma is a variant of Lemma 2.3 from [12], where Γ was assumed to be trace class.

Proof. We shall actually prove that if $\widetilde{D}(k+i\varepsilon)\equiv\sqrt{\varepsilon}(\tilde{L}^*-z)^{-1}\sqrt{\widetilde{V}}\xrightarrow{s}0$ as $\varepsilon\to 0$ for a.e. $k\in\mathbf{R}$, then $D(k+i\varepsilon)\xrightarrow{s}0$ for a.e. $k\in\mathbf{R}$ as well. Taking into account the assumed factorization of Γ we find that

$$\sqrt{\varepsilon}(\tilde{L}^*-z)^{-1}\sqrt{\Gamma}\xrightarrow{s}0$$

for a.e. $k\in\mathbf{R}$. Let us show that $\sqrt{\varepsilon}(L^*-z)^{-1}\sqrt{\Gamma}\xrightarrow{s}0$ for a.e. k. We have,

$$\sqrt{\varepsilon}(L^*-z)^{-1}\sqrt{\Gamma}G(z)=\sqrt{\varepsilon}(\tilde{L}^*-z)^{-1}\sqrt{\Gamma},$$

where

$$G(z)=I+i\sqrt{\Gamma}(\tilde{L}^*-z)^{-1}\sqrt{\Gamma}.$$

Let $V=\operatorname{Im} L$. Then the following calculation shows that $G(z)$ is a contraction for all $z\in\mathbf{C}_+$,

$$\begin{aligned} I-G^*(z)G(z) &= I-\big(I-i\sqrt{\Gamma}(\tilde{L}-\bar{z})^{-1}\sqrt{\Gamma}\big)\big(I+i\sqrt{\Gamma}(\tilde{L}^*-z)^{-1}\sqrt{\Gamma}\big) \\ &= i\sqrt{\Gamma}(\tilde{L}-\bar{z})^{-1}((\tilde{L}^*-z)-(\tilde{L}-\bar{z})+i\Gamma)(\tilde{L}^*-z)^{-1}\sqrt{\Gamma} \\ &= \sqrt{\Gamma}(\tilde{L}-\bar{z})^{-1}(2\varepsilon+2V+\Gamma)(\tilde{L}^*-z)^{-1}\sqrt{\Gamma}. \end{aligned} \tag{6}$$

Since the right-hand side is obviously non-negative, we conclude that $G(z)$ is a contractive analytic operator-function in $\mathbf{C}_+$. We shall now show that this function admits a scalar multiple, that is, there exists a scalar contractive analytic function, $g\not\equiv 0$, such that $G(z)\Omega(z)=g(z)I$ for a certain bounded analytic function Ω. Let $\{X_N\}_{N=1}^{\infty}$ be an arbitrary family of finite-dimensional subspaces in H such that $X_{N-1}\subset X_N$ and $\bigvee_N X_N=H$. We shall use the subscript N with operators to denote their truncation to X_N, $B_N:=P_NB|_{X_N}$, where P_N is the orthogonal projection on X_N. Define $g_N(z)=\det G_N(z)$. Our aim is to show that there exists a subsequence of N such that $g_N(z)$ converges when $N\to\infty$ along the subsequence for all $z\in\mathbf{C}_+$ and $g(z)=\lim g_N(z)$ is the scalar multiple. Notice that the operator $G(z)$ is boundedly invertible for $\operatorname{Im} z$ large enough, for $G(z)\to I$ in the operator norm when $\operatorname{Im} z\to+\infty$ since $\tilde{L}$ has a bounded imaginary part. Fix an arbitrary $z_0\in\mathbf{C}_+$ such that $\|I-G(z_0)\|\le\frac{1}{2}$, so that $T=G(z_0)$ is boundedly invertible. Obviously, $|\det T_N|\le 1$ since T is a contraction, and T_N is boundedly invertible for all N. Fix arbitrarily a subsequence of N such that $\det T_N$ converges. To keep notation to a minimum, we do not introduce the corresponding subscript and will always assume that $N\to\infty$ along this subsequence. Then, $g(z)=\lim g_N(z)$ is the scalar multiple. Indeed, the limit exists for all $z\in\mathbf{C}_+$: we have

$$g_N=\det G_N=\det T_N\cdot\det\,(I_N+T_N^{-1}(G_N-T_N));$$

then,

$$\begin{aligned} G(z)-T &= i\sqrt{\Gamma}((\tilde{L}^*-z)^{-1}-(\tilde{L}^*-z_0)^{-1})\sqrt{\Gamma} \\ &= i(z-z_0)\sqrt{\Gamma}(\tilde{L}^*-z)^{-1}\cdot(\tilde{L}^*-z_0)^{-1}\sqrt{\Gamma}\in\mathbf{S}^1 \end{aligned}$$

since $\sqrt{\Gamma}(\tilde{L}^*-z)^{-1}$ is Hilbert–Schmidt, which is easily seen to be equivalent to the assumption (5); hence, $G_N-T_N\to G-T$ in the $\mathbf{S}^1$-norm, and therefore

$$\det\,(I_N+T_N^{-1}(G_N-T_N))\to\det\,(I+T^{-1}(G-T)).$$

Since $\det T_N$ converges by construction, we obtain that the limit $g(z)$ exists. Obviously, the limit is a bounded function of z in $\mathbf{C}_+$ because $|g_N(z)|\le 1$ for all N. An application of the Montel theorem shows that $g(z)$ is an analytic function.

Let us check that $g\not\equiv 0$. If $\det(I+T^{-1}(G(z)-T))=0$, then $\ker G(z)$ is non-trivial which is only possible for a discrete set of values of $z\in\mathbf{C}_+$ since $G(z)$ is a compact operator. It remains to show that $\lim\det T_N\neq 0$. Indeed, consider $|\det T_N|^2=\det(T_N^*T_N)$. It follows from (6) that $I-T^*T\in\mathbf{S}^1$, and a calculation similar to (6) gives

$$I_N-T_N^*T_N=P_N\sqrt{\Gamma}(\tilde L-\bar z_0)^{-1}\big(2\varepsilon+2\widetilde V-\sqrt{\Gamma}P_N\sqrt{\Gamma}\big)(\tilde L^*-z_0)^{-1}\sqrt{\Gamma}|_{X_N}.$$

This shows that the right-hand side (extended to $X_N^{\perp}$ by zero) converges in the trace norm to $I-T^*T$. Therefore, $\lim\det(T_N^*T_N)=\det T^*T\neq 0$ since $\ker T$ is trivial by the choice of z_0.

Now, $\Omega(z)=g(z)G^{-1}(z)$ satisfies $\|\Omega(z)\|\le 1$ since $|g_N(z)|\|G_N^{-1}(z)\|\le 1$ for each N and all z such that $g_N(z)\neq 0$, and thus $g(z)$ is a scalar multiple.

It follows from the existence of a scalar multiple and the Fatou theorem that the strong boundary values, $G^{-1}(k)=s-\lim_{\varepsilon\searrow 0}G^{-1}(k+i\varepsilon)$, of the function G^{-1} exist for a.e. $k\in\mathbf{R}$. We infer that for a.e. $k\in\mathbf{R}$,

$$\sqrt{\varepsilon}(L^*-z)^{-1}\sqrt{\Gamma}=\sqrt{\varepsilon}(\tilde L^*-z)^{-1}\sqrt{\Gamma}G^{-1}(z)\xrightarrow{s}0,\ \text{as } \varepsilon\searrow 0.$$

One can now verify the condition (4) for the operator L. We have,

$$\sqrt{\varepsilon}(L^*-z)^{-1}\sqrt{V}=\sqrt{\varepsilon}(\tilde L^*-z)^{-1}\sqrt{V}-i\sqrt{\varepsilon}(L^*-z)^{-1}\sqrt{\Gamma}Q(z), \tag{7}$$

where $Q(z)=\sqrt{\Gamma}(\tilde L^*-z)^{-1}\sqrt{V}$ is a bounded analytic function in $\mathbf{C}_+$. The latter follows from a calculation similar to (6) giving that $I-G(z)G^*(z)$ equals the right-hand side of (6) with $(\tilde L-\bar z)^{-1}$ and $(\tilde L^*-z)^{-1}$ swapped, and therefore $Q(z)Q^*(z)\le I-G(z)G^*(z)\le I$. The first term on the right-hand side of (7) converges strongly to zero by the assumption, and an application of the Fatou theorem to the function $Q(z)$ shows that the second term converges strongly to zero as well. □

In the next section we establish triviality of the absolutely continuous subspace for an operator L by verifying the condition (4) for an operator $\tilde L$ obeying the assumptions of this lemma. Since the *proof* of the lemma consists of demonstrating that the condition (4) for the operator L is satisfied if it is satisfied for the operator $\tilde L$, the final result then does not depend on the “only if” part of the criterion of Proposition 1.3.

2. The operator corresponding to l_Q

Let $\Theta(\,\cdot\,,z)$ and $\Phi(\,\cdot\,,z)$ be the solutions of the initial value problems

$$l_{Q^*}\Theta = z\Theta, \quad \Theta(0,z) = \begin{pmatrix}1\\0\end{pmatrix},$$
$$l_{Q^*}\Phi = z\Phi, \quad \Phi(0,z) = \begin{pmatrix}0\\1\end{pmatrix}.$$

We also let for $h\in\mathbf{C}$,

$$\Psi_h(x,z) = \Theta(x,z) + h\Phi(x,z).$$

It is standard results that the functions $\Theta(x,z)$ and $\Phi(x,z)$ are entire functions of z for all x and are continuous in z uniformly in $x\in I$ for any compact interval I.

Lemma 2.1. *For each z in the upper half-plane $\mathbf{C}_+$, there exists a unique (up to scalar multiples) solution of the differential equation $l_{Q^*}\Psi = z\Psi$ which lies in $L^2(\mathbf{R}_+,\mathbf{C}^2)$, and which can be chosen to be of the form*

$$\Psi(x,z) = \Theta(x,z) + m(z)\Phi(x,z). \tag{8}$$

Proof. Multiplying the equation $l_{Q^*}y = zy$ by J we obtain an equation of the form $y' = Ay$ in which A has zero diagonal. The fundamental matrix of this equation therefore has a constant determinant; it follows that any two linearly independent solutions, say

$$u = \begin{pmatrix}u_1\\u_2\end{pmatrix} \text{ and } v = \begin{pmatrix}v_1\\v_2\end{pmatrix}$$

satisfy

$$W[u,v] \equiv u_1(x)v_2(x) - u_2(x)v_1(x) = \text{const.} \neq 0.$$

Integrating this identity in x and applying the Schwarz inequality shows that at most one of the solutions lies in $L^2(\mathbf{R}_+,\mathbf{C}^2)$. This is a version of a standard argument: see, e.g., Levitan and Sargsjan [5, Chapter 13, §7].

Let $L_{\min}$ be the closure of the operator defined by the differential expression l_Q on $C_0^\infty(\mathbf{R}_+,\mathbf{C}^2)$ in the space $L^2(\mathbf{R}_+,\mathbf{C}^2)$. Integration by parts shows that $L_{\min}$ is a dissipative operator. Since $L_{\min}$ is obviously not maximal dissipative, $\dim\ker(L_{\min}^* - zI)$ is either 1 or 2 for all $z\in\mathbf{C}_+$. By the argument of the previous paragraph, dimension 2 is impossible. Thus there exists exactly one solution Ψ of the equation $l_{Q^*}y = zy$ in L^2.

To show that Ψ can be chosen of the form (8) it remains to note that $\Phi(\,\cdot\,,z)\notin L^2(\mathbf{R}_+,\mathbf{C}^2)$ for all $z\in\mathbf{C}_+$: if it were, for some z, then $-z$ would be an eigenvalue in the lower half-plane of a dissipative operator corresponding to the differential expression $-l_{Q^*}$. $\square$

Definition 2.2. Let $h\in i\mathbf{R}$ be fixed. By L we denote the operator in the Hilbert space $H=L^2(\mathbf{R}_+,\mathbf{C}^2)$ given by the differential expression l_Q with domain

$$D(L)=\left\{y=\begin{pmatrix}y_1\\y_2\end{pmatrix}\in H\;\middle|\; y\in AC_{\mathrm{loc}}(\mathbf{R}_+),\ l_Q y\in H \text{ and } y_2(0)=hy_1(0)\right\}.$$

Using Lemma 2.1 it is easy to see that L is a maximal dissipative operator, and so $\sigma(L)\subset\overline{\mathbf{C}}_+$. As the following example shows, this operator is not completely non-selfadjoint in general.

Example. We construct a potential Q with $\operatorname{Im} Q\neq 0$, such that L has a real eigenvalue. The boundary condition will be $y_1(0)=0$, corresponding to $h=\infty$.

Let $X>0$ be fixed and let y be given on $[0,X]$ by

$$y(x)=\begin{pmatrix}0\\1\end{pmatrix},\quad 0\le x\le X.$$

Fix a real non-zero k and choose $q_2=k$ on $[0,X]$; let q_1 satisfy $\operatorname{Im} q_1\not\equiv 0$ and be otherwise arbitrary on $[0,X]$. Clearly y satisfies the differential equation $l_Q y=ky$ on $[0,X]$ together with the boundary condition.

For $x>X$, let $y_2(x)=1/\cosh(x-X)$ and $q_1(x)\equiv 0$. Define $y_1(x)$ for $x>X$ by

$$y_1(x)=\frac{iy_2'(x)}{k}=-\frac{i\sinh(x-X)}{k\cosh^2(x-X)}.$$

Obviously, y_1 and y_2 are continuous at $x=X$ and decay exponentially for large x; therefore $y\in D(L)$. The equation $iy_2'=(k-q_1)y_1$ is satisfied for all x by construction. If we now choose q_2 for $x>X$ so that the equation $iy_1'=(k-q_2)y_2$ holds, then y will be an eigenfunction of L. We thus let

$$q_2=k+k^{-1}\frac{y_2''}{y_2}.$$

The function q_2 is asymptotically a real constant at infinity and so Q satisfies all our hypotheses.

The following assertion shows that the selfadjoint part of L is irrelevant in proving Theorem 1.

Proposition 2.3. *If* $\operatorname{Im} Q\not\equiv 0$ *then the a.c. subspace of the operator* L_0*, the selfadjoint part of* L*, is trivial.*

Proof. Suppose that $H_{\rm ac}(L_0)\neq\{0\}$. By the abstract theory [6], H_0 is a reducing subspace of $\operatorname{Re} L$, and L_0 coincides with the restriction of $\operatorname{Re} L$ to it. Since $\operatorname{Re} L$ has simple spectrum, it follows that there exists a compact set Ω, $|\Omega|>0$, such that $\{0\}\neq\operatorname{Ran} P_\Omega\subset H_0$, where P_Ω is the spectral projection of the a.c. part of $\operatorname{Re} L$ for the set Ω. Let us denote with the superscript r the m-function and the Cauchy problem solution Ψ_h corresponding to the potential $\operatorname{Re} Q$. Without loss of generality, one can assume that $m^r(k)$ is bounded on Ω. It follows from the spectral theorem for the operator L_0 (see, e.g., [5, Chapter 11]) that any vector of the form $\int_\Omega \Psi_h^r(x,k)g(k)\operatorname{Re} m^r(k)\,dk$ with a bounded g is in $\operatorname{Ran} P_\Omega$. Arguing in the contrapositive form, assume that $\operatorname{Im} q_1\not\equiv 0$. Since $H_0\perp\operatorname{Ran} V$, then there exists an $X\in\mathbf{R}$ such that for any $f\in D(L_0)$ the first component $f_1(X)=0$, in particular, $\int_\Omega \Psi_{h,1}^r(X,k)g(k)\operatorname{Re} m_r(k)\,dk=0$ for any bounded g. Now, choosing g to be a properly normalized indicator (characteristic) function of the interval $(k_0-\varepsilon, k_0+\varepsilon)$, sending ε to 0 and taking into account the analyticity of Ψ, we infer that $\Psi_{h,1}^r(X,k_0)=0$ for any Lebesgue point of the set $\Omega\cap\{k:\operatorname{Re} m^r(k)\neq 0\}$. Since the set of such points has positive measure, this means that a selfadjoint operator generated by the differential expression $l_{\operatorname{Re} Q}$ on the interval $(0,X)$ by the same boundary condition at zero and the condition $y_1(X)=0$ has uncountably many eigenvalues – a contradiction implying that $\operatorname{Im} q_1\equiv 0$. In a similar way, we show that $\operatorname{Im} q_2$ vanishes. □

We shall require an expression for the action of the resolvent $(L^*-z)^{-1}$ on compactly supported vectors. Notice that $m(z)\neq h$ for any $z\in\mathbf{C}_+$, for otherwise $\Psi(x,z)$ would be an eigenfunction of L^*, and z would be an eigenvalue. Let $E=\operatorname{diag}(1,-1)$. Then a straightforward calculation taking into account that $\det(\Psi_h,\Psi)=m(z)-h$ shows that for any compactly supported $u\in H$

$$(L^*-z)^{-1}u= \tag{9}$$
$$\frac{i}{m(z)-h}\biggl(\Psi_h(x,z)\int_x^\infty \Psi^T(s,z)Eu(s)\,ds+\Psi(x,z)\int_0^x \Psi_h^T(s,z)Eu(s)\,ds\biggr).$$

Lemma 2.4. *The function $m(z)$ defined in Lemma 2.1 is analytic in $\mathbf{C}_+$. For all $z\in\mathbf{C}_+$ it satisfies $\operatorname{Re} m(z)<0$. Moreover, for all z with $\varepsilon=\operatorname{Im} z>0$,*

$$\begin{aligned}-\operatorname{Re} m(z)&=\varepsilon\int_0^\infty (|\Psi_1(x,z)|^2+|\Psi_2(x,z)|^2)\,dx\\ &\quad+\int_0^\infty (\operatorname{Im} q_1(x)|\Psi_1(x,z)|^2+\operatorname{Im} q_2(x)|\Psi_2(x,z)|^2)\,dx,\end{aligned}\tag{10}$$

and therefore

$$\limsup_{\varepsilon\searrow 0}\varepsilon\|\Psi(\,\cdot\,,k+i\varepsilon)\|_H^2\le -\operatorname{Re} m(k)$$

whenever the limit $m(k)=\lim_{\varepsilon\searrow 0} m(k+i\varepsilon)$ exists.

Proof. It follows from (9) that for any compactly supported $u,v\in H$,

$$F(z)\equiv\langle (L^*-z)^{-1}u,v\rangle=\frac{Am(z)+B}{m(z)-h},$$

where A and B are complex constants depending on u and v. Then, $B\neq -hA$ for a suitable choice of u and v, since otherwise the fact that $F(z)\to 0$ when $\operatorname{Im} z\to\infty$ would imply that $F(z)\equiv 0$ for any compactly supported $u,v\in H$, and therefore $(L^*-z)^{-1}=0$. We now infer that $m(z)$ is analytic in $\mathbf{C}_+$ from the analyticity of $F(z)$. Alternatively, the analyticity of m can be proved through the nesting circles analysis in the same way as in the case of a dissipative second order Schrödinger operator [14].

We shall now establish (10). The differential equations satisfied by $\Psi_1(x,k+i\varepsilon)$ and $\Psi_2(x,k+i\varepsilon)$ are

$$\begin{aligned} i\Psi_2'+(\operatorname{Re} q_1(x)-i\operatorname{Im} q_1(x)-k-i\varepsilon)\Psi_1&=0,\\ i\Psi_1'+(\operatorname{Re} q_2(x)-i\operatorname{Im} q_2(x)-k-i\varepsilon)\Psi_2&=0.\end{aligned}$$

Multiply the first equation by $\overline{\Psi}_1$ and the complex conjugate of the second equation by Ψ_2, subtract and obtain

$$\begin{aligned}-i(\operatorname{Im} q_1+\varepsilon)|\Psi_1|^2-i(\operatorname{Im} q_2+\varepsilon)|\Psi_2|^2+(\operatorname{Re} q_1-k)|\Psi_1|^2&\\ -(\operatorname{Re} q_2-k)|\Psi_2|^2+i(\overline{\Psi}_1\Psi_2)'&=0.\end{aligned}$$

Now on taking imaginary parts and integrating over $[0,X]$,

$$\operatorname{Re}\big(\overline{\Psi_1(X)}\Psi_2(X)\big)=\int_0^X ((\operatorname{Im} q_1+\varepsilon)|\Psi_1|^2+(\operatorname{Im} q_2+\varepsilon)|\Psi_2|^2)\,dx+\operatorname{Re} m(k+i\varepsilon).$$

This shows that the left-hand side has a limit, finite or infinite, when $X\to\infty$. This limit must be zero, because $\Psi\in L^2$, and therefore, $\overline{\Psi}_1\Psi_2\in L^1$. The result follows. □

In order to use Proposition 1.3 we shall need an asymptotic of $(L^*-z)^{-1}u$ when $\varepsilon\searrow 0$ for compactly supported u, given by the following lemma.

Lemma 2.5. *For any $u\in L^2(\mathbf{R}_+,\mathbf{C}^2)$ having compact support and $z=k+i\varepsilon$, $\varepsilon>0$,*

$$(11)\qquad \begin{aligned}(L^*-z)^{-1}u&=\beta_z[u]\Psi(x,z)+r(x,z),\\ \beta_z[u]&=\frac{i}{m(z)-h}\int_0^\infty \Psi_h^T(s,z)Eu(s)\,ds,\end{aligned}$$

where $r(x,z)$ satisfies $\limsup_{\varepsilon\searrow 0}\|r(\,\cdot\,,z)\|_H<\infty$ for all k such that the finite boundary value $m(k)=\lim_{\varepsilon\searrow 0} m(k+i\varepsilon)$ exists and $m(k)\neq h$.

Proof. Let $X<\infty$ be such that u is supported on $[0,X]$, and consider the expression (9) for $(L^*-z)^{-1}u$. Notice that the first term in the brackets vanishes for $x>X$, and therefore $r=(L^*-z)^{-1}u-\beta_z\Psi$ is supported on $[0,X]$ for any $z\in\mathbf{C}_+$. Then, the standard perturbation theory for initial value problems implies that $\Psi_h(x,z)$ and $\Psi(x,z)$ converge to $\Psi_h(x,k)$ and $\Psi(x,k)$ when $\varepsilon\searrow 0$ uniformly in $x<X$ for all k such that the finite boundary value $m(k)$ exists. Combining these facts, we obtain that $r(x,z)$ converges when $\varepsilon\searrow 0$ uniformly in $x<X$, provided that the common factor in the right-hand side of (9) converges to a finite limit. The result follows. □

3. Proof of the main result

Define the non-negative functions r_1 and r_2 by

$$r_1(x)^2 := \operatorname{Im} q_1(x) \quad \text{and} \quad r_2(x)^2 := \operatorname{Im} q_2(x).$$

Then the operator $(\operatorname{Im} L)^{1/2}$ is given by multiplication by the matrix

$$R(x)=\begin{pmatrix} r_1(x) & 0 \\ 0 & r_2(x)\end{pmatrix}. \tag{12}$$

Lemma 3.1. *Let $k\in\mathbf{R}$ be such that $m(k)$ exists finitely and is not equal to h. If $R\Psi_h(\cdot,k)\notin L^2(\mathbf{R}_+,\mathbf{C}^2)$, then $D(k+i\varepsilon)\xrightarrow{s}0$ as $\varepsilon\searrow 0$.*

Proof. Observe that the linear set

$$\mathcal{D}=\left\{u\in L^2(\mathbf{R}_+,\mathbf{C}^2)\;\middle|\; \begin{array}{l} u \text{ is compactly supported and} \\ \int_0^\infty \Psi_h^T(s,k)ER(s)u(s)\,ds=0 \end{array}\right\}$$

is dense in $L^2(\mathbf{R}_+,\mathbf{C}^2)$ if $R(\cdot)\Psi_h(\cdot,k)\notin L^2(\mathbf{R}_+,\mathbf{C}^2)$. Since the function $D(z)$ is bounded in $\mathbf{C}_+$ by the remark after Proposition 1.3, it suffices to prove

$$\lim_{\varepsilon\to 0} D(k+i\varepsilon)u=0 \tag{13}$$

for $u\in\mathcal{D}$. From Lemma 2.5, we have for any $u\in\mathcal{D}$, with $z=k+i\varepsilon$,

$$D(k+i\varepsilon)u=\varepsilon^{1/2}\beta_z[Ru]\Psi(\cdot,z)+o(1), \quad \text{as } \varepsilon\searrow 0,$$

where the o-symbol refers to the L^2-norm. Now, Lemma 2.4 shows that $\varepsilon^{1/2}\|\Psi(k+i\varepsilon)\|_H$ is bounded above when $\varepsilon\searrow 0$. Since $m(k)\neq h$, one can pass to the limit $\varepsilon\searrow 0$ in the expression (11) for β_z to find that $\beta_{k+i\varepsilon}$ converges to a multiple of $\int_0^\infty \Psi_h^T(s,k)ER(s)u(s)\,ds$ which is zero because $u\in\mathcal{D}$. Combining these, we obtain (13). □

Note that the complement of the set of all $k\in\mathbf{R}$ for which $m(k)$ exists and is not equal to h is a set of (Lebesgue) measure 0, by the uniqueness theorem for Nevanlinna functions. Taking into account the criterion of Proposition 1.3, we arrive at the following result.

Corollary 3.2. *If* $R\Psi_h(\,\cdot\,,k)\notin L^2(\mathbf{R}_+,\mathbf{C}^2)$ *for Lebesgue almost all* $k\in\mathbf{R}$, *then*(3) $H_{\mathrm{ac}}(L')=\{0\}$.

Thus, our main result will be established if we prove that

$$R(\,\cdot\,)\Psi_h(\,\cdot\,,k)\notin L^2(\mathbf{R}_+,\mathbf{C}^2) \text{ for a.a. } k\in\mathbf{R}$$

if $\operatorname{Im} Q\notin L^1$. Before doing this, we would like to indicate a simple argument showing that H_{ac} is trivial when $\operatorname{Im} Q\notin L^1$ for a class of potentials Q. Notice first that if $m(k)$ exists finitely, then the solution $\Psi(x,k)=\lim_{\varepsilon\searrow 0}\Psi(x,k+i\varepsilon)$ to $l_{Q^*}\Psi=k\Psi$ exists. As is shown in the asymptotic theory of linear differential systems [3], if the potential Q decays at infinity in a sufficiently regular way the asymptotics of the solutions of the equation $l_{Q^*}\Psi=k\Psi$ can be calculated explicitly. In particular, an application of the general theory in the situation under consideration gives the following.

Proposition 3.3. ([14, Theorem 1.8.3]) *Let* $\operatorname{Re} Q=0$, *and suppose that* $Q\to 0$, $Q'\in L^1$ *and* $Q\in L^2$. *Then for any* $k\in\mathbf{R}$ *there exist two solutions,* $\Psi_\pm$, *of the equation* $l_{Q^*}\Psi=k\Psi$ *with the asymptotics of the form* (*the Wentzel–Kramers–Brillouin (WKB) asymptotic*)

$$\Psi_\pm\sim\exp\left[\pm i\left(kx+\frac{1}{2}\int^x \operatorname{tr}Q(\xi)\,d\xi\right)\right]\begin{pmatrix}1\\ \mp 1\end{pmatrix},\quad \text{as } x\to\infty. \tag{14}$$

Assume that the asymptotics (14) holds for a.e. $k\in\mathbf{R}$, and that $\operatorname{Im} Q\notin L^1$. Then, obviously, the solution Ψ_- is growing, $|\Psi_-|\sim\exp\left(\frac{1}{2}\int^x \operatorname{tr}(\operatorname{Im} Q)\,d\xi\right)$, while Ψ_+ decays. Notice that if the asymptotics holds for a given k, and a solution v to $l_{Q^*}\Psi=k\Psi$ satisfies $Rv\in L^2$, then v must be a multiple of Ψ_+, for

$$\begin{aligned}\int^\infty \|R\Psi_-\|^2\,dx &\sim \int^\infty \operatorname{tr}(\operatorname{Im} Q(x))\exp\left(C\int^x \operatorname{tr}(\operatorname{Im} Q(\xi))\,d\xi\right)dx\\ &=\exp\left(C\int^\infty \operatorname{tr}(\operatorname{Im} Q(x))\,dx\right)=\infty.\end{aligned}$$

It follows that if (14) holds for a $k\in\mathbf{R}$ such that $m(k)$ exists and $R\Psi_h(\,\cdot\,,k)\in L^2$, then the solutions $\Psi_h(\,\cdot\,,k)$ and $\Psi(\,\cdot\,,k)$ must coincide. This is, however, only possible on

(3) Recall that L' denotes the completely non-selfadjoint part of the operator L.

the set of k such that $m(k)=h$, which has zero measure. Therefore $R\Psi_h(\,\cdot\,,k)\notin L^2$ for a.e. k. Applying Corollary 3.2, we thus obtain the main result for potentials Q such that the asymptotics (14) holds for a.e. $k\in\mathbf{R}$. Notice that although the conditions of Proposition 3.3 can be sharpened, the WKB asymptotics cannot be justified for generic potentials, and the argument above has nothing to do with the actual proof of Theorem 1. Moreover, even in the case of Schrödinger operators it is well known [10] that the WKB asymptotic may fail for a.e. positive value of the spectral parameter.

We shall establish Theorem 1 by successively reducing it to a partial case which we now proceed to prove.

Proposition 3.4. *If $r_1r_2\notin L^1(\mathbf{R}_+)$ then $R\Psi_h(\,\cdot\,,k)\notin L^2$ for a.e. $k\in\mathbf{R}$.*

Proof. We prove the result in the contrapositive form. Assume that $R\Psi_h(\,\cdot\,,k)\in L^2$ for a certain real k such that $m(k)\neq h$. Multiply the Wronskian identity $\det(\Psi_h(x,k),\Psi(x,k))=m(k)-h\neq 0$ by r_1r_2 and integrate in x to obtain

$$\text{const.}\int^N r_1(x)r_2(x)\,dx\le\int^N|r_1\Psi_{h,1}|\,|r_2\Psi_2|\,dx+\int^N|r_2\Psi_{h,2}|\,|r_1\Psi_1|\,dx.$$

Now $r_1\Psi_{h,1}\in L^2(\mathbf{R}_+)$ and $r_2\Psi_{h,2}\in L^2(\mathbf{R}_+)$ by the choice of k, while $r_1\Psi_1\in L^2(\mathbf{R}_+)$ and $r_2\Psi_2\in L^2(\mathbf{R}_+)$ by Lemma 2.4. By the Schwarz inequality it follows that $r_1r_2\in L^1(\mathbf{R}_+)$. $\square$

In particular, this proposition shows that if $r_1r_2\notin L^1(\mathbf{R}_+)$, then $H_{\mathrm{ac}}(L')=\{0\}$ in view of Corollary 3.2.

We would now like to reduce the problem under consideration to a problem where only one of the entries of $\operatorname{Im} Q$ is non-zero. We start with a Gronwall-type lemma comparing the solutions Ψ and Ψ_h corresponding to different $\operatorname{Im} Q$ at large x.

Lemma 3.5. *Let $k\in\mathbf{R}$ be such that $m(k)$ exists finitely, and $m(k)\neq h$. Let the function*

$$\widehat{Q}(x)=\begin{pmatrix}\hat q_1(x) & 0\\ 0 & \hat q_2(x)\end{pmatrix}$$

satisfy $0\le\operatorname{Im}\widehat{Q}(x)\le\operatorname{Im} Q(x)$ and $\operatorname{Re}\widehat{Q}(x)=\operatorname{Re} Q(x)$ for a.e. x. If $R(\,\cdot\,)\Psi_h(\,\cdot\,,k)\in L^2(\mathbf{R}_+,\mathbf{C}^2)$, then the equation $l_{\widehat{Q}^*}y=ky$ has solutions $\widehat{\Psi}^{(1)}$ and $\widehat{\Psi}^{(2)}$ such that

$$\begin{aligned}\widehat{\Psi}^{(1)}(x,k)&=\Psi_h(x,k)+o(|\Psi_h(x,k)|+|\Psi(x,k)|),\quad &&\text{as } x\to\infty,\\ \widehat{\Psi}^{(2)}(x,k)&=\Psi(x,k)+o(|\Psi_h(x,k)|+|\Psi(x,k)|),\quad &&\text{as } x\to\infty.\end{aligned}\tag{15}$$

Proof. Let the function $B(x)\geq 0$ be defined by $B^2=\operatorname{Im}(Q-\widehat{Q})$, and Y denote the 2×2-matrix (Ψ_h, Ψ). Write $(\widehat{\Psi}^{(1)}, \widehat{\Psi}^{(2)})=YZ$. Elementary manipulations show that the matrix Z must satisfy the differential equation

$$JYZ' = -iB^2 YZ. \tag{16}$$

As the argument from the beginning of the proof of Lemma 2.1 shows([4]), the determinant of the matrix Y is constant in x, $\det Y=m(k)-h\neq 0$. Now, let $J_0=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. A straightforward calculation shows that

$$Y^T J_0 Y = (\det Y) J_0.$$

Thus, $(JY)^{-1}=J_0^{-1}Y^T J_0 J^{-1}/\det Y$ and hence from (16), together with the fact that $J_0J^{-1}=iE=i\operatorname{diag}(1,-1)$,

$$Z' = -\frac{i}{\det Y} J_0^{-1} Y^T J_0 J^{-1} B^2 YZ = -\frac{1}{\det Y} J_0 (BY)^T E(BY)Z. \tag{17}$$

It follows from Lemma 2.4 that

$$\limsup_{\varepsilon\searrow 0} \int_0^\infty \|R(x)\Psi(x,z)\|^2\,dx$$

is finite when $m(k)$ exists finitely. Hence, by the Fatou lemma $R\Psi(\,\cdot\,,k)\in L^2(\mathbf{R}_+,\mathbf{C}^2)$. Together with the hypothesis of the current lemma this implies that $RY\in L^2$, and, moreover, $BY\in L^2$, since $B(x)\leq R(x)$ a.e. Thus, $(BY)^T E(BY)$ is summable, and by the Levinson theorem (see, e.g., [3, Theorem 1.3]) it follows that (17) possesses a solution Z satisfying

$$Z(x) = I + o(1), \quad \text{as } x\to\infty, \tag{18}$$

and (15) is proved. □

Let $\widehat{R}=(\operatorname{Im}\widehat{Q})^{1/2}$. The asymptotic (15) shows, in particular, that in the situation of the lemma, $\widehat{R}\widehat{\Psi}\in L^2$ for any solution $\widehat{\Psi}$ to $l_{\widehat{Q}^*}y=ky$. Applying Corollary 3.2, we now obtain the following assertion.

Corollary 3.6. *Let L_j, $j=1,2$, denote the operators defined by the same boundary conditions as L but given by*

$$L_j y = l_{Q_j} y,$$

([4]) This is the first time that we in an essential way use the fact that we deal with diagonal $\operatorname{Im} Q$.

where

$$Q_1(x)=\begin{pmatrix} q_1(x) & 0 \\ 0 & \operatorname{Re} q_2(x)\end{pmatrix} \quad \textit{and} \quad Q_2(x)=\begin{pmatrix} \operatorname{Re} q_1(x) & 0 \\ 0 & q_2(x)\end{pmatrix}.$$

Let $R_j=(\operatorname{Im} Q_j)^{1/2}$ *and* $\Psi_{h,j}(x,k)$ *be the solutions to* $l_{Q_j^*}y=ky$ *with the Cauchy data* $y(0)=(1,h)^T$. *If at least one of the operators* L_j *satisfies the condition*

$$R_j\Psi_{h,j}(\,\cdot\,,k)\notin L^2 \textit{ for a.e. } k\in\mathbf{R}$$

(*and so the corresponding subspace* $H_{\mathrm{ac}}(L'_j)$ *is trivial*), *then* $H_{\mathrm{ac}}(L')$ *is trivial.*

Proof of Theorem 1. In view of Corollary 3.6 it is sufficient to show that $R\Psi_h(\,\cdot\,,k)\notin L^2$ for a.e. $k\in\mathbf{R}$ for the operator L corresponding to a potential Q such that $\operatorname{Im} q_1\equiv 0$ and $\operatorname{Im} q_2\notin L^1$. Since $\operatorname{Im} q_2\notin L^1$ we have $r_2\notin L^2$ and so by the uniform boundedness principle there exists a function $d\in L^2$ such that $r_2 d\notin L^1$. Obviously we can choose d to be non-negative. Let L_d denote the operator with the potential

$$Q_d=\begin{pmatrix} \operatorname{Re} q_1+id^2 & 0 \\ 0 & q_2(x)\end{pmatrix},$$

and let $R_d=(\operatorname{Im} Q_d)^{1/2}$, and $\Psi_h^d(x,k)$ be the solution of the differential equation $l_{Q_d^*}\Psi_h=z\Psi_h$ with the initial condition

$$\Psi_h^d(0,z)=\begin{pmatrix}1\\ h\end{pmatrix}.$$

Since $r_2 d\notin L^1$, we have $R_d\Psi_h^d(\,\cdot\,,k)\notin L^2$ for a.e. $k\in\mathbf{R}$ by Proposition 3.4, and therefore $H_{\mathrm{ac}}(L'_d)=\{0\}$. We now wish to apply Lemma 1.4 with $\tilde{L}=L_d$ so that Γ is given by multiplication by $\operatorname{diag}(d^2,0)$. We only have to check that $(L-z)^{-1}-(L_d-z)^{-1}\in \mathbf{S}^1$, since all the other assumptions of the lemma are satisfied trivially. Let A_0 be the operator corresponding to the potential $Q\equiv 0$. The fact that $d\in L^2$ and the assumption that $\operatorname{Re} Q$ is bounded imply that, in the notation of Lemma 1.4, the operator $\sqrt{\Gamma}(A_0-z)^{-1}$ is Hilbert–Schmidt for any non-real z since the integral matrix kernel $K(x,s)$ of this operator satisfies

$$\|K(x,s)\|_{\mathrm{Mat}_2(\mathbf{C})}\le C d(x)\exp(-\varepsilon|x-s|), \quad \varepsilon=|\operatorname{Im} z|>0,$$

for all x and s, where C is a constant depending on z only. Applying the resolvent identity repeatedly, we now have (for any non-real $z\in\rho(L_d)\cap\rho(L)$)

$$(L-z)^{-1}-(L_d-z)^{-1}=\Xi_1(z)(A_0-z)^{-1}\sqrt{\Gamma}\cdot\sqrt{\Gamma}(A_0-z)^{-1}\Xi_2(z),$$

where $\Xi_{1,2}(z)$ are bounded operators. This formula shows that its left-hand side is of the trace class. The result follows. □

Apparently, Theorem 1 cannot be obtained in this way without a condition on the behavior of $\operatorname{Re} Q$ at infinity, which guarantees that $(\operatorname{Re} L-z)^{-1}-(A_0-z)^{-1}$ is a bounded operator and makes it possible to apply the resolvent identity to show that the difference of the resolvents of L and L_d is trace class.

Concluding remarks

Notice that the general matrix differential operator L corresponding to the differential expression

$$l_V := J\frac{d}{dx}+V(x)$$

with the same boundary conditions as above and arbitrary bounded matrix potential

$$V=\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}$$

is unitarily equivalent to an operator with a diagonal potential Q if $v_{12}=v_{21}$ and is real. The equivalence is given by a unitary gauge transformation U (see [5, pp. 48–49]) defined by

$$Uy=\exp\left(i\int_0^x v_{12}(t)\,dt\right)y.$$

Then $U^*LU=L_Q$, where L_Q is the operator corresponding to the differential expression l_Q with

$$Q(x)=\begin{pmatrix} v_{11}(x) & 0 \\ 0 & v_{22}(x) \end{pmatrix}.$$

Since the operator U transforms the absolutely continuous subspaces of L_Q into that of L, we have $H_{\mathrm{ac}}(L)=UH_{\mathrm{ac}}(L_Q)$.

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