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Spectral Analysis of the One-Speed Transport Operator and the Functional Model*

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Introduction

The present paper deals with the spectral analysis of the non-self-adjoint one-speed transport operator acting in the Hilbert space L^2 of distribution functions. For various geometric situations, this operator is known to have a rich essential spectrum [1]. Therefore, it seems reasonable to attempt to apply the functional model [2, 3, 4] to the spectral analysis of it. We shall study the particle transport operator for a slab of multiplying medium with isotropic scattering surrounded by a perfect absorber. The phase space of this problem is $\Gamma = \mathbb{R} \times \Omega$, $\Omega \equiv [-1,1]$. For a particle at a point $(x,\mu) \in \Gamma$, the number $x \in \mathbb{R}$ is the position and $\mu \in \Omega$ is the cosine of the angle between the particle velocity and the coordinate axis. The absorption of particles and the production of secondaries are described by the total cross-section $\sigma > 0$, the mean number of secondaries per collision $c\colon \mathbb{R} \to \mathbb{R}_+$, and the collision operator $K \in \mathbf{B}L^2(\Omega)$. It is assumed that $c \in L^\infty(\mathbb{R})$.

Let us describe the results of the paper. It was shown in [5] that the spectrum of the transport operator for the case in which c is proportional to the indicator function of an interval consists of finitely many eigenvalues lying on the imaginary axis and the essential spectrum that fills the real axis. We prove the same result for arbitrary compactly supported $c \in L^\infty(\mathbb{R})$ and give an estimate of the Birman–Schwinger type for the dimension of the subspace corresponding to the discrete spectrum. Our derivation of the spectrum essentially follows [5] but is somewhat simplified and uses modern terminology. Next, we show that the essential spectrum of the transport operator is absolutely continuous. The corresponding component T_{ess} of the operator is similar to a self-adjoint operator for the case in which $c \notin \mathscr{E}$ for a certain singular set $\mathscr{E} \subset L^\infty(\mathbb{R})$. If $c \in \mathscr{E}$, then the transport operator is shown to have a unique point of spectral singularity at 0. In this case, the component T_{ess} is similar to the orthogonal sum of a self-adjoint operator and an operator with spectrum of finite multiplicity $\mathscr{M}$, which is calculated in terms of c. For the spectral component of the transport operator corresponding to a neighborhood of the spectral singularity, we also give an estimate of the angle between the corresponding invariant subspaces. To derive these results, we single out some invariant subspaces of the model operator using the spectral decomposition of the operator $\Delta = 1 - S^*S$, where S is the characteristic function. The estimate of the angle between these subspaces is reduced to the estimate of the function S^{-1} on the real axis.

We point out that the functional model is not involved in the statement of results about the transport operator and is only used as a tool for their derivation. We also note that the appearance of the spectral singularity in our problem is completely nonpathological. Namely, it turns out that for any positive compactly supported $c \in L^\infty$, the function κc belongs to $\mathscr{E}$ for some values of the constant κ. This distinguishes the transport operator from another example of an operator with spectral singularities known in mathematical physics, namely, the Schrödinger operator with complex potential [6, 7], where the spectral singularities appear *ad hoc* for specially constructed potentials.

For the reader's convenience, in Sec. 1 we give a brief description of the functional model of a dissipative operator. The model is used in the symmetric form [3, 4]. Then we describe the construction of invariant

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subspaces corresponding to the absolutely continuous spectrum. The proofs of results in the abstract part of the paper (Sec. 1) are omitted for lack of space.

1. The Functional Model

Let L be a closed dissipative operator with bounded imaginary part $V = \operatorname{Im} L$ such that $\sigma_{\mathrm{ess}}(L) \subset \mathbb{R}$, and let $E = \overline{\operatorname{Ran} V}$. The *characteristic function* of L is the contractive analytic function $S(z)\colon E \to E$, $z \in \mathbb{C}_+$, defined by the formula

$$S(z) = I + 2i\sqrt{V}(L^* - z)^{-1}\sqrt{V}, \qquad z \in \mathbb{C}_+.$$

By the Fatou theorem [2], S has the boundary values $S(k) \equiv S(k+i0)$ on the real axis for almost all $k \in \mathbb{R}$. For $z \in \mathbb{C}_+ \cap \rho(L)$, the characteristic function has the bounded inverse $S^{-1}(z) = I - 2i\sqrt{V}(L-z)^{-1}\sqrt{V}$. Let $\mathscr{X} = L^2\begin{pmatrix} I & S^* \\ S & I \end{pmatrix}$ be the Hilbert space obtained by the closure of the linear manifold $L^2(\mathbb{R}, E \oplus E)$ in the metric given by the weight $\begin{pmatrix} I & S^* \\ S & I \end{pmatrix}$. We define a subspace $\mathscr{K} \subset \mathscr{X}$ as follows:

$$\mathscr{K} = \left\{ \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \in \mathscr{X} : \tilde{g} + S^* g \in H^2_-(E),\ S\tilde{g} + g \in H^2_+(E) \right\}.$$

Here the $H^2_\pm(E)$ are the Hardy classes of E-valued functions f analytic in $\mathbb{C}_\pm$, respectively, and satisfying $\sup_{\varepsilon>0} \int_{\mathbb{R}} \|f(k \pm i\varepsilon)\|^2_E \, dk < \infty$. Let $\mathscr{U}_t$ be the unitary shift group in $\mathscr{X}$ given by $(\mathscr{U}_t f)(k) = e^{ikt} f(k)$. Then the completely non-self-adjoint part of L is unitarily equivalent to the generator of the contraction semigroup $Z_t = P_{\mathscr{K}} \mathscr{U}_t|_{\mathscr{K}}$, where $P_{\mathscr{K}}$ is the orthogonal projection on $\mathscr{K}$ in $\mathscr{X}$. This generator is called the *functional model* of the operator L. Note that since S is a contraction, it follows that $\Delta(k) = I - S^*(k)S(k) \geqslant 0$ for almost all $k \in \mathbb{R}$, which allows us to define the space $L^2(\mathbb{R}; \Delta)$ as the closure of $L^2(\mathbb{R}, E)$ in the metric given by the weight Δ. In what follows, $(\,\cdot\,,\,\cdot\,)$ stands for the angle between subspaces of a Hilbert space.

We use theorems on dissipative operators corresponding to theorems stated for contractions in [2] without special explanations.

Throughout the paper, an invariant subspace of an operator is understood as a regular invariant subspace, that is, a subspace $\mathscr{H}$ is called an invariant subspace of the operator L if $\overline{(L-z)^{-1}\mathscr{H}} = \mathscr{H}$ for all $z \in \rho(L)$. Using the functional calculus for the operator L, one can show that every regular invariant subspace of it is invariant in the following natural sense: (i) $\operatorname{Dom}(L) \cap \mathscr{H} = (L-z)^{-1}\mathscr{H}$, $z \in \rho(L)$; (ii) $L(\operatorname{Dom}(L) \cap \mathscr{H}) \subset \mathscr{H}$. This allows us to define the closed densely defined restriction $L_{\mathscr{H}} = L|_{\mathscr{H}}$.

In the following, we introduce some invariant subspaces of the completely non-self-adjoint part of L in terms of its functional model. In doing so, we omit the operator that accomplishes the unitary equivalence between the completely non-self-adjoint part of L and its functional model.

We define the absolutely continuous subspace $\mathscr{N}_e \subset \mathscr{K}$ of the operator L [3] as the closure of the set $\widetilde{\mathscr{N}_e}$ of smooth vectors:

$$\mathscr{N}_e = \overline{\widetilde{\mathscr{N}_e}}, \qquad \widetilde{\mathscr{N}_e} \equiv \left\{ P_{\mathscr{K}} \begin{pmatrix} \tilde{g} \\ -S\tilde{g} \end{pmatrix},\ \tilde{g} \in L^2(\mathbb{R}; \Delta) \right\}.$$

Then $\mathscr{N}_e$ coincides with the invariant subspace of L corresponding to the canonical factorization $S = S_i S_e$ [2] of the characteristic function. We say that the spectrum of L is absolutely continuous if $L = L|_{H_0} \oplus L|_{\mathscr{N}_e}$, where H_0 is an invariant subspace of L such that $L|_{H_0}$ is a self-adjoint operator with absolutely continuous spectrum in the sense of spectral theory [12].

Let us define a *wave operator* $W\colon L^2(\mathbb{R}; \Delta) \to \mathscr{K}$ by setting $W\colon \tilde{g} \mapsto P_{\mathscr{K}}\left(\begin{smallmatrix} \tilde{g} \\ -S\tilde{g} \end{smallmatrix}\right)$. As is shown in [4], $W = \operatorname{s-lim}_{t \to +\infty} e^{iLt} J e^{-iA_0 t}$, where $J\colon L^2(\mathbb{R}; \Delta) \to \mathscr{K}$ is an isometric identification operator, which can be given by an explicit formula in terms of the model, and A_0 is the operator of multiplication by the independent variable in $L^2(\mathbb{R}; \Delta)$. The main property of smooth vectors is expressed by the intertwining relation

$$(L-z)^{-1} W = W (A_0 - z)^{-1}, \qquad z \in \rho(L). \tag{1}$$

The following lemma describes a class of invariant subspaces of the operator L in $\mathscr{N}_e$.

Let $\{X(k)\}$ be a measurable family of subspaces of E defined for almost all $k \in \mathbb{R}$, and let $X = \{f \in L^2(\mathbb{R}, E) : f(k) \in X(k)$ for almost all $k\}$. We set $H = \overline{X}$ in $L^2(\mathbb{R}; \Delta)$. We define the *multiplicity of the spectrum* of an operator A as the number $m(A) = \inf \dim \mathscr{N}$, where $\mathscr{N}$ ranges over the generating subspaces of A.

Lemma 1. *$\mathscr{H} = \overline{WH}$ is an invariant subspace of the operator L. If $S(k)$ is invertible a.e. on the real axis in the wide sense, then*

$$m(L_{\mathscr{H}}) = \operatorname*{ess\,sup}_{k\in\mathbb{R}} \dim \Delta(k)X(k). \tag{2}$$

If $\operatorname{ess\,sup}_{k\in\mathbb{R}} \|(S(k)|_{\Delta(k)X(k)})^{-1}\| < \infty$, then $W|_H$ has a bounded inverse and $L_{\mathscr{H}}$ is similar to a self-adjoint operator.

Note that a special case of formula (2) is contained in [8].

Given X, one can always construct a second invariant subspace $\mathscr{H}^{\sim}$ of L such that $\mathscr{N}_e = \overline{\mathscr{H} \dotplus \mathscr{H}^{\sim}}$ by applying Lemma 1 to the family $\{X(k)^{\perp}\}$. However, in general, the angle $(\mathscr{H}, \mathscr{H}^{\sim})$ can hardly be estimated. Let us describe how to choose $\{X(k)\}$ so that $(\mathscr{H}, \mathscr{H}^{\sim})$ could be estimated in terms of S.

We introduce the operator $D(k) = S^*(k)S(k)$. Given a measurable function $k \mapsto \gamma_k$, $\gamma_k \in [0,1]$, let $P_1(k)$ and $P_2(k)$ be the spectral projections of $D(k)$ for the intervals $[0, \gamma_k)$ and $[\gamma_k, 1]$ respectively, and let $X_i(k) = \operatorname{Ran} P_i(k)$, $i = 1, 2$, so that $E = X_1(k) \oplus X_2(k)$, since $0 \leqslant D(k) \leqslant I$. We set $X_i = \{f \in L^2(\mathbb{R}, E) : f(k) \in X_i(k)$ for almost all $k \in \mathbb{R}\}$, $i = 1, 2$. By construction, the pair of orthogonal subspaces $X_1, X_2 \subset L^2(\mathbb{R}, E)$ reduces Δ. Let $H_i = \overline{X}_i$, $i = 1, 2$, in $L^2(\mathbb{R}; \Delta)$, so that $L^2(\mathbb{R}; \Delta) = H_1 \oplus H_2$. By Lemma 1, $\mathscr{H}_i = \overline{WH_i}$, $i = 1, 2$, are invariant subspaces of the operator L.

Lemma 2. *The following inequality holds*:

$$\sin(\mathscr{H}_1, \mathscr{H}_2) \geqslant \operatorname*{ess\,inf}_{k\in\mathbb{R}} \|(S(k)|_{X_2(k)})^{-1}\|^{-1}. \tag{3}$$

2. The Linear Transport Operator

In what follows, $\chi_M(\,\cdot\,)$ is the indicator of a set $M \subset \mathbb{R}$, $\mathbf{1}$ is the indicator of the set Ω, f_∞ is the L^∞-norm of a function f, $\|\cdot\|_2$ is the norm in the Hilbert–Schmidt class, $U_\delta(z) = \{z' \in \mathbb{C} : |z - z'| < \delta\}$, $\omega_\delta = U_\delta(0) \cap \mathbb{C}_+$, and $d\mu$ is the Lebesgue measure on Ω. We introduce the class $L_0^+ = \{d \in L^\infty(\mathbb{R}) : d(x) \geqslant 0$ a.e. and there exists an a such that $d(x) = 0$ for almost all x with $|x| > a\}$.

The evolution of the distribution function in the Hilbert space $H = L^2(\Gamma)$ is described by the Boltzmann equation [1]

$$-i\partial_t u = Lu.$$

The generator L, which is called the one-speed transport operator, acts in H by the formula*

$$L = i\mu\partial_x + i\sigma(1 - c(x)K), \qquad K = \frac{1}{2}\int_\Omega \cdot \, d\mu'$$

on the domain $\mathscr{D} = \{f \in H : f(\,\cdot\,, \mu)$ is absolutely continuous for almost all $\mu \in \Omega$ and $\mu\partial_x f \in H\}$ of its real part $L_0 = i\mu\partial_x$. The imaginary part of L is bounded. The operator L_0 corresponds to the evolution $U_t = \exp iL_0 t$, $(U_t f)(x, \mu) = f(x - \mu t, \mu)$, of distribution functions in vacuum.

Instead of L, it is convenient to deal with the dissipative operator $T = L^* + i\sigma = i\mu\partial_x + i\sigma c(x)K$. Without loss of generality, one can set $\sigma = 1$; then $V = \operatorname{Im} T = c(x)K$. The subspace $\operatorname{Ran} K$ in H is naturally identified with the space $L^2(\mathbb{R})$ of functions of the variable x, $KH = L^2(\mathbb{R}) \otimes \mathbf{1} \simeq L^2(\mathbb{R})$.

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*We do not distinguish between the operator K in $L^2(\Omega)$ and the operator $I \otimes K$ in $L^2(\mathbb{R} \times \Omega) = L^2(\mathbb{R}) \otimes L^2(\Omega)$ in our notation.

3. Spectral Analysis of the Transport Operator

From now on we assume that the slab has a finite width, i.e., $c \in L_0^+$. Depending on the context, we use $\sqrt{c}$ either for the operator of multiplication by the function $\sqrt{c(x)}$ in $KH = L^2(\mathbb{R})$ or for the corresponding element of $E \equiv \overline{\operatorname{Ran} V} \subset KH$.

Let $R(z) = (T-z)^{-1}$ and $R_0(z) = (L_0 - z)^{-1}$. By multiplying the resolvent identity $R(z) - R_0(z) = -iR_0(z)VR(z)$ by $\sqrt{V}$, we obtain

$$R(z) = R_0(z) - iR_0(z)\sqrt{V}(I + i\sqrt{V}R_0(z)\sqrt{V})^{-1}\sqrt{V}R_0(z). \tag{4}$$

Now we see that $\sigma_+(T) \equiv \sigma(T) \cap \mathbb{C}_+ = \{z : -1 \in \sigma(Q(z))\}$, where $Q(z) = i\sqrt{V}R_0(z)\sqrt{V}|_{\overline{\operatorname{Ran} V}}$, $z \in \mathbb{C}_+$, is an operator in the space $\overline{\operatorname{Ran} V} \subset L^2(\mathbb{R})$. The kernel of the operator $Q(z)$ has the form* (see [5]) $-\frac{1}{2}\sqrt{c(x)}E(-iz|x-y|)\sqrt{c(y)}$, where $E(s) = \int_1^\infty e^{-st}\,dt/t$ for $\operatorname{Re} s > 0$. The function $E(s)$ admits the representation

$$E(s) = -\ln s - \gamma + \theta(s), \tag{5}$$

where $\theta(s) = -\sum_{m=1}^\infty (-s)^m/(m!\,m)$ is an entire function and γ is the Euler constant. Since $E(s) = O(\ln|s|)$ as $s \to 0$ and c is compactly supported, it follows that $Q(z)$ is of the Hilbert–Schmidt class for each $z \in \mathbb{C}_+$. Let $\{\eta_n(z)\}_{n=1}^\infty$, $|\eta_n| \geqslant |\eta_{n+1}| > 0$, be the eigenvalues of the operator $Q(z)$. Using (5), one can represent $Q(z)$ as

$$Q(z) \equiv \widetilde{Q}(z) + \tfrac{1}{2}\Theta(z), \tag{6}$$

where $\widetilde{Q}(z)$ is the operator with kernel $\frac{1}{2}\sqrt{c(x)}(\ln(-iz|x-y|) + \gamma)\sqrt{c(y)}$; thus, $\Theta(z)$ is an entire function and $\widetilde{Q}(z)$ is an analytic function in $\mathscr{O} = \mathbb{C} \setminus \{-it,\, t \geqslant 0\}$. According to this formula, $Q(z)$ admits an analytic continuation from $\mathbb{C}_+$ to $\mathscr{O}$.

Theorem 1. *Suppose that $c \in L_0^+$. Then the nonreal spectrum $\sigma_+(T)$ of the operator T consists of finitely many eigenvalues lying on the imaginary axis. Moreover, T has no associated vectors. The dimension $N(c)$ of the subspace corresponding to the discrete spectrum satisfies the estimate*

$$N(c) \leqslant 1 + \frac{1}{4}\iint \ln^2|x-y|\, c(x)c(y)\,dx\,dy. \tag{7}$$

The essential spectrum of T coincides with the real axis: $\sigma_{\text{ess}}(T) = \mathbb{R}$.

Proof. The operator $\sqrt{V}R_0(z)$ is compact, since so is $Q(z)$ and

$$\operatorname{Re} Q(z) = \frac{i}{2}(\sqrt{V}(R_0(z) - R_0(\overline{z}))\sqrt{V}) = -\operatorname{Im} z\sqrt{V}R_0(z)\cdot R_0(\overline{z})\sqrt{V}.$$

By the resolvent identity, it follows that the difference $R(z) - R_0(z)$ is compact for $z \in \rho(T)$, $\operatorname{Im} z \neq 0$. By the Weyl theorem [12], one concludes that $\sigma_+(T)$ is discrete in $\mathbb{C}_+$ and $\sigma_{\text{ess}}(T) = \mathbb{R}$

Next, $\eta_n(z) \notin \mathbb{R}$ for all n (and, in particular, $-1 \notin \sigma(Q(z))$) provided that $k \equiv \operatorname{Re} z \neq 0$. It suffices to show that the operator $\operatorname{Im} Q(z)$ is strictly positive or negative for $k \neq 0$. Consider the operator $\Xi(z) = iKR_0(z)|_{KH}$ acting in the space KH. In the Fourier representation with respect to the variable x, the operator $\operatorname{Im}\Xi(z)$ acts as the multiplication by the function

$$r_z(p) = \frac{1}{4p}\ln\frac{(p-k)^2 + (\operatorname{Im} z)^2}{(p+k)^2 + (\operatorname{Im} z)^2}. \tag{8}$$

Clearly, $r_z(p) < 0$ or $-r_z(p) < 0$ and hence $\operatorname{Im}\Xi(z) < 0$ or $-\operatorname{Im}\Xi(z) < 0$ depending on $\operatorname{sign} k$. Since $\operatorname{Im} Q(z) = \sqrt{c}\operatorname{Im}\Xi(z)\sqrt{c}|_E$, it follows that $\operatorname{Im} Q(z) < 0$ or $-\operatorname{Im} Q(z) < 0$. Therefore, $\sigma_+(T) \subset i\mathbb{R}$.

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*This integral kernel is actually calculated in [5, §2] for the case $c = c_\infty \chi_{[-a,a]}$. The result for arbitrary $c \in L_0^+$ is obtained by an obvious transformation. Below we use some auxiliary estimates on Q obtained in [9, 10] for the case $c = c_\infty \chi_{[-a,a]}$ when deriving the spectrum, since their proofs in the general case are absolutely the same.

Let $z = i\varepsilon$, $\varepsilon > 0$. In the Fourier representation, $\Xi(i\varepsilon) = \Xi^*(i\varepsilon)$ acts as the multiplication by the function

$$\xi_\varepsilon(p) = -\frac{\varepsilon}{2}\int_{-1}^{1}\frac{d\mu}{|p\mu + i\varepsilon|^2} = -\frac{1}{|p|}\arctan\frac{|p|}{\varepsilon}\,.$$

Thus, $Q(i\varepsilon) = \sqrt{c}\,\Xi(i\varepsilon)\sqrt{c}|_E$ is a negative operator and a monotone increasing function of ε. Consider $Q'(i\varepsilon) = -i\sqrt{c}\,(d\Xi(i\varepsilon)/d\varepsilon)\sqrt{c}|_E$. In the Fourier representation, the operator in parentheses acts as the multiplication by the function $(p^2+\varepsilon^2)^{-1}$. Thus $iQ'(i\varepsilon) > 0$. According to standard arguments of perturbation theory, this implies that $(I + Q(z))^{-1}$ has simple poles at the points of $\sigma_+(T)$, and the same is true of $R(z)$ by (4). Hence T has no associated vectors.

We shall now estimate $N(c)$. Let $N_\varepsilon(c)$ be the rank of the spectral projection corresponding to the interval $[i\varepsilon, i\infty)$. Since $\sigma_+(T)$ is discrete in $\mathbb{C}_+$, it follows that $N_\varepsilon(c) < \infty$ for $\varepsilon > 0$. By the monotonicity and continuity of the functions $\eta_n(i\varepsilon)$, we have $N_\varepsilon(c) \equiv \#\{s \geqslant \varepsilon : \ker(I + Q(is)) \neq 0\} = \#\{\tau \in (0,1] : \ker(I + \tau Q(i\varepsilon)) \neq 0\}$, counting multiplicity. According to (6),

$$2Q(i\varepsilon) = Q^0 + \Theta(i\varepsilon) + (\gamma + \ln\varepsilon)\mathbf{P}_1 \equiv 2Q_1(i\varepsilon) + (\gamma + \ln\varepsilon)\mathbf{P}_1, \tag{9}$$

where $\mathbf{P}_1 = \langle\,\cdot\,,\sqrt{c}\rangle\sqrt{c}$, $\langle\,\cdot\,,\,\cdot\,\rangle$ is the inner product in $L^2(\mathbb{R})$ and Q^0 is the operator with kernel $\sqrt{c(x)c(y)}\ln|x-y|$. Since $\operatorname{rank}\mathbf{P}_1 = 1$, it follows that

$$\#\{\tau \in (0,1] : \ker(I + \tau Q(i\varepsilon)) \neq 0\} \leqslant 1 + \#\{\tau \in (0,1] : \ker(I + \tau Q_1(i\varepsilon)) \neq 0\} \leqslant 1 + \|Q_1(i\varepsilon)\|_2^2.$$

We now use the following estimate for $\Theta(z)$ (see (5), (6)):

$$\|\Theta(z)\|_2 \leqslant \sum_1^\infty \frac{\|A_m\|_2|z|^m}{m!\,m} \leqslant c_1\sum_1^\infty\frac{(2a\,|z|)^m}{m!\,m} \leqslant c_1\sum_1^\infty\frac{(2a\,|z|)^m}{m!} = c_1(e^{2a|z|}-1), \tag{10}$$

where $c_1 = \int_{\mathbb{R}} c(x)\,dx$ and A_m is the operator with kernel $\sqrt{c(x)}\,|x-y|^m\sqrt{c(y)}$, which satisfies the estimate $\|A_m\|_2^2 = \iint |x-y|^{2m}c(x)c(y)\,dx\,dy \leqslant (2a)^{2m}c_1^2$. This gives

$$N_\varepsilon(c) \leqslant 1 + \|Q_1(i\varepsilon)\|_2^2 = 1 + \tfrac14\|Q^0\|_2^2 + O(\varepsilon).$$

Since $N(c) = \lim_{\varepsilon\downarrow 0} N_\varepsilon(c)$, we arrive at (7). $\square$

Note that $\sigma_+(T)$ is not empty for any nonzero function $c \in L_0^+$, since $\|Q(it)\| \to \infty$ as $t \to 0$ and $\|Q(it)\| \to 0$ as $t \to \infty$.

The proof of the finiteness of $N(c)$ given in [5, 9] for the case $c = c_\infty\chi_{[-a,a]}$ is based on the same considerations (the monotonicity of $Q(i\varepsilon)$ and the finite rank of the divergent term in the asymptotics of $Q(i\varepsilon)$ at 0) but does not contain the estimate (7) of multiplicity, which is apparently new.

Let us decompose the space H into a linear sum of invariant subspaces corresponding to the components of the spectrum $\sigma(T)$, $H = H_d \dot{+} H_{\mathrm{ess}}$, where $H_d = \mathscr{P}_d H$, $H_{\mathrm{ess}} = (I - \mathscr{P}_d)H$, and $\mathscr{P}_d$ is the Riesz projection corresponding to $\sigma_+(T)$. By Theorem 1, the subspace H_d is finite-dimensional, and so $(H_d, H_{\mathrm{ess}}) > 0$. An estimate of the angle (H_d, H_{ess}) for small c is given by the following assertion (we omit the proof).

Proposition 1. *Let c be such that the integral in (7) is less than 4 and hence $\dim H_d = 1$. Then*

$$\sin(H_d, H_{\mathrm{ess}}) \geqslant \frac{e^{-\gamma}}{4+\pi^2}\,\frac{1}{ac_\infty}\exp\Big(-\frac{2}{c_1}\Big).$$

We now proceed to study the component $T_{\mathrm{ess}} = T|_{H_{\mathrm{ess}}}$ by means of the functional model.

Note that $\operatorname{Re} Q(z) \leqslant 0$ for $z \in \mathbb{C}_+$ by the definition of the operator Q, and hence $I - Q(z)$ has a bounded inverse, $\|(I - Q(z))^{-1}\| \leqslant 1$. For $z \in \mathbb{C}_+$, the characteristic function $S(z)$ of the operator T is expressed in terms of Q as follows [11]:

$$S(z) = \frac{I + Q(z)}{I - Q(z)}. \tag{11}$$

We see that S admits an analytic extension into an open set containing $\mathbb{R}\setminus 0$. The following lemma reduces, to some extent, the study of the operator-valued function S to that of a scalar analytic function. Recall that a scalar analytic function f is called a *scalar multiple* [2] of S if there exists a bounded analytic function Π in $\mathbb{C}_+$ such that $f(z)I = S(z)\Pi(z) = \Pi(z)S(z)$ for $z \in \mathbb{C}_+$.

Lemma 3. *S has a scalar multiple.*

Proof. It follows from (11) that $T(z) \equiv S(z) - I$, as well as $Q(z)$, belongs to the class $\mathbf{S}^2$ for every $z \in \mathbb{C}_+$. Let us show that $\|T(z)\|_2$ is uniformly bounded in $\mathbb{C}_+$. Indeed, by (6),

$$T(z) = (A + \Theta(z))(I - Q(z))^{-1} + M(z),$$

where $\operatorname{rank} M(z) = 1$ and $\|A\|_2 < \infty$. Since $\|T(z)\| \leqslant 1 + \|S(z)\| \leqslant 2$, we have $\|M(z)\|_2 = \|M(z)\| \leqslant 2 + \|A\| + C_\delta|z| \leqslant C_{1,\delta}$ for $|z| \leqslant \delta$. It follows that $\sup_{z\in\omega_\delta} \|T(z)\|_2 \leqslant \widetilde{C}_\delta < \infty$, since $\|\Theta(z)\|_2 \leqslant C_\delta|z|$ for $|z| \leqslant \delta$ by (10). Lemma 5 in [10] states, in our notation, that*

$$\|Q(z)\|_2^2 \leqslant \frac{C_\varepsilon}{1 + |\operatorname{Re} z|} \tag{12}$$

for $z \in \Pi_\varepsilon \cap \{z : |\operatorname{Re} z| \geqslant 1\}$ for any $\varepsilon > 0$, where $\Pi_\varepsilon = \{z : 0 \leqslant \operatorname{Im} z \leqslant \varepsilon\}$. Choosing an arbitrary $\varepsilon \in (0, 1]$ and taking $\delta > \sqrt{2}$, we obtain $\|T(z)\|_2 \leqslant C < \infty$ for $z \in \Pi_\varepsilon$. Then, by the obvious inequality $|E(s)| \leqslant E(\operatorname{Re} s) \leqslant E(\operatorname{Re} s_0)$, which is valid for $0 < \operatorname{Re} s_0 \leqslant \operatorname{Re} s$, we have $\|Q(z)\|_2 \leqslant \|Q(i\varepsilon)\|_2 < \infty$ for $z \in \mathbb{C}_+ \setminus \Pi_\varepsilon$. Combining these estimates, we obtain $\sup_{z\in\mathbb{C}_+} \|T(z)\|_2 < \infty$. Let $m(z) \equiv \det(I - T^2(z))$. We now use the following fact, essentially proved in [13]. If $A(\,\cdot\,)$ is an $\mathbf{S}^2$-valued function on a domain $\mathscr{D} \subset \mathbb{C}$ analytic in the operator norm and satisfying $\sup_{z\in\mathscr{D}} \|A(z)\|_2 < \infty$, then $s(z) = \det(I - A^2(z))$ is a scalar multiple of $S(z) = I + A(z)$ provided that $s(z) \not\equiv 0$. Since $T(\,\cdot\,)$ is obviously analytic, we conclude that m is a scalar multiple of S. $\square$

The estimate (12) was used in [10] for the derivation of some pointwise asymptotics of solutions of the Boltzmann equation. However, the derivation itself contains an error. Namely, the logarithmic estimate for $(I + Q(z))^{-1}$ in the vicinity of 0 given by Lemma 6 in [10] is false in general.

Let $\mathscr{B}_c = \{k_n = \lim_{\varepsilon\downarrow 0} \eta_n(i\varepsilon) : |k_n| < \infty\}$. We define the *singular set* $\mathscr{E} = \{c \in L_0^+ : -1 \in \mathscr{B}_c\}$. Note that for any $c \in L_0^+$ the function κc belongs to $\mathscr{E}$ for the infinite discrete set $\{-k_n^{-1},\ k_n \in \mathscr{B}_c\}$ of values of the constant $\kappa > 0$. The following lemma explains why we consider the set $\mathscr{E}$.

Lemma 4. 1. *If $c \in \mathscr{E}$, then $S(i\rho)f(\rho) \to 0$ as $\rho \to 0$ for some function $f\colon (0, 1) \to \{u \in E, \|u\| = 1\}$.*
2. *If $c \notin \mathscr{E}$, then $\sup_{z\in\omega_\delta} \|S^{-1}(z)\| < \infty$ for sufficiently small $\delta > 0$.*

Proof. 1. By the definition of the set $\mathscr{E}$, there exists a normalized eigenfunction φ_ρ of $Q(i\rho)$, $Q(i\rho)\varphi_\rho = \eta(\rho)\varphi_\rho$, such that $\eta(\rho) \to -1$ as $\rho \to 0$. It suffices to take $f(\rho) = \varphi_\rho$.

2. By (11), we have $S^{-1}(z) = -I + 2(I + Q(z))^{-1}$ for $z \in \mathbb{C}_+ \setminus \sigma_+$, and thus the desired estimate is equivalent to the finiteness of $\sup_{z\in\omega_\delta} \|(I + Q(z))^{-1}\|$. Suppose, on the contrary, that $(I + Q(z_n))\varphi_n \to 0$ as $z_n \to 0$ for some sequence $\{\varphi_n\}$, $\|\varphi_n\| = 1$. Then $\langle \operatorname{Im} Q(z_n)\varphi_n, \varphi_n\rangle \to 0$. Taking into account the fact that $\operatorname{Im} Q(z)$ is sign-definite for $\operatorname{Re} z \neq 0$ and $\operatorname{Im} Q(z) = 0$ for $\operatorname{Re} z = 0$, we see that $\operatorname{Im} Q(z_n)\varphi_n \to 0$ and hence $(I + \operatorname{Re} Q(z_n))\varphi_n \to 0$. By (6), $\operatorname{Re} Q(z) - Q(i|z|) \to 0$ in the operator norm as $z \to 0$. It follows that $(I + Q(i|z_n|))\varphi_n \to 0$, which contradicts the assumption that $c \notin \mathscr{E}$. $\square$

Thus, for $c \in \mathscr{E}$ the function $S^{-1}(\,\cdot\,)$ is unbounded in a neighborhood of 0. A priori there are two possible causes of this behavior, namely, the presence of a "singular" inner factor in the canonical factorization of the function S and/or a spectral singularity at the point 0. Following [3], we say that a point $k \in \mathbb{R}$ is a *spectral singularity* if $\sup_{z\in U_\delta(k)\cap\mathbb{C}_+} \|S_e^{-1}(z)\| = \infty$ for all $\delta > 0$. In Proposition 2 below, we show that only the second possibility can occur. Let us first obtain an explicit description of the set $\mathscr{E}$.

Lemma 5. *$S(\,\cdot\,)$ is continuous at 0 in the operator norm, and*

$$c \in \mathscr{E} \iff \ker\Bigl(I - \widetilde{Q}_0^2 - \frac{2}{\vartheta_c}\langle\,\cdot\,, \sqrt{c}\rangle\sqrt{c}\Bigr) \neq 0, \tag{13}$$

where $\widetilde{Q}_0$ is the integral operator with kernel $\frac{1}{2}\sqrt{c(x)}\ln(|x-y|/(2a))\sqrt{c(y)}$ and $\vartheta_c = \langle (I - \widetilde{Q}_0)^{-1}\sqrt{c}, \sqrt{c}\rangle$.

*This estimate was essentially established in the course of the proof. The statement of Lemma 5 in [10] is weakened to the estimate of the operator norm of $Q(z)$.

Proof. Since $\|\Theta(z)\| = o(1)$ as $z \to 0$, it follows from (6) that $(I - \widetilde{Q}(z))^{-1}$ exists and is bounded for $z \in \omega_\delta$ provided that δ is sufficiently small. We see from the identity

$$(I - Q(z))^{-1} - (I - \widetilde{Q}(z))^{-1} = -(I - Q(z))^{-1}\frac{\Theta(z)}{2}(I - \widetilde{Q}(z))^{-1} \tag{14}$$

that S is continuous at 0 whenever so is $(I - \widetilde{Q}(z))^{-1}$, since $S(z) = -I + 2(I - Q(z))^{-1}$ by (11). We now use Lemma 7 in [9], which states, in our notation, that $\widetilde{Q}_0 < 0$, and hence the operator $I - \widetilde{Q}_0$ has a bounded inverse and $\vartheta_c \neq 0$. Set $\alpha(z) = \frac{1}{2}\ln(-2ae^{\gamma}iz)$. A straightforward computation from the formula $\widetilde{Q}(z) = \widetilde{Q}_0 + \alpha(z)\mathbf{P}_1$ gives

$$(I - \widetilde{Q}(z))^{-1} = (I - \widetilde{Q}_0)^{-1} + \frac{\alpha(z)}{1 - \alpha(z)\vartheta_c}(I - \widetilde{Q}_0)^{-1}\mathbf{P}_1(I - \widetilde{Q}_0)^{-1}. \tag{15}$$

Thus, S is continuous at 0, and

$$S(0) = \frac{I + \widetilde{Q}_0}{I - \widetilde{Q}_0} - \frac{2}{\vartheta_c}(I - \widetilde{Q}_0)^{-1}\mathbf{P}_1(I - \widetilde{Q}_0)^{-1}.$$

Multiplying this equality on the left and right by $I - \widetilde{Q}_0$, we obtain (13), since $c \in \mathscr{E}$ if and only if $\ker S(0) \neq 0$. ☐

Proposition 2. *The spectrum of the component T_{ess} of the operator T is purely absolutely continuous.*

Proof. Note that the scalar multiple m admits an analytic continuation into an open neighborhood of $\mathbb{R} \setminus 0$, since so does $Q(z)$ as an $\mathbf{S}^2$-valued function. This means that the singular component m_s in the canonical factorization of m has the form $m_s(z) = e^{i\mu_1 z - \mu_2 i/z}$ with some $\mu_1, \mu_2 \geqslant 0$.

We see from (15) that $(I - \widetilde{Q}(z))^{-1}$ is a bounded analytic function in $U_\tau(0) \cap \mathscr{O}$, $\tau = (2ae^{\gamma})^{-1}$. Solving identity (14) with $z \in \omega_\tau$ for $(I - Q(z))^{-1}$ and taking into account the fact that $\|\Theta(z)\| = o(1)$ as $z \to 0$, we find that $(I - Q(z))^{-1}$, and therefore $S(z)$, admits a bounded analytic continuation into $U_\delta(0) \cap \mathscr{O}$ for sufficiently small δ. Arguing as in the beginning of the proof of Lemma 3, we see that m admits an analytic continuation into $U_\delta(0) \cap \mathscr{O}$, which is bounded in view of the inequality $|\det(I - A^2)| \leqslant \exp(\|A\|_2^2)$, $A \in \mathbf{S}^2$. It follows that $\mu_2 = 0$. Indeed, otherwise an application of the Carlson theorem [12] to the function $g(1/z)$, where $g(z) = e^{\mu_2 i/(2z)}m(z)$, implies $g \equiv 0$.

Next, $\|Q(i\varepsilon)\|_2 \to 0$ as $\varepsilon \to \infty$ by the Lebesgue theorem. Hence $m(i\varepsilon) \to 1$ as $\varepsilon \to \infty$. This implies $\mu_1 = 0$. Finally, $m(z) = 0$ for $z \in \mathbb{C}_+$ if and only if $z \in \sigma_+$. Indeed, $m(z) = \det(S(z)(2 + S(z)))$ and thus $m(z) = 0$ if either $\ker S(z) \neq 0$ or $\ker(2 + S(z)) \neq 0$. The latter is impossible, since $\|S(z)\| \leqslant 1$. It follows that the canonical factorization of m has the form $m = bm_e$, where m_e is an outer function and b is the finite Blaschke product corresponding to the set σ_+. Let $S = B\widetilde{S}$ be the canonical factorization of S. The factor $\widetilde{S}$ coincides with the characteristic function of the operator $\widetilde{T}_{\mathrm{ess}} = T|_{\mathscr{N}_e}$, where $\mathscr{N}_e$ is the absolutely continuous subspace of the operator T [3]. Since m_e is a scalar multiple of $\widetilde{S}$ by [2, Proposition V.6.4], we have $\sigma(\widetilde{T}_{\mathrm{ess}}) \subset \mathbb{R}$, and so $\widetilde{T}_{\mathrm{ess}} \subset T_{\mathrm{ess}}$.

It remains to note that $\mathscr{N}_e = H_{\mathrm{ess}} \ominus H_0$. Indeed, since $(H_d, H_{\mathrm{ess}}) > 0$, we have $H \ominus H_0 = \mathscr{N}_e \dotplus \mathscr{N}_i$, where $\mathscr{N}_i$ is the inner subspace of T (see [3]). In our case $\mathscr{N}_i = H_d$, and the desired equality follows. ☐

Thus, the point 0 is a spectral singularity of the operator T if $c \in \mathscr{E}$. According to the Nagy–Foiaş criterion [2], this means that if $c \in \mathscr{E}$, then T_{ess} is not similar to a self-adjoint operator.

Proposition 3. *If $c \notin \mathscr{E}$, then T_{ess} is similar to a self-adjoint operator.*

Proof. First, we note that $\sup_{z \in X}\|\widetilde{S}^{-1}(z)\| < \infty$ for any compact set $X \subset \mathbb{C}_+$, since $\sigma(T_{\mathrm{ess}}) \subset \mathbb{R}$. Next, $\|Q(z)\| \leqslant \|Q(i\,\mathrm{Im}\,z)\| \to 0$ as $\mathrm{Im}\,z \to \infty$. From this and the estimate (12), we conclude that $S(z) \to I$ in the operator norm as $z \to \infty$ in $\mathbb{C}_+$ uniformly in $\arg z$. By passing to the limit as $\mathrm{Im}\,z \to 0$ in (8), we see that, depending on $\operatorname{sign} k$, either $\mathrm{Im}\,Q(k) < 0$ or $-\mathrm{Im}\,Q(k) < 0$, and so $\ker S(k) = 0$ for $k \neq 0$. Combining these assertions, we see that $\sup_{z \in \mathbb{C}_+ \setminus \omega_\delta}\|\widetilde{S}^{-1}(z)\| < \infty$, since $\widetilde{S}^{-1} = S^{-1}B$ with B a contraction. Note that this conclusion does not depend on the assumption that $c \notin \mathscr{E}$. Finally, taking into account item 2 of Lemma 4, we obtain $\sup_{z \in \mathbb{C}_+}\|\widetilde{S}^{-1}(z)\| < \infty$, and the result follows from the Nagy-Foiaş criterion. ☐

At the abstract level, one can always construct a decomposition $T_{\mathrm{ess}} = T_1 \dotplus T_2$ of T_{ess} into a linear sum such that the component T_2 is similar to a self-adjoint operator by setting $\gamma_k \equiv \beta$, $\beta \in (0,1)$, in the construction of Sec. 1. If $S(z) \in I + \mathbb{S}^\infty$ and the function S is continuous up to the real axis and at infinity in the operator norm, as is the case in our problem, then the multiplicity of the spectrum of T_1 is finite for any $\beta \in (0,1)$. Note that if S fails to be continuous at least at one point, then it may happen that $m(T_1) = \infty$ for all $\beta > 0$. Let $\mathscr{M} = \dim\ker S(0)$. Since the point 0 is the unique spectral singularity of T for $c \in \mathscr{E}$, it follows that $m(T_1) = \mathscr{M}$ for sufficiently small β, and this number is the minimum possible. At the same time, an effective estimate of the angle between the corresponding invariant subspaces does not follow from abstract arguments. As is shown in Theorem 2 below, which summarizes the results pertaining to the case $c \in \mathscr{E}$, one can choose the function γ_k in our problem in such a way that $m(T_1) = \mathscr{M}$, while for the spectral component of T_{ess} corresponding to a neighborhood of the spectral singularity the angle can be estimated in terms of the operator Q.

Let χ_δ be the operator of multiplication by the indicator function of the set $\mathbb{R} \setminus [-\delta, \delta]$, $\delta \neq 0$, in $L^2(\mathbb{R}; \Delta)$. Then $\mathscr{P}_\delta = I - W\chi_\delta W^{-1}$ is the spectral projection of the operator T for the interval $[-\delta, \delta]$ in the following sense: (i) $\mathscr{P}_\delta$ is a bounded operator in $\mathscr{N}_e$ and $\mathscr{P}_\delta^2 = \mathscr{P}_\delta$; (ii) $\mathscr{P}_\delta$ commutes with $R(z)$ for all $z \in \rho(T)$ and thus $\operatorname{Ran}\mathscr{P}_\delta$ is an invariant subspace of the operator T; (iii) $\sigma(T|_{\operatorname{Ran}\mathscr{P}_\delta}) = [-\delta, \delta]$; (iv) any invariant subspace $\mathscr{G} \subset \mathscr{N}_e$ of the operator T such that $\sigma(T|_{\mathscr{G}}) \subset [-\delta, \delta]$ is contained in $\operatorname{Ran}\mathscr{P}_\delta$. Properties (i)–(iii) follow from Lemma 1 and relation (1). Property (iv) follows, say, from the existence of a scalar multiple [2, Theorem VII.6.2].

Theorem 2. *If $c \in \mathscr{E}$, then for all sufficiently small $\delta \neq 0$ the operator T_{ess} can be represented as the linear sum $T_{\mathrm{ess}} = T_1 \dotplus T_2$ of operators T_1 and T_2 acting in invariant subspaces $\mathscr{H}_1$ and $\mathscr{H}_2$ such that*

(1) *T_1 has the spectrum $\sigma(T_1) = [-\delta, \delta]$ of multiplicity $\mathscr{M} = \dim\ker S(0)$;*

(2) *T_2 is similar to a self-adjoint operator and $\sigma(T_2) = \mathbb{R}$;*

(3) *$(\mathscr{H}_1, \mathscr{H}_2) > 0$. The angle $(\mathscr{H}_1, \mathscr{P}_\delta\mathscr{H}_2)$ admits the estimate*

$$\sin(\mathscr{H}_1, \mathscr{P}_\delta\mathscr{H}_2) \geqslant p/\sqrt{2}, \qquad p = \operatorname{dist}(0, \sigma(S(0)) \setminus \{0\}).$$

Proof. Choose a $\delta \neq 0$ such that the rank of spectral projection of the operator $D(k)$ for the interval $[0, p^2/2)$ is equal to $\mathscr{M}$ for all $k \in [-\delta, \delta]$. We define the subspaces $X_1(k), X_2(k) \subset E$ according to the construction of Sec. 1 with $\gamma_k = \frac{1}{2}p^2\chi_{[-\delta,\delta]}(k)$. By Lemma 1, $\mathscr{H}_i = \overline{WX_i}$, $i = 1, 2$, are invariant subspaces of T, $m(T_1) = \mathscr{M}$, and the restriction $T_2 = T|_{\mathscr{H}_2}$ is similar to a self-adjoint operator. Indeed, $S|_{X_2}$ has a bounded inverse, since $\sup_{k\in[-\delta,\delta]} \|(S(k)|_{X_2(k)})^{-1}\| \leqslant \sqrt{2}/p < \infty$ and $\sup_{|k|\geqslant\delta} \|S^{-1}(k)\| < \infty$ (see the proof of Proposition 3). □

Using the estimate (10) and formulas (14), (15), we can readily estimate δ from below. The result is as follows.

Remark. The assertions of items (1)–(3) of Theorem 2 hold for

$$\delta < \frac{1}{2a} \exp\left(- \frac{64\Upsilon^2}{p^2} \frac{ac_2^2}{c_1} - \gamma \right),$$

where $c_2 = \|c\|_{L^2(\mathbb{R})}$ and $\Upsilon = \left(\int_{-1}^{1}\int_{-1}^{1} \ln^4|x-y|\,dx\,dy \right)^{1/4}$.

4. Concluding Remarks

(a) In the limit case of our model, when $c(x) \equiv c_\infty$ for all $x \in \mathbb{R}$, the spectrum of the operator T can readily be calculated with the use of the Fourier transform with respect to x. The result is $\sigma(T) = \mathbb{R} \cup [ic_\infty, 0)$. This fact is natural in the sense that for arbitrary $c \in L_0^+$ we have $N(c_\varepsilon) \to \infty$ for $c_\varepsilon(x) = c(\varepsilon x)$ as $\varepsilon \to 0$.

Note that in this case T is not a spectral operator (see [14]) *for all* $c_\infty > 0$. Indeed, it can be shown that although the spectral projection $\mathscr{P}_\alpha$ of T for the interval $[i\alpha, ic_\infty]$ is bounded for $\alpha > 0$, $\|\mathscr{P}_\alpha\| \to \infty$ as $\alpha \to 0$, that is, T does not possess property (B) of spectral operators.

(b) The subcriticality condition $c_\infty \leqslant 1$ is necessary and sufficient for the dissipativity of L. At the same time, the operator T is dissipative for any nonnegative $c \in L^\infty$. Thus, for $c_\infty > 1$ one can use (7) as a rough estimate of the number of exponentially increasing modes for the supercritical transport problem.

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