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ORDINARY DIFFERENTIAL EQUATIONS

A Generalized Normal Form and Formal Equivalence of Two-Dimensional Systems with Quadratic Zero Approximation: IV

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Abstract— We continue the study of invertible formal transformations of two-dimensional autonomous systems of differential equations with zero approximation represented by homogeneous polynomials of degree 2 and with perturbations in the form of power series without terms of order < 3. In the regular case, we consider systems that have the canonical form $(\alpha x_1^2 - \operatorname{sgn}\alpha\, x_2^2, x_1x_2)$ with $\alpha \neq 0$ as the zero approximation.

For such systems, we obtain resonance equations in closed form and use them to prove the theorem on the formal equivalence of systems and establish a generalized normal form to which any original system can be reduced by an invertible change of variables.

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The present paper is an immediate continuation of [1–3]; therefore, we preserve the notation and continue the numbering of sections, formulas, theorems, remarks, corollaries, and examples.

12. SYSTEMS WITH ZERO APPROXIMATION $(\alpha x_1^2 - \operatorname{sgn}\alpha\, x_2^2, x_1x_2)$

0^0. Consider system (1) whose unperturbed part belongs to the regular case and has the form $(a_1, 0, c_1)(0, 2b_2, 0)$, where $a_1c_1 < 0$ and $b_2 \neq 0$. Then the linear change of variables (12_1)

$$((2b_2)^{-1}, 0)(0, (2|b_2c_1|)^{-1/2})$$

reduces it to a system with unperturbed part $(\alpha, 0, -\operatorname{sgn}\alpha)(0, 1, 0)$, that is, the canonical form (15_2); here $\alpha = a_1/(2b_2) \neq 0$. Therefore, let us study the system

$$\dot x_1 = \alpha x_1^2 - \widetilde\alpha x_2^2 + X_1(x_1, x_2), \qquad \dot x_2 = x_1x_2 + X_2(x_1, x_2) \qquad (\alpha \neq 0,\ \widetilde\alpha = \operatorname{sgn}\alpha). \tag{151}$$

Let the change of variables (2) $x_i = y_i + h_i(y_1, y_2)$ with $h_i = \sum_{p=2}^{\infty} h_i^{(p)}(y_1, y_2)$ $(i = 1, 2)$ bring system (151) to system (3) of the form $\dot y_1 = \alpha y_1^2 - \widetilde\alpha y_2^2 + Y_1(y_1, y_2)$, $\dot y_2 = y_1y_2 + Y_2(y_1, y_2)$, where

$$Y_i = \sum_{p=2}^{\infty} Y_i^{(p+1)}(y_1, y_2), \qquad Z_i^{(q)}(z_1, z_2) = \sum_{s=0}^{q} Z_i^{(s,q-s)} z_1^s z_2^{q-s}.$$

By differentiating (2) according to (151) and (3), by matching the coefficients of $y_1^s y_2^{p+1-s}$ $(0 \le s \le p+1,\ p \ge 2)$, and by using the notation (4)

$$\widehat Y_i^{(s,p+1-s)} = \widetilde Y_i^{(s,p+1-s)} - Y_i^{(s,p+1-s)}$$

(where the $\widetilde Y_i^{(s,p+1-s)}$ are known), we obtain a system of the form (5),

$$\begin{aligned}(\alpha(s-3) + p - s + 1)h_1^{(s-1,p-s+1)} - \widetilde\alpha(s+1)h_1^{(s+1,p-s-1)} + 2\widetilde\alpha h_2^{(s,p-s)} &= \widehat Y_1^{(s,p-s+1)},\\ (\alpha(s-1) + p - s)h_2^{(s-1,p-s+1)} - \widetilde\alpha(s+1)h_2^{(s+1,p-s-1)} - h_1^{(s,p-s)} &= \widehat Y_2^{(s,p-s+1)}.\end{aligned} \tag{152}$$

For $p = 2$, system (152) is consistent provided that the following equations hold:

$$\begin{gathered}2(\alpha-1)Y_2^{(0,3)}+2\widetilde{\alpha}Y_2^{(2,1)}-Y_1^{(1,2)}=\widetilde{c},\qquad Y_1^{(3,0)}=\widetilde{c},\\ \widetilde{\alpha}(1-\alpha)Y_1^{(0,3)}-Y_1^{(2,1)}-2(1-\alpha)Y_2^{(1,2)}+2\widetilde{\alpha}Y_2^{(3,0)}=\widetilde{c};\\ \text{if}\quad \alpha=1/2,\quad\text{then}\quad Y_2^{(3,0)}=\widetilde{c}\quad (h_1^{(1,1)}\ \text{is arbitrary}).\end{gathered}\tag{153}$$

Likewise, the following resonance equations should hold for $p = 3$:

$$\begin{gathered}\alpha^2Y_1^{(0,4)}-Y_1^{(4,0)}-\widetilde{\alpha}\alpha^2Y_2^{(1,3)}-\alpha Y_2^{(3,1)}=\widetilde{c},\\ (\alpha+1)(3\alpha-1)Y_1^{(1,3)}+3\widetilde{\alpha}(3\alpha-1)Y_1^{(3,1)}-3(2\alpha+1)Y_2^{(4,0)}\\ +\widetilde{\alpha}(1-2\alpha)(3\alpha-1)Y_2^{(2,2)}+(\alpha+1)(3\alpha-1)(3-2\alpha)Y_2^{(0,4)}=\widetilde{c};\\ \text{if}\quad \alpha=2,\quad\text{then}\quad 4Y_1^{(2,2)}+5Y_1^{(4,0)}-2Y_2^{(3,1)}=\widetilde{c}\quad (h_2^{(0,3)}\ \text{is arbitrary}).\end{gathered}\tag{154}$$

In (153), (154), and all forthcoming resonance equations of the form $(a, Y) = \widetilde{c}$, the constant $\widetilde{c}$ is known and, by (7), is equal to $(a, \widetilde{Y})$.

Now for the indices $p \geq 4$ and s, it is convenient to introduce the expansions

$$p = 2r + \mu \quad (r \geq 2,\ \mu \in \{0, 1\}), \qquad s = 2\tau + \mu + \nu \quad (-(\nu + \mu)/2 \leq \tau \leq r,\quad \nu \in \{0, 1\}).$$

Then for arbitrary $r \geq 2$, system (152) splits into two systems

$$\begin{aligned}&(\alpha(2\tau+\mu+\nu-3)+2(r-\tau)+1-\nu)h_1^{(2\tau+\mu+\nu-1,2(r-\tau)-\nu+1)}\\ &\quad-\widetilde{\alpha}(2\tau+\mu+\nu+1)h_1^{(2\tau+\mu+\nu+1,2(r-\tau)-\nu-1)}+2\widetilde{\alpha}h_2^{(2\tau+\mu+\nu,2(r-\tau)-\nu)}=\widehat{Y}_1^{(2\tau+\mu+\nu,2(r-\tau)-\nu+1)},\\ &(\alpha(2\tau+\mu+\nu-1)+2(r-\tau)-\nu)h_2^{(2\tau+\mu+\nu-1,2(r-\tau)-\nu+1)}\\ &\quad-\widetilde{\alpha}(2\tau+\mu+\nu+1)h_2^{(2\tau+\mu+\nu+1,2(r-\tau)-\nu-1)}-h_1^{(2\tau+\mu+\nu,2(r-\tau)-\nu)}=\widehat{Y}_2^{(2\tau+\mu+\nu,2(r-\tau)-\nu+1)}.\end{aligned}\tag{155^μ}$$

(The coefficients are zero if one of the superscripts is less than zero.)

By setting $\nu = 1$ in (155_1^μ) and $\nu = 0$ in (155_2^μ) and then vice versa, for each $p = 2r + \mu$ from system (155^μ), we obtain two independent systems

$$\begin{aligned}&(\alpha(2\tau+\mu-2)+2(r-\tau))h_1^{(2\tau+\mu,2(r-\tau))}\\ &\qquad-\widetilde{\alpha}(2\tau+\mu+2)h_1^{(2\tau+\mu+2,2(r-\tau)-2)}+2\widetilde{\alpha}h_2^{(2\tau+\mu+1,2(r-\tau)-1)}\\ &\qquad=\widehat{Y}_1^{(2\tau+\mu+1,2(r-\tau))}\qquad(-(1+\mu)/2\leq\tau\leq r),\\ &(\alpha(2\tau+\mu-1)+2(r-\tau))h_2^{(2\tau+\mu-1,2(r-\tau)+1)}\\ &\qquad-\widetilde{\alpha}(2\tau+\mu+1)h_2^{(2\tau+\mu+1,2(r-\tau)-1)}-h_1^{(2\tau+\mu,2(r-\tau))}\\ &\qquad=\widehat{Y}_2^{(2\tau+\mu,2(r-\tau)+1)}\qquad(0\leq\tau\leq r);\end{aligned}\tag{156^μ}$$

$$\begin{aligned}&(\alpha(2\tau+\mu-3)+2(r-\tau)+1)h_1^{(2\tau+\mu-1,2(r-\tau)+1)}\\ &\qquad-\widetilde{\alpha}(2\tau+\mu+1)h_1^{(2\tau+\mu+1,2(r-\tau)-1)}+2\widetilde{\alpha}h_2^{(2\tau+\mu,2(r-\tau))}\\ &\qquad=\widehat{Y}_1^{(2\tau+\mu,2(r-\tau)+1)}\qquad(0\leq\tau\leq r),\\ &(\alpha(2\tau+\mu)+2(r-\tau)-1)h_2^{(2\tau+\mu,2(r-\tau))}\\ &\qquad-\widetilde{\alpha}(2\tau+\mu+2)h_2^{(2\tau+\mu+2,2(r-\tau)-2)}-h_1^{(2\tau+\mu+1,2(r-\tau)-1)}\\ &\qquad=\widehat{Y}_2^{(2\tau+\mu+1,2(r-\tau))}\qquad(-(1+\mu)/2\leq\tau\leq r).\end{aligned}\tag{157^μ}$$

In systems (156^0) and (156^1), we set $\widehat{Y}^r_{i,\tau} = \widehat{Y}_i^{(2\tau+2-i+\mu,2(r-\tau)-1+i)}$, $\widehat{Y}^r_i = (\widehat{Y}^r_{i,(i-2)\mu},\dots,\widehat{Y}^r_{i,r})$, $h^r_{i,\tau} = h_i^{(2\tau+1-i+\mu,2(r-\tau)-1+i)}$, and $h^r_i = (h^r_{i,(i-1)(1-\mu)},\dots,h^r_{i,r})$ ($i=1,2$; $\mu=0,1$). By passing to the new notation and by substituting $h^r_{1,\tau}$ from the second subsystem in (156^μ) into the first one, we obtain the system

$$c_\tau h^r_{2,\tau} + a_\tau h^r_{2,\tau+1} + b_\tau h^r_{2,\tau+2} = Y^r_{0,\tau} \qquad (\tau=-\mu,\dots,r),$$

where

$$\begin{aligned} a_\tau &= -\widetilde{\alpha}(2\alpha(4\tau^2+2\tau+(4\tau+2)\mu)+(2r-2\tau-1)(4\tau+3+2\mu)-3),\\ b_\tau &= (2\tau+2+\mu)(2\tau+3+\mu),\\ c_\tau &= (\alpha(2\tau-2+\mu)+2(r-\tau))(\alpha(2\tau-1+\mu)+2(r-\tau)),\\ Y^r_{0,\tau} &= \widehat{Y}^r_{1,\tau} + (\alpha(2\tau-2+\mu)+2(r-\tau))\widehat{Y}^r_{2,\tau} - \widetilde{\alpha}(2\tau+2+\mu)\widehat{Y}^r_{2,\tau+1}, \end{aligned}$$

or the system

$$\Theta^r h^r_2 = Y^r_0, \tag{158^μ}$$

where Θ^r is the tridiagonal $(r+1+\mu)\times(r+\mu)$ matrix with entries $\theta_{\tau,\tau}=a_\tau$ ($\tau=-\mu,\dots,r-1$), $\theta_{\tau,\tau+1}=b_\tau$ ($\tau=-\mu,\dots,r-2$), and $\theta_{\tau,\tau-1}=c_\tau$ ($\tau=1-\mu,\dots,r$) and Y^r_0 is the vector $Y^r_0 = (Y^r_{0,-\mu},\dots,Y^r_{0,r})$; here we assume that $\widehat{Y}^r_{2,-1},\widehat{Y}^r_{2,r+1}=0$.

In systems (157^0) and (157^1), we set $\widehat{Y}^r_{i,\tau} = \widehat{Y}_i^{(2\tau-1+i+\mu,2(r-\tau)+2-i)}$, $\widehat{Y}^r_i = (\widehat{Y}^r_{i,(1-i)\mu},\dots,\widehat{Y}^r_{i,r})$, $h^r_{i,\tau} = h_i^{(2\tau-2+i+\mu,2(r-\tau)+2-i)}$, and $h^r_i = (h^r_{i,(2-i)(1-\mu)},\dots,h^r_{i,r})$.

By substituting $2\widetilde{\alpha}h^r_{2,\tau}$ from the first subsystem in (157^μ) into the second one, we obtain the system

$$c_\tau h^r_{1,\tau} + a_\tau h^r_{1,\tau+1} + b_\tau h^r_{1,\tau+2} = Y^r_{0,\tau} \qquad (\tau=-\mu,\dots,r),$$

where

$$\begin{aligned} a_\tau &= -\widetilde{\alpha}(2\alpha(4\tau^2+2\tau-1+(4\tau+2)\mu)+(2r-2\tau-1)(4\tau+3+2\mu)-2),\\ b_\tau &= (2\tau+2+\mu)(2\tau+3+\mu),\\ c_\tau &= (\alpha(2\tau-3+\mu)+2(r-\tau)+1)(\alpha(2\tau+\mu)+2(r-\tau)-1),\\ Y^r_{0,\tau} &= (\alpha(2\tau+\mu)+2(r-\tau)-1)\widehat{Y}^r_{1,\tau} - \widetilde{\alpha}(2\tau+2+\mu)\widehat{Y}^r_{1,\tau+1} - 2\widetilde{\alpha}\widehat{Y}^r_{2,\tau}, \end{aligned}$$

or the system

$$\Theta^r h^r_1 = Y^r_0, \tag{159^μ}$$

where Θ^r and Y^r_0 have the same structure as in system (158^μ) and $\widehat{Y}^r_{1,-1},\widehat{Y}^r_{1,r+1}=0$.

1^0. Let us study system (158^0). To annihilate $c_1,\dots,c_r$, we introduce numbers d_τ by the formulas

$$d_0 = a_0 \neq 0, \qquad d_\tau = a_\tau - c_\tau b_{\tau-1}/d_{\tau-1} \quad \text{if} \quad d_{\tau-1}\neq 0 \quad (1\le\tau\le r-1). \tag{160_1}$$

The following two cases are possible:
(i) there exists a $\breve{\tau}$ ($1\le\breve{\tau}\le r-1$) such that $d_0,\dots,d_{\breve{\tau}-1}\neq 0$ and $d_{\breve{\tau}}=0$;
(ii) $d_0,\dots,d_{r-1}\neq 0$. In the latter case, we set $\breve{\tau}=r$.

Lemma 3. *For the numbers d_τ in (160_1), one has the closed-form expression*

$$d_\tau = -\widetilde{\alpha}(2\tau+3)(2\alpha\tau+2(r-\tau)-2), \tag{161_1}$$

where $\tau=0,\dots,\breve{\tau}$ in case (i) *and $\tau=0,\dots,r-1$ in case* (ii).

We split the set of pairs (α,r) with $\alpha\neq 0$ and $r\ge 2$ into two families

$$\{\alpha,r\}^1_1 = \{-k/l,\ (k+l)n+1\}_{k,l,n\in\mathbb{N}}, \quad \tau_1 = ln; \qquad \{\alpha,r\}^0_1 = \{(\alpha,r)\notin\{\alpha,r\}^1_1\}.$$

Lemma 4. *If $(\alpha,r)\in\{\alpha,r\}^1_1$, then case* (i) *with $\breve{\tau}=\tau_1$ holds for d_τ in* $(161)_1$; *if $(\alpha,r)\in\{\alpha,r\}^0_1$, then case* (ii) *holds and $\breve{\tau}=\tau_0=r$.*

System (158^0) can be reduced by the Gauss method to the form

$$\Theta_d^r h_2^r = Y_d^r, \tag{162_1}$$

where

$$\Theta_d^r = \begin{pmatrix} d_0 & b_0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & d_1 & \ddots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & d_{\breve\tau-1} & b_{\breve\tau-1} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & b_{\breve\tau} & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & c_{\breve\tau+1} & a_{\breve\tau+1} & b_{\breve\tau+1} & \ddots & \vdots & 0 \\ 0 & 0 & \dots & 0 & 0 & c_{\breve\tau+2} & a_{\breve\tau+2} & \ddots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \ddots & a_{r-2} & b_{r-2} \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \ddots & c_{r-1} & a_{r-1} \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & c_r \end{pmatrix},$$

Y_d^r has the entries $Y_{d,0}^r = Y_{0,0}^r$, $Y_{d,\tau}^r = Y_{0,\tau}^r - (c_\tau/d_{\tau-1})Y_{d,\tau-1}^r$ ($\tau = 1,\dots,\breve\tau$), and $Y_{d,\tau}^r = Y_{0,\tau}^r$ ($\tau = \breve\tau+1,\dots,r$), and the numbers a_τ, b_τ, c_τ, and d_τ are defined in (158^0) and (161_1). Obviously,

$$Y_{d,\tau}^r = \sum_{j=0}^{\tau}(-1)^{\tau-j}Y_{0,j}^r \prod_{\nu=j+1}^{\tau} c_\nu/d_{\nu-1} \qquad (\tau = 0,\dots,\breve\tau).$$

The first $\breve\tau$ equations in system (162_1) are uniquely solvable for $h_{2,1}^r,\dots,h_{2,\breve\tau}^r$, and the $\breve\tau$th equation has the form

$$0\cdot h_{2,\breve\tau}^r + 0\cdot h_{2,\breve\tau+1}^r + b_{\breve\tau}h_{2,\breve\tau+2}^r = Y_{d,\breve\tau}^r \qquad (h_{2,r+1}^r, h_{2,r+2}^r = 0). \tag{163_1}$$

In case (ii), $\breve\tau = r$, Θ_d^r is a bidiagonal matrix with zero last row, and Eq. (163_1), which is the last in (162_1), has the form $0\cdot h_{2,r}^r = Y_{d,r}^r$.

In case (i), let us single out the last $r-\breve\tau = kn+1 \ge 2$ equations in (162_1):

$$\Theta_d^{r+}h_2^{r+} = Y_d^{r+}, \tag{164_1}$$

where Θ_d^{r+} is a tridiagonal upper triangular matrix with main diagonal $c_{\breve\tau+1},\dots,c_r$,

$$h_2^{r+} = (h_{2,\breve\tau+1}^r,\dots,h_{2,r}^r), \qquad Y_d^{r+} = (Y_{0,\breve\tau+1}^r,\dots,Y_{0,r}^r).$$

Let us split the pairs (α, r) into three disjoint families in a different way,

$$\{\alpha,r\}_1^{1c} = \{\alpha,r\}_1^1, \qquad \tau_1^c = ln+1;$$
$$\{\alpha,r\}_1^{2c} = \{-2k/(2l-1),(2n-1)(k+l)-n+1\}_{k,l,n\in\mathbb{N}}, \qquad \tau_2^c = 2ln-l-n+1;$$
$$\{\alpha,r\}_1^{0c} = \{(\alpha,r)\notin\{\alpha,r\}_1^{1c}\cup\{\alpha,r\}_1^{2c}\}.$$

Set $c_\tau = c'_\tau c''_\tau$, where $c'_\tau = \alpha(2\tau-2)+2(r-\tau)$ and $c''_\tau = \alpha(2\tau-1)+2(r-\tau)$.

Lemma 5. *If* $(\alpha,r)\in\{\alpha,r\}_1^{\nu c}$ ($\nu = 1,2$), *then* $c_\tau = 0$ *in* (158^0) *only for* $\tau = \tau_\nu^c$, *and if* $(\alpha,r)\in\{\alpha,r\}_1^{0c}$, *then* $c_1,\dots,c_r \ne 0$.

Corollary 19. *In system* (158^0), *one has* $c_1, \dots, c_r \neq 0$ *except that* $c'_{\tau_1+1} = 0$ *if* $(\alpha, r) \in \{\alpha, r\}_1^1$ *and* $c''_{2ln-l-n+1} = 0$ *if* $(\alpha, r) \in \{\alpha, r\}_1^{2c} \subset \{\alpha, r\}_1^0$.

Consequently, in case (i), one can introduce the matrix $G = \{g_{\tau j}\}_{\tau, j=\check{\tau}+1}^r$ with entries

$$\begin{aligned} &g_{\tau j} = 0 \qquad \forall \tau = \check{\tau}+1, \dots, r \qquad (\check{\tau}+1 \le j \le \tau - 1) \qquad (g_{\check{\tau}+1\check{\tau}} = 0), \\ &g_{\tau\tau} = 1, \qquad g_{\tau j} = -(g_{\tau j-1} a_{j-1} + g_{\tau j-2} b_{j-2})/c_j \qquad (\tau + 1 \le j \le r). \end{aligned} \tag{165$_1$}$$

Then $G\Theta_d^{r+} = \{g_{\tau j-2} b_{j-2} + g_{\tau j-1} a_{j-1} + g_{\tau j} c_j\}_{\tau, j=\check{\tau}+1}^r = \operatorname{diag}\{c_{\check{\tau}+1}, \dots, c_r\}$, and system (164_1) multiplied by the matrix G on the left is equivalent to the system

$$c_\tau h_{2,\tau}^r = \sum_{j=\tau}^r g_{\tau j} Y_{0,j}^r \qquad (\tau = \check{\tau}+1, \dots, r). \tag{166$_1$}$$

We return to Eq. (163_1) and substitute there the closed-form expression for $Y_{d,\check{\tau}}^r$ in (162_1) and, in case (i), the expression for $h_{2,\check{\tau}+2}^r$ in (166_1) $(c_{\check{\tau}+2} \neq 0)$; then

$$0 \cdot h_{2,\check{\tau}+1}^r = \sum_{j=0}^{\check{\tau}} (-1)^{\check{\tau}-j} \prod_{\nu=j+1}^{\check{\tau}} \frac{c_\nu}{d_{\nu-1}} Y_{0,j}^r - \frac{b_{\check{\tau}}}{c_{\check{\tau}+2}} \sum_{j=\check{\tau}+2}^r g_{\check{\tau}+2\,j} Y_{0,j}^r. \tag{167$_1$}$$

In (167_1), we express $Y_{0,j}^r$ via $\widehat{Y}_{i,j}^r$ $(i = 1, 2)$. To this end, we introduce the constants

$$\begin{aligned} u_j &= \widetilde{\alpha}^{\check{\tau}-j} \prod_{\nu=j+1}^{\check{\tau}} (\alpha(2\nu-1) + 2(r-\nu))/(2\nu+1), \\ v_j &= (-2\alpha + 2(r-j)) u_j/(2j+1) \qquad (j = 0, \dots, \check{\tau}), \\ u_{\check{\tau}+1} &= 0, \qquad v_{\check{\tau}+1} = -2\widetilde{\alpha}(\check{\tau}+1), \\ u_j &= -b_{\check{\tau}} g_{\check{\tau}+2j}/c_{\check{\tau}+2}, \\ v_j &= -b_{\check{\tau}}((\alpha(2j-2) + 2(r-j)) g_{\check{\tau}+2\,j} - 2\widetilde{\alpha} j g_{\check{\tau}+2\,j-1})/c_{\check{\tau}+2} \qquad (j = \check{\tau}+2, \dots, r); \end{aligned} \tag{168$_1$}$$

then, by (158^0) and (161_1), $u_j = (-1)^{\check{\tau}-j} \prod_{\nu=j+1}^{\check{\tau}} c_\nu/d_{\nu-1}$, $v_j = ((2j-2)\alpha + 2(r-j)) u_j - 2j\widetilde{\alpha} u_{j-1}$ $(j = 0, \dots, \check{\tau})$, $u_{\check{\tau}} = 1$, and $v_{\check{\tau}} = (-2\alpha + 2(r-\check{\tau}))/(2\check{\tau}+1)$.

Now, by (158^0), in (167_1), we have

$$\sum_{j=0}^{\check{\tau}} u_j Y_{0,j}^r = \sum_{j=0}^{\check{\tau}} u_j \widehat{Y}_{1,j}^r + \sum_{j=0}^{\check{\tau}+1} v_j \widehat{Y}_{2,j}^r$$

and

$$\sum_{j=\check{\tau}+2}^r g_{\check{\tau}+2\,j} Y_{0,j}^r = \sum_{j=\check{\tau}+2}^r g_{\check{\tau}+2\,j} (\widehat{Y}_{1,j}^r + (\alpha(2j-2) + 2(r-j)) \widehat{Y}_{2,j}^r) - \sum_{j=\check{\tau}+3}^{r+1} 2\widetilde{\alpha} j g_{\check{\tau}+2\,j-1} \widehat{Y}_{2,j}^r.$$

As a result, relation (167_1) acquires the form

$$\sum_{j=0}^{\check{\tau}} (u_j \widehat{Y}_{1,j}^r + v_j \widehat{Y}_{2,j}^r) + v_{\check{\tau}+1} \widehat{Y}_{2,\check{\tau}+1}^r + \sum_{j=\check{\tau}+2}^r (u_j \widehat{Y}_{1,j}^r + v_j \widehat{Y}_{2,j}^r) = 0. \tag{169$_1$}$$

If $(\alpha, r) \in \{\alpha, r\}_1^1$, then, by Lemma 4 and Corollary 19, $\check{\tau} = \tau_1$ and $c_{\tau_1+1} = 0$; therefore, (166_1) with $\tau = \tau_1 + 1$ provides the additional resonance relation $0 \cdot h_{2,\tau_1+1}^r = \sum_{j=\tau_1+1}^r g_{\tau_1+1j} Y_{0,j}^r$, which, by analogy with (169_1), can be represented in the form

$$\sum_{j=\tau_1+1}^r (u_j^1 \widehat{Y}_{1,j}^r + v_j^1 \widehat{Y}_{2,j}^r) = 0, \qquad (\alpha, r) \in \{\alpha, r\}_1^1 \qquad (h_{2,\tau_1+1}^r \text{ is arbitrary}), \tag{170$_1$}$$

where $u_j^1 = g_{\tau_1+1j}$, $v_j^1 = 2(\alpha(j-1) + r - j) g_{\tau_1+1j} - 2\widetilde{\alpha} j g_{\tau_1+1\,j-1}$, and $g_{\tau j}$ is given by (165_1).

Let us rewrite relations (169_1) and (170_1) in terms of the coefficients of system (3).

By definition (168_1), we have $u_j, v_j = 0$ for $0 \le j \le \breve{\tau} - 1$ provided that

$$\prod_{\nu=j+1}^{\breve{\tau}} (\alpha(2\nu - 1) + 2(r - \nu)) = 0$$

(i.e., $c''_{j+1} \cdots c''_{\breve{\tau}} = 0$). In addition, $v_j = 0$ $(0 \le j \le \breve{\tau})$ if $-2\alpha + 2(r - j) = 0$, i.e., if $j = r - \alpha$.

Let us introduce the family

$$\{\alpha, r\}_1^v = \{n, m\}_{m,n\in\mathbb{N},\, m\ge 2} \subset (\{\alpha, r\}_1^0 \backslash \{\alpha, r\}_1^{2c}).$$

Then $v_j = 0 \Leftrightarrow (\alpha, r) \in \{\alpha, r\}_1^v$ and $j = m - n$. By using Corollary 19, one can readily transform Eq. (169_1) in case (ii) for $\breve{\tau} = r$.

Let $(\alpha, r) \in \{\alpha, r\}_1^{2c} \subset \{\alpha, r\}_1^0$; i.e., $\alpha = -2k/(2l-1)$ and $r = (2n-1)(k+l)-n+1$ $(k, l, n \in \mathbb{N})$. Then $\tau_2^c = 2ln - l - n + 1$ $(1 \le \tau_2^c \le r - 1)$; therefore, $u_j, v_j \ne 0$ $(j = \tau_2^c, \ldots, r)$, since they do not contain $c''_{\tau_2^c} = 0$, and relation (169_1) has the form

$$\sum_{j=\tau_2^c}^{r} (u_j Y_1^{(2j+1, 2(r-j))} + v_j Y_2^{(2j, 2(r-j)+1)}) = \widetilde{c}. \qquad (169_1^{2c})$$

If $(\alpha, r) \in \{\alpha, r\}_1^0 \backslash \{\alpha, r\}_1^{2c}$, then Eq. ($169_1$) acquires the form

$$\sum_{j=0}^{r} (u_j Y_1^{(2j+1, 2(r-j))} + v_j Y_2^{(2j, 2(r-j)+1)}) = \widetilde{c} \qquad (r \ge 2), \qquad (169_1^0)$$

and all u_j, v_j are nonzero except for $v_{m-n} = 0$ for $(\alpha, r) \in \{\alpha, r\}_1^v$.

Let $(\alpha, r) \in \{\alpha, r\}_1^1 = \{\alpha, r\}_1^{1c}$ $(\breve{\tau} = \tau_1)$. Let us estimate u_j and v_j in (169_1) and (170_1) by introducing the following recursive sequence for $\tau = \tau_1 + 1, \ldots, r - 1$:

$$f_{\tau\tau} = -a_\tau, \qquad f_{\tau j} = -a_j - b_{j-1} c_j / f_{\tau j-1} \qquad (\tau + 1 \le j \le r - 1) \quad \text{for} \quad f_{\tau j-1} \ne 0. \qquad (171_1)$$

By induction over j, one can show that

$$g_{\tau\, j} = g_{\tau\, j-1} f_{\tau\, j-1} / c_j \qquad (j = \tau + 1, \ldots, r, \quad g_{\tau\tau} = 1) \qquad (172_1)$$

in (165_1) for $\breve{\tau} = \tau_1$.

Since $\alpha = -k/l$, $r = (k + l)n + 1$, and $\tau_1 = ln$, we have

$$c_\tau = (2n(k + l) + 2 - 2\tau - k(2\tau - 2)/l)(2n(k + l) + 2 - 2\tau - k(2\tau - 1)/l)$$

in (158^0), and $c_\tau = 0$ for $\tau = ln + (k + 2l)/(2k + 2l)$, $ln + 1$; therefore, $c_\tau > 0$ for $\tau \ge \tau_1 + 2$.

To estimate the numbers $f_{\tau j}$ from below, we introduce $\xi_j = -2(j + 1)(\alpha(2j + 1) + 2(r - j) - 2)$ $(j = \tau_1 + 1, \ldots, r - 1)$. Since $\xi_j = 0$ for $j = -1$, $ln - k/(2k + 2l)$, we have $\xi_j > 0$ for $j \ge ln$. By induction, $f_{\tau j} > \xi_j$ $(\tau = \tau_1 + 1, \tau_1 + 2,\ j = \tau, \ldots, r - 1)$. By virtue of ($165_1$) and ($172_1$), we have $g_{\tau j} > g_{\tau\, j-1} \xi_{j-1} / c_j > 0$ $(j = \tau + 1, \ldots, r)$ for such τ; consequently, $u_j < 0$ and $u_j^1 > 0$.

Since $c'_j = 0$ for $j = \tau_1 + 1$, we have $v^1_{\tau_1+1} = 0$ and $c'_j, \alpha(2j - 1) + 2(r - j) < 0$ for $j \ge \tau_1 + 2$ in (170_1); therefore, the expression

$$(\alpha(2j - 2) + 2(r - j))g_{\tau j} - 2\widetilde{\alpha} j g_{\tau\, j-1}$$

occurring in v_j in (168_1) and in v_j^1 in (170_1) is negative.

As a result, if $(\alpha, r) \in \{\alpha, r\}_1^1$, then Eq. (170_1) acquires the form

$$u^1_{\tau_1+1} Y_1^{(2\tau_1+3,2(r-\tau_1)-2)} + \sum_{j=\tau_1+2}^{r} (u^1_j Y_1^{(2j+1,2(r-j))} + v^1_j Y_2^{(2j,2(r-j)+1)}) = \widetilde{c}, \qquad (170^1_1)$$

where $u^1_j > 0$ and $v^1_j < 0$ for all j; next, $u_j < 0$ and $v_j > 0$ for $j = \tau_1 + 2, \ldots, r$ in Eq. (169_1), and Eq. (169_1) acquires the form

$$\sum_{j=0}^{\tau_1} (u_j Y_1^{(2j+1,2(r-j))} + v_j Y_2^{(2j,2(r-j)+1)}) + v_{\tau_1+1} Y_2^{(2(\tau_1+1),2(r-\tau_1)-1)} + \sum_{j=\tau_1+2}^{r} (u_j Y_1^{(2j+1,2(r-j))} + v_j Y_2^{(2j,2(r-j)+1)}) = \widetilde{c}. \qquad (169^1_1)$$

2^0. Let us study system (158^1). We split the pairs (α, r) $(\alpha \neq 0,\ r \geq 2)$ into three disjoint sets and introduce the corresponding constants τ_ν:

$$\{\alpha, r\}_2^1 = \{-2k/(2l-1), (2n-1)(k+l) - n + 1\}_{k,l,n \in M_2^0 \cup M_2^1},$$
$$M_2^0 = \{k \geq 2,\ l = 0,\ n = 1\}, \qquad M_2^1 = \{k, l, n \in \mathbb{N}\}, \qquad \tau_1 = 2ln - l - n + 1;$$
$$\{\alpha, r\}_2^2 = \{-k/l, (k+l)n\}_{k,l,n \in \mathbb{N}}, \qquad \tau_2 = ln;$$
$$\{\alpha, r\}_2^0 = \{(\alpha, r) \notin \{\alpha, r\}_2^1 \cup \{\alpha, r\}_2^2\}, \qquad \tau_0 = -1.$$

Lemma 6. *If* $(\alpha, r) \in \{\alpha, r\}_2^\nu$ $(\nu = 1, 2)$, *then* $c_{\tau_\nu} = 0$ *and* $c_\tau \neq 0$ $(0 \leq \tau \leq r,\ \tau \neq \tau_\nu)$ *in* Θ^r *in* (158^1). *If* $(\alpha, r) \in \{\alpha, r\}_2^0$, *then* $c_0, \ldots, c_r \neq 0$.

Corollary 20. *If* $(\alpha, r) \in \{\alpha, r\}_2^2$, *then* $c_\tau > 0$ $(0 \leq \tau \leq \tau_2 - 1)$.

We single out the last $r - \tau_\nu + 1$ equations in (158^1) into the separate subsystem

$$\Theta_\nu^{r+} h_2^{r+} = Y_0^{r+} \qquad (\nu = 0, 1, 2), \qquad (164_2)$$

where Θ_ν^{r+} is a tridiagonal upper triangular matrix with main diagonal $c_{\tau_\nu}, \ldots, c_r$,

$$h_2^{r+} = (h^r_{2,\tau_\nu}, \ldots, h^r_{2,r}), \qquad Y_0^{r+} = (Y^r_{0,\tau_\nu}, \ldots, Y^r_{0,r}).$$

If $\nu = 0$, then system (164_2) coincides with system (158^1); only the matrix Θ_0^{r+} has the additional zero first column, which generates the pseudoelements $c_{-1} = 0$ and corresponds to $\tau_0 = -1$ and $h^r_{2,-1}$.

We multiply system (164_2) on the left by the matrix G defined in (165_1), where $\check{\tau} = \tau_\nu - 1$. Then $c_\tau h^r_{2,\tau} = \sum_{j=\tau}^r g_{\tau j} Y^r_{0,j}$ $(\tau = \tau_\nu, \ldots, r)$.

By expressing the components $Y^r_{0,j}$ via $\widehat{Y}^r_{i,j}$ in accordance with (158^1), we obtain the relations

$$c_\tau h^r_{2,\tau} = \sum_{j=\tau}^{r} (g_{\tau j} \widehat{Y}^r_{1,j} + ((\alpha(2j-1) + 2(r-j)) g_{\tau j} - \widetilde{\alpha}(2j+1) g_{\tau j-1}) \widehat{Y}^r_{2,j}).$$

By Lemma 6, only c_{τ_ν} are zero in these relations; therefore, we have

$$\sum_{j=\tau_\nu}^{r} (u^\nu_j \widehat{Y}^r_{1,j} + v^\nu_j \widehat{Y}^r_{2,j}) = 0, \qquad (\alpha, r) \in \{\alpha, r\}_2^\nu \qquad (\nu = 0, 1, 2) \qquad (169_2)$$

only for $\tau = \tau_\nu$, where $u^\nu_j = g_{\tau_\nu j}$ and $v^\nu_j = (\alpha(2j-1)+2(r-j))g_{\tau_\nu j} - \widetilde{\alpha}(2j+1)g_{\tau_\nu j-1}$ $(j = \tau_\nu, \ldots, r)$, the $g_{\tau j}$ are given by (165_1) with $\breve{\tau} = \tau_\nu - 1$, and the h^r_{2,τ_ν}, $\nu = 1, 2$, are not subjected to any constraints.

If $\nu = 0$ $(\tau_0 = -1)$, then $h^r_{2,-1}$ and $\widehat{Y}^r_{2,-1}$ are missing, and relation (169_2) has the form $\sum_{j=-1}^r u^0_j \widehat{Y}^r_{1,j} + \sum_{j=0}^r v^0_j \widehat{Y}^r_{2,j} = 0$ and guarantees the solvability of system (158^1).

Let $\nu = 1, 2$. Then the first $\tau_\nu + 1$ equations with $\tau_\nu + 1$ unknowns $h^r_{2,0}, \ldots, h^r_{2,\tau_\nu}$ $(0 \le \tau_\nu \le r)$ remain unsolved in system (158^1). For their solvability, it suffices to show that $\det \Theta^{r-}_\nu \ne 0$, where Θ^{r-}_ν is the tridiagonal matrix with main diagonal $a_{-1}, \ldots, a_{\tau_\nu - 1}$, subdiagonal $c_0, \ldots, c_{\tau_\nu-1}$, and superdiagonal $b_{-1}, \ldots, b_{\tau_\nu-2}$ $(\nu = 1, 2)$.

The Gauss method can be used to transform the matrix Θ^{r-}_ν into the bidiagonal matrix $\breve{\Theta}^{r-}_\nu$ with main diagonal $e_{-1}, \ldots, e_{\tau_\nu-1}$ and subdiagonal $c_0, \ldots, c_{\tau_\nu-1}$ by the recursive formulas

$$e_{\tau_\nu-1} = a_{\tau_\nu-1}, \qquad e_\tau = a_\tau - b_\tau c_{\tau+1}/e_{\tau+1} \qquad (\tau = \tau_\nu - 2, \ldots, -1) \tag{160_2}$$

provided that $e_{\tau_\nu-1}, \ldots, e_0 \ne 0$, where the entries a_τ, b_τ, and c_τ are defined in (158^1).

Lemma 7. *In the matrix* $\breve{\Theta}^{r-}_\nu$ $(\nu = 1, 2)$, *the diagonal entries* $e_{\tau_\nu-1}, \ldots, e_{-1}$ *given by formulas* (160_2) *are nonzero.*

Proof. Let $(\alpha, r) \in \{\alpha, r\}^1_2$. If $k, l, n \in M^0_2$, then $(\alpha, r) \in \{2k, k\}_{k\ge2}$, $\tau_1 = 0$, and $\breve{\Theta}^{r-}_1 = a_{-1}$. Therefore, $e_{-1} = a_{-1} = -2k + 2 \ne 0$.

If $k, l, n \in M^1_2$, then $\alpha < 0$ and e_τ in (160_2) admits the closed-form expression

$$e_\tau = (2\tau + 1)(2\alpha\tau + 2(r - \tau)) \qquad (\tau = \tau_1 - 1, \ldots, -1). \tag{161_2}$$

It follows from the relation $e_\tau = 0$ that $\tau = r/(1-\alpha) = 2ln - l - n + (k+l-1)/(2k+2l-1) \notin \mathbb{Z}$ for $k, l, n \in \mathbb{Z}$. Therefore, $e_{\tau_1-1}, \ldots, e_{-1} \ne 0$ in (161_2).

Let $(\alpha, r) \in \{\alpha, r\}^2_2$. Then $\alpha < 0$. Let us show that $e_{\tau_2-1}, \ldots, e_0 > 0$. Set

$$\zeta_\tau = (2\tau + 1)(\alpha(2\tau - 1) + 2(r - \tau)).$$

Then $\zeta_\tau > 0 \Leftrightarrow \tau < \tau_2 + k/(2l + 2k)$ for $\tau \ge 0$ and, by induction, $e_\tau > \zeta_\tau$ $(\tau = \tau_2 - 1, \ldots, 0)$.

For e_{-1}, we set $\eta_\tau = (2\tau + 2)(\alpha(2\tau - 1) + 2(r - \tau))$. Then $\eta_\tau > \zeta_\tau > 0$ for $0 \le \tau \le \tau_2 - 1$ and $\eta_{-1} = 0$. By induction, $e_\tau < \eta_\tau$ $(\tau = \tau_2 - 1, \ldots, -1)$. The proof of the lemma is complete.

Let $(\alpha, r) \in \{\alpha, r\}^1_2$, $k, l, n \in M^0_2$. Let us estimate u^1_j and v^1_j in (169_2) $(j = \tau_1, \ldots, r)$. In (158^1), we have $c_\tau = (4k\tau - 2\tau)(4k\tau + 2(k - \tau))$; i.e., $c_\tau = 0$ for $\tau = -k/(2k-1), 0$; consequently, $c_\tau > 0$ for $\tau \ge 1 = \tau_1 + 1$. By (171_1), formula (172_1) $g_{\tau_1 j} = g_{\tau_1 j-1} f_{\tau_1 j-1}/c_j$ holds for $j = \tau_1 + 1, \ldots, r$ such that $f_{\tau_1 j-1} \ne 0$.

To estimate $f_{\tau_1 j}$, set

$$\xi_j = (2j + 3)(\alpha(2j + 1) + 2(r - j) - 2) > 0,$$
$$\eta_j = (2j + 3)(\alpha(2j + 2) + 2(r - j) - 2) > 0$$

for $j \ge 0$. By induction, $\xi_j < f_{\tau_1 j} < \eta_j$ $(j = \tau_1, \ldots, r - 1)$. Then

$$g_{\tau_1 \tau_1} = 1, \qquad 0 < g_{\tau_1 j-1}\xi_{j-1}/c_j < g_{\tau_1 j} < g_{\tau_1 j-1}\eta_{j-1}/c_j \qquad (j = \tau_1 + 1, \ldots, r).$$

Therefore, first, $u^1_j > 0$, and second, since $\varkappa_2 = \alpha(2j-1) + 2(r-j) = 0$ for $j = \tau_1$, we have $v^1_{\tau_1} = 0$ in (169_2). If $j \ge 1 = \tau_1 + 1$, then $\varkappa_2 > 0$ and $v^1_j < g_{\tau_1 j-1}(\varkappa_2 \eta_{j-1}/c_j - 2j - 1) = 0$. Therefore, if $(\alpha, r) \in \{\alpha, r\}^1_2$, $k, l, n \in M^0_2$, then all $u^1_j > 0$ and all $v^1_j < 0$ except for $v^1_{\tau_1} = 0$, and relation (169_2) has the form

$$u^1_{\tau_1} Y_1^{(2\tau_1+2, 2(r-\tau_1))} + \sum_{j=\tau_1+1}^r \left(u^1_j Y_1^{(2j+2, 2(r-j))} + v^1_j Y_2^{(2j+1, 2(r-j)+1)}\right) = \widetilde{c}. \tag{169^1_2}$$

Let us show that relation (169_2^1) also holds for $k, l, n \in M_2^1$. Since $\tau_1 = 2ln - l - n + 1$, we have $c_\tau > 0$ for $\tau \geq \tau_1 + 1$.

To estimate $f_{\tau_1 j}$, we set $\zeta_j = -(2j+3)(\alpha(2j+2)+2(r-j)-2) > 0$ for $j \geq \tau_1 - 1$. By induction, $f_{\tau_1 j} > \zeta_j$ $(j = \tau_1, \dots, r-1)$. Then $u_j^1 > 0$, since $g_{\tau_1 \tau_1} = 1$ and $g_{\tau_1 j} > g_{\tau_1 j-1}\zeta_{j-1}/c_j > 0$ $(j = \tau_1 + 1, \dots, r)$. Since $\varkappa_2 = 0$ for $j = \tau_1$, we have $v_{\tau_1}^1 = 0$. For $j \geq \tau_1 + 1$, we have $\varkappa_2 < 0$ and $v_j^1 < g_{\tau_1 j-1}(\varkappa_2 \zeta_{j-1}/c_j + 2j + 1) = 0$.

Now let $(\alpha, r) \in \{\alpha, r\}_2^2$ $(\tau_2 = ln)$. Let us estimate u_j^2 and v_j^2 in (169_2) for $j = \tau_2, \dots, r$. In (158^1), we have

$$c_\tau = 2((1-2\tau)k/l + 2n(k+l) - 2\tau)(-k\tau/l + n(k+l) - \tau) > 0$$

for $\tau \geq \tau_2 + 1$, since $c_\tau = 0$ for $\tau = ln,\ ln + k/(2l+2k)$.

By induction, $f_{\tau j}$ in (171_1) with $\tau = \tau_2$ satisfies the formula

$$f_{\tau_2 j} = -(2j+4)(\alpha(2j+1) + 2(r-j) - 2) \qquad (j = \tau_2, \dots, r).$$

Since $f_{\tau_2 j} = 0$ for $j = -2,\ ln - (2l+k)/(2l+2k)$, we have $f_{\tau_2 j} > 0$ for $j \geq ln = \tau_2$.

By (172_1), in (169_2), we have $u_j^2 > 0$ $(j = \tau_2, \dots, r)$, $v_{\tau_2}^2 = \alpha(2\tau_2 - 1) + 2(r - \tau_2) = k/l > 0$, and $2\alpha - 2r + 2j,\ 2\alpha j + 2(r-j) < 0$ for $j = \tau_2 + 1, \dots, r$; therefore,

$$v_j^2 = g_{\tau_2 j-1}(2\alpha - 2r + 2j)/(2\alpha j + 2(r-j)) > 0.$$

As a result, if $(\alpha, r) \in \{\alpha, r\}_2^2$, then $u_j^2, v_j^2 > 0$, and relation (169_2) acquires the form

$$\sum_{j=\tau_2}^{r} (u_j^2 Y_1^{(2j+2, 2(r-j))} + v_j^2 Y_2^{(2j+1, 2(r-j)+1)}) = \widetilde{c}. \qquad (169_2^2)$$

Let $(\alpha, r) \in \{\alpha, r\}_2^0$. Then $u_{-1}^0 = 1$ in (169_2), but $v_{-1}^0 = 2r - 3\alpha + 2 = 0$ for $\alpha = (2r+2)/3$. If $j = 1$, then, by (165_1),

$$\begin{aligned} u_1^0 = g_{-1\,1} &= -(g_{-1\,0} a_0 + b_{-1})/c_1 \\ &= (-2\alpha + 3\alpha r + 3r^2 - 9r + 4)/(r(2r-\alpha)(\alpha + 2r - 2)(\alpha + r - 1)) = 0 \end{aligned}$$

for $\alpha = -(3r^2 - 9r + 4)/(3r - 2)$; i.e., the factors u_j^0 and v_j^0 can be zero. As a result, Eq. (169_2) has the form

$$Y_1^{(0, 2r+2)} + \sum_{j=0}^{r} (u_j^0 Y_1^{(2j+2, 2(r-j))} + v_j^0 Y_2^{(2j+1, 2(r-j)+1)}) = \widetilde{c}. \qquad (169_2^0)$$

3^0. Let us study system (159^0). We split the pairs (α, r) $(\alpha \neq 0,\ r \geq 2)$ into three disjoint sets and introduce the corresponding constants τ:

$$\begin{aligned} &\{\alpha, r\}_3^1 = \{-(2k-1)/(2l-1), (2n-1)(k+l-1)+1\}_{k,l,n \in M_3^0 \cup M_3^1}, \\ &\qquad \tau_1 = 2ln - l - n + 2, \quad \text{where} \quad M_3^0 = \{k \geq 2,\ l = 0,\ n = 1\}, \quad M_3^1 = \{k, l, n \in \mathbb{N}\}; \\ &\{\alpha, r\}_3^2 = \{-(2k-1)/(2l), (2n-1)(k+l) - n + 1\}_{k,l,n \in M_3^1 \cup M_3^2}, \\ &\qquad \tau_2 = l(2n-1), \quad \text{where} \quad M_3^2 = \{k = 0,\ l \geq 2,\ n = 1\}; \\ &\{\alpha, r\}_3^0 = \{(\alpha, r) \notin \{\alpha, r\}_3^1 \cup \{\alpha, r\}_3^2\}, \qquad \tau_0 = 0. \end{aligned}$$

Lemma 8. *If* $(\alpha, r) \in \{\alpha, r\}_3^\nu$ $(\nu = 1, 2)$, *then in the matrix* Θ^r *in* (159^0), *one has* $c_{\tau_\nu} = 0$ *and* $c_\tau \neq 0$ $(1 \leq \tau \leq r,\ \tau \neq \tau_\nu)$. *If* $(\alpha, r) \in \{\alpha, r\}_3^0$, *then* $c_1, \dots, c_r \neq 0$.

Corollary 21. *If* $(\alpha, r) \in \{\alpha, r\}_3^1$ *and* $k, l, n \in M_3^1$, *then* $c_\tau > 0$ *for* $\tau = 1, \dots, \tau_1 - 2$ *and* $c_{\tau_1 - 1} < 0$; *if* $(\alpha, r) \in \{\alpha, r\}_3^2$, *then* $c_\tau > 0$ *for* $\tau = 1, \dots, \tau_2 - 1$.

We single out the last $r-\tau_\nu+1$ equations in (159^0) into the separate subsystem

$$\Theta_\nu^{r+}h_1^{r+}=Y_0^{r+}\quad(\nu=0,1,2),\tag{164_3}$$

where Θ_ν^{r+} is a tridiagonal upper triangular matrix with main diagonal $c_{\tau_\nu},\dots,c_r$,

$$h_1^{r+}=(h_{1,\tau_\nu}^r,\dots,h_{1,r}^r),\qquad Y_0^{r+}=(Y_{0,\tau_\nu}^r,\dots,Y_{0,r}^r).$$

For $\nu=0$, system (164_3) essentially coincides with system (159^0).

By treating (164_3) by analogy with system (164_2), we obtain the relations

$$c_\tau h_{1,\tau}^r=\sum_{j=\tau}^r(((2\alpha j+2(r-j)-1)g_{\tau j}-2\widetilde{\alpha}jg_{\tau j-1})\widehat{Y}_{1,j}^r-2\widetilde{\alpha}g_{\tau j}\widehat{Y}_{2,j}^r),$$

which imply that

$$\sum_{j=\tau_\nu}^r(u_j^\nu\widehat{Y}_{1,j}^r+v_j^\nu\widehat{Y}_{2,j}^r)=0,\qquad(\alpha,r)\in\{\alpha,r\}_3^\nu\qquad(\nu=0,1,2),\tag{169_3}$$

where $u_j^\nu=(2\alpha j+2(r-j)-1)g_{\tau_\nu j}-2\widetilde{\alpha}jg_{\tau_\nu j-1}$, $v_j^\nu=-2\widetilde{\alpha}g_{\tau_\nu j}$ $(j=\tau_\nu,\dots,r)$, the $g_{\tau j}$ are given by (165_1) with $\breve{\tau}=\tau_\nu-1$, and the h_{1,τ_ν}^r, $\nu=1,2$, are not subjected to any constraints.

If $\nu=0$ $(\tau_0=0)$, then $h_{1,0}^r$ is lacking and relation (169_3) acquires the form

$$\sum_{j=0}^r u_j^0\widehat{Y}_{1,j}^r+\sum_{j=0}^r v_j^0\widehat{Y}_{2,j}^r=0$$

and ensures the solvability of system (158_3).

Let $\nu=1,2$. Then the first τ_ν equations with τ_ν unknowns $h_{1,1}^r,\dots,h_{1,\tau_\nu}^r$ $(1\le\tau_\nu\le r)$ remain unsolved in system (158_3).

Let us show that $\det\Theta_\nu^{r-}\ne0$, where Θ_ν^{r-} is the tridiagonal matrix with main diagonal $a_0,\dots,a_{\tau_\nu-1}$, subdiagonal $c_1,\dots,c_{\tau_\nu-1}$, and superdiagonal $b_0,\dots,b_{\tau_\nu-2}$.

The Gauss method can be used to reduce the matrix Θ_ν^{r-} to the bidiagonal matrix $\breve{\Theta}_\nu^{r-}$ with diagonals $d_0,\dots,d_{\tau_\nu-1}$ and $b_0,\dots,b_{\tau_\nu-2}$ by the recursive formulas

$$d_0=a_0,\qquad d_\tau=a_\tau-b_{\tau-1}c_\tau/d_{\tau-1}\qquad(\tau=1,\dots,\tau_\nu-1)\tag{160_3}$$

provided that $d_0,\dots,d_{\tau_\nu-2}\ne0$, where a_τ, b_τ, and c_τ are defined in (159^0).

Lemma 9. *In the matrix $\breve{\Theta}_\nu^{r-}$ $(\nu=1,2)$, the diagonal entries $d_0,\dots,d_{\tau_\nu-1}$ given by (160_3) are nonzero.*

Proof. Let $(\alpha,r)\in\{\alpha,r\}_3^1$. If $k,l,n\in M_3^0$, then $(\alpha,r)\in\{2k-1,k\}_{k\ge2}$, $\tau_1=1$, and $\breve{\Theta}_1^{r-}=a_0$. Therefore, $d_0=a_0=3-2k\ne0$.

Let $k,l,n\in M_3^1$. Set

$$\zeta_\tau=(2\tau+2)(\alpha(2\tau-1)+2(r-\tau)-1)(2r-2\tau-2\alpha-3)/(2r-2\tau-2\alpha-1)>0$$

for $0\le\tau\le\tau_1-2$. Let us show that $d_\tau>\zeta_\tau$. Since

$$d_0-\zeta_0=(4r^2-2\alpha(2r-1)-1)/(2r-2\alpha-1)>0,$$

we have the inductive assumption. Suppose that $d_{\tau-1}>\zeta_{\tau-1}$. By virtue of ($160_3$),

$$d_\tau>a_\tau-b_{\tau-1}c_\tau/\zeta_{\tau-1},$$

since, by Corollary 21, $b_{\tau-1}c_\tau > 0$ $(\tau = 1, \dots, \tau_1 - 2)$. But

$$\begin{aligned}
a_\tau - b_{\tau-1}c_\tau/\zeta_{\tau-1} = \zeta_\tau \\
\Leftrightarrow\quad & 8\alpha\tau^2 + 4\alpha\tau - 2\alpha + 8r\tau + 6r - 8\tau^2 - 10\tau - 5 \\
& - ((2\tau+1)(2\alpha\tau + 2(r-\tau) - 1)(2r - 2\tau - 2\alpha + 1) \\
& + (2\tau+2)(2r - 2\tau - 2\alpha - 3)(\alpha(2\tau - 1) + 2(r-\tau) - 1))/(2r - 2\tau - 2\alpha - 1) = 0 \\
\Leftrightarrow\quad & -2 + (4r - 4\tau - 4\alpha - 2)/(2r - 2\tau - 2\alpha - 1) = 0;
\end{aligned}$$

i.e., $d_\tau > \zeta_\tau > 0$ for $0 \le \tau \le \tau_1 - 2$.

Since $b_{\tau_1-2}c_{\tau_1-1} < 0$, we have

$$\begin{aligned}
d_{\tau_1-1} &= a_{\tau_1-1} - b_{\tau_1-2}c_{\tau_1-1}/d_{\tau_1-2} < a_{\tau_1-1} - b_{\tau_1-2}c_{\tau_1-1}/\zeta_{\tau_1-2} \\
&= 4\alpha(\tau_1-1)^2 + 2\alpha\tau_1 - 4\alpha + 4r\tau_1 - 4(\tau_1-1)^2 - 10\tau_1 + 4 \\
&\quad - (8\alpha(\tau_1-1)^2 + 12\alpha\tau_1 - 8\alpha)/(2r - 2\tau_1 - 2\alpha + 1) = 0
\end{aligned}$$

after the substitution of α and r.

As a result, if $(\alpha, r) \in \{\alpha, r\}_3^1$ and $k, l, n \in M_3^1$, then $d_\tau \ne 0$ $(\tau = 0, \dots, \tau_1 - 1)$.

Let $(\alpha, r) \in \{\alpha, r\}_3^2$. For $k, l, n \in M_3^1$, we set $\eta_\tau = (2\tau + 3)(2\tau\alpha + 2(r-\tau) - 2) > 0$ for $0 \le \tau \le 2ln - l - 1 = \tau_2 - 1$. By induction, $d_\tau > \eta_\tau$. For $k, l, n \in M_3^2$, we set

$$\xi_\tau = -(2\tau+3)(2\tau\alpha + 2\alpha + 2(r-\tau) - 3) \le 0$$

for $0 \le \tau \le l - 1 = \tau_2 - 1$. By induction, we have $d_\tau < \xi_\tau$. The proof of the lemma is complete.

Let $(\alpha, r) \in \{\alpha, r\}_3^1$, $k, l, n \in M_3^0$ $(\tau_1 = 1)$. Let us estimate u_j^ν and v_j^ν in (169_3) $(j = \tau_1, \dots, r)$. In (159^0), we have

$$c_\tau = ((2k-1)(2\tau-3) + 2k - 2\tau + 1)(2\tau(2k-1) + 2k - 2\tau - 1) > 0$$

for $\tau \ge 2$. By (171_1), formula (172_1) $g_{\tau_1 j} = g_{\tau_1\, j-1} f_{\tau_1\, j-1}/c_j$ holds for $j = \tau_1 + 1, \dots, r$ such that $f_{\tau_1\, j-1} \ne 0$.

To estimate $f_{\tau_1 j}$, we set $\xi_j = (2j+3)(\alpha(2j-1) + 2(r-j) - 1) > 0$ for $j \ge 1$. By induction, $f_{\tau_1 j} > \xi_j$ $(j = \tau_1, \dots, r-1)$. Then $g_{\tau_1\tau_1} = 1$ and $g_{\tau_1 j} > g_{\tau_1\, j-1}\xi_{j-1}/c_j > 0$ $(j = \tau_1 + 1, \dots, r)$. Since $2\alpha j + 2(r-j) - 1 > 0$ for $j \ge 0$, we have $u_j^1 > g_{\tau_1\, j-1}((2\alpha j + 2(r-j) - 1)\xi_{j-1}/c_j - 2j) = g_{\tau_1\, j-1}$ in (169_3).

Therefore, if $(\alpha, r) \in \{\alpha, r\}_3^1$, $k, l, n \in M_3^0$, i.e., if $\alpha = 2r - 1$, then all $u_j^1 > 0$ and $v_j^1 < 0$ in (169_3), and relation (169_3) acquires the form

$$\sum_{j=\tau_1}^{r} \left(u_j^1 Y_1^{(2j, 2(r-j)+1)} + v_j^1 Y_2^{(2j+1, 2(r-j))}\right) = \widetilde{c}. \tag{169_3^1}$$

Let us show that the same equation can be obtained for $k, l, n \in M_3^1$ with the only difference that $u_j^1 < 0$ and $v_j^1 > 0$. Since $\tau_1 = 2ln - l - n + 2$, we have

$$c_\tau = 0 \quad \text{for} \quad \tau = \tau_1 - 2 + (2k + 4l - 3)/(4k + 4l - 4),\ \tau_1;$$

therefore, $c_\tau > 0$ for $\tau \ge \tau_1 + 1$. To estimate $f_{\tau_1 j}$, we set $\zeta_j = -(2j+2)(\alpha(2j-1) + 2(r-j) - 1) > 0$ for $j \ge \tau_1$. By induction, $f_{\tau_1 j} > \zeta_j$ $(j = \tau_1, \dots, r-1)$ and $g_{\tau_1 j} > 0$ $(j = \tau_1, \dots, r)$. Since $2\alpha j + 2(r-j) - 1 < 0$ for $j \ge \tau_1$, we have $u_j^1 < g_{\tau_1\, j-1}(\zeta_{j-1}(2\alpha j + 2(r-j) - 1)/c_j + 2j) = 0$ in (169_3).

Now let $(\alpha, r) \in \{\alpha, r\}_3^2$ and $k, l, n \in M_3^1$. Then $\tau_2 = l(2n-1)$ and $v_{\tau_2}^2 = 2$ in (169_3); if $j = \tau_2 + 1$, then, by (165_1), $v_{\tau_2+1}^2 = 2g_{\tau_2\,\tau_2+1} = -2a_{\tau_2}/c_{\tau_2+1} = 0$ for $k, l, n = 1$; i.e., u_j^2 and v_j^2 can be zero. Here Eq. (169_3) has the form

$$\sum_{j=\tau_2}^{r} \left(u_j^2 Y_1^{(2j, 2(r-j)+1)} + v_j^2 Y_2^{(2j+1, 2(r-j))}\right) = \widetilde{c}. \tag{$169_3^{2_1}$}$$

Let $(\alpha, r) \in \{\alpha, r\}_3^2$ and $k, l, n \in M_3^2$; i.e., $\alpha = 1/(2r)$. Then $\tau_2 = r$, $u_r^2 = 2\alpha r - 1 = 0$, $v_r^2 = -2\widetilde{\alpha} g_{rr} = -2$, and Eq. (169_3) acquires the form

$$-2Y_2^{(2r+1,0)} = \widetilde{c}. \qquad (169_3^{2_2})$$

Let $(\alpha, r) \in \{\alpha, r\}_3^0$. Then $u_0^0, v_0^0 \neq 0$ in (169_3). But, by (165_1), $v_1^0 = -2\widetilde{\alpha} g_{01} = 2\widetilde{\alpha} a_0/c_1 = 0$ for $\alpha = (6r-5)/2$. Equation (169_3) has the form

$$\sum_{j=0}^{r} (u_j^0 Y_1^{(2j, 2(r-j)+1)} + v_j^0 Y_2^{(2j+1, 2(r-j))}) = \widetilde{c}. \qquad (169_3^0)$$

4^0. Let us study system (159^1). Let us introduce a recursive sequence d_τ by setting

$$d_{-1} = a_{-1}, \qquad d_\tau = a_\tau - c_\tau b_{\tau-1}/d_{\tau-1} \quad \text{for} \quad d_{\tau-1} \neq 0 \quad (0 \leq \tau \leq r-1). \qquad (160_4)$$

The following two cases are possible:
(i) there exists a $\breve{\tau}$ $(-1 \leq \breve{\tau} \leq r-1)$ such that $d_{-1}, \ldots, d_{\breve{\tau}-1} \neq 0$ and $d_{\breve{\tau}} = 0$;
(ii) $d_{-1}, \ldots, d_{r-1} \neq 0$. In the latter case, we set $\breve{\tau} = r$.

Lemma 10. *For the elements d_τ in (160_4), we have the closed-form expression*

$$d_\tau = -\widetilde{\alpha}(2\tau+3)\frac{(2\alpha\tau + 2(r-\tau) - 1)(-2\alpha + 2(r-\tau) - 3)}{-2\alpha + 2(r-\tau) - 1}, \qquad (161_4)$$

where $\tau = -1, \ldots, \breve{\tau}$ in case (i) *and $\tau = -1, \ldots, r-1$ in case* (ii).

We split the set of pairs (α, r) with $\alpha \neq 0$ and $r \geq 2$ into three families

$$\{\alpha, r\}_4^1 = \{-(2k-1)/(2l), (k+l)(2n-1) - n + 1\}_{k,l,n\in\mathbb{N}}, \qquad \tau_1 = l(2n-1);$$
$$\{\alpha, r\}_4^2 = \{(2\ell-1)/2, m\}_{\ell, m\in\mathbb{Z}_+,\, m\geq\max\{\ell,2\}}, \qquad \tau_2 = m - \ell - 1;$$
$$\{\alpha, r\}_4^0 = \{(\alpha, r) \notin \{\alpha, r\}_4^1 \cup \{\alpha, r\}_4^2\}, \qquad \tau_0 = r.$$

Using the fact that $\{\alpha, r\}_4^1 \cap \{\alpha, r\}_4^2 \neq \varnothing$, we introduce two more families

$$\{\alpha, r\}_4^{2_1} = \{\alpha, r\}_4^1 \cap \{\alpha, r\}_4^2, \qquad \{\alpha, r\}_4^{2_2} = \{\alpha, r\}_4^2 \backslash \{\alpha, r\}_4^{2_1}.$$

Lemma 11. *One has*

$$\{\alpha, r\}_4^{2_1} = \{-1/2, (3k-1)(2n-1) - n + 1\}_{k,n\in\mathbb{N}}, \qquad \tau_{2_1} = (3k-1)(2n-1) - n;$$
$$\{\alpha, r\}_4^{2_2} = \{-1/2, m\}_{m\in M_4} \cup \{(2\ell-1)/2, m\}_{\substack{\ell, m\in\mathbb{N}\\ m\geq\max\{\ell,2\}}},$$

where $M_4 = \mathbb{N}\backslash(\{1\} \cup \{(3k-1)(2n-1) - n + 1\}_{k,n\in\mathbb{N}})$.

Lemma 12. *If $(\alpha, r) \in \{\alpha, r\}_4^1$, then case* (i) *with $\breve{\tau} = \tau_1$ holds for the elements d_τ in (161_4). If $(\alpha, r) \in \{\alpha, r\}_4^{2_2}$, then one has case* (i) *with $\breve{\tau} = \tau_2$. If $(\alpha, r) \in \{\alpha, r\}_4^0$, then one has case* (ii) *with $\breve{\tau} = \tau_0 = r$.*

The Gauss method can be used to transform system (159^1) into the system

$$\Theta_d^r h_1^r = Y_d^r, \qquad (162_4)$$

where Θ_d^r is the matrix in (162_1) with dimension increased by unity by addition of the first row starting from the entries d_{-1} and b_{-1} and the vector Y_d^r has components $Y_{d,-1}^r = Y_{0,-1}^r$,

$Y^r_{d,\tau} = Y^r_{0,\tau} - (c_\tau/d_{\tau-1})Y^r_{d,\tau-1}$ ($\tau = 0,\ldots,\breve{\tau}$), and $Y^r_{d,\tau} = Y^r_{0,\tau}$ ($\tau = \breve{\tau}+1,\ldots,r$); the elements a_τ, b_τ, c_τ, and d_τ are defined in (159^1) and (161_4); obviously, $Y^r_{d,\tau} = \sum_{j=-1}^{\tau}(-1)^{\tau-j}Y^r_{0,j}\prod_{\nu=j+1}^{\tau} c_\nu/d_{\nu-1}$ ($\tau = -1,\ldots,\breve{\tau}$).

The first $\breve{\tau}+1$ equations in system (162_4) are uniquely solvable for $h^r_{1,0},\ldots,h^r_{1,\breve{\tau}}$, and the equation with index $\breve{\tau}$ has the form

$$0\cdot h^r_{1,\breve{\tau}} + 0\cdot h^r_{1,\breve{\tau}+1} + b_{\breve{\tau}}h^r_{1,\breve{\tau}+2} = Y^r_{d,\breve{\tau}} \qquad (h^r_{1,\,r+1}, h^r_{1,\,r+2} = 0). \tag{163_4}$$

In case (ii), $\breve{\tau} = r$ and Θ^r_d is a bidiagonal matrix with zero last row; Eq. (163_4), that is, the last equation in (162_4), has the form $0\cdot h^r_{1,r} = Y^r_{d,r}$.

In case (i), we single out the last $r-\breve{\tau}\ge 1$ equations in (162_4); they form the system

$$\Theta^{r+}_d h^{r+}_1 = Y^{r+}_d, \tag{164_4}$$

which is similar to system (164_1).

We split the pairs (α, r) into five disjoint sets in a different way:

$$\begin{aligned}
&\{\alpha,r\}^{1c}_4 = \{(2k+1)/2, k\}_{k\ge 2}, \qquad \tau^c_1 = 0;\\
&\{\alpha,r\}^{2c}_4 = \{\alpha,r\}^1_4, \qquad \tau^c_2 = 2ln - l + 1;\\
&\{\alpha,r\}^{3c}_4 = \{(1-2k)/(2l-1), (k+l-1)(2n-1)\}_{k,l,n\in\mathbb{N}}, \qquad \tau^c_3 = 2ln - l - n;\\
&\{\alpha,r\}^{4c}_4 = \{1/(2l+1), l\}_{l\ge 2}, \qquad \tau^c_4 = r;\\
&\{\alpha,r\}^{0c}_4 = \{(\alpha,r)\notin \cup_{\nu=1}^4\{\alpha,r\}^{\nu\,c}_4\}.
\end{aligned}$$

Lemma 13. *If* $(\alpha,r)\in\{\alpha,r\}^{\nu\,c}_4$ ($\nu = 1,\ldots,4$), *then* $c_\tau = 0$ *in* (159^1) *only for* $\tau = \tau^c_\nu$. *If* $(\alpha,r)\in\{\alpha,r\}^{0c}_4$, *then* $c_0,\ldots,c_r\ne 0$.

Corollary 22. *In* (159^1), *one has* $c_0,\ldots,c_r\ne 0$ *except for the following cases*: $c'_{\tau_1+1} = 0$ *if* $(\alpha,r)\in\{\alpha,r\}^1_4$; $c'_0 = 0$ *if* $(\alpha,r)\in\{\alpha,r\}^{1c}_4\subset\{\alpha,r\}^0_4$; $c''_{2ln-l-n} = 0$ *if* $(\alpha,r)\in\{\alpha,r\}^{3c}_4\subset\{\alpha,r\}^0_4$; $c''_r = 0$ *if* $(\alpha,r)\in\{\alpha,r\}^{4c}_4\subset\{\alpha,r\}^0_4$, *where* $c_\tau = c'_\tau c''_\tau$, $c'_\tau = \alpha(2\tau-2)+2(r-\tau)+1$, *and* $c''_\tau = \alpha(2\tau+1)+2(r-\tau)-1$.

By multiplying (164_4) on the left by the matrix G in (165_1), we obtain the system

$$c_\tau h^r_{1,\tau} = \sum_{j=\tau}^{r} g_{\tau j}Y^r_{0,j} \quad (\tau = \breve{\tau}+1,\ldots,r). \tag{166_4}$$

By substituting the closed-form expression for $Y^r_{d,\breve{\tau}}$ in (162_4) into (163_4) and, in case (i) for $\breve{\tau}\le r-2$, also $h^r_{1,\breve{\tau}+2}$ in (166_4) ($c_{\breve{\tau}+2}\ne 0$), we obtain the relations

$$0\cdot h^r_{1,\breve{\tau}+1} = \sum_{j=-1}^{\breve{\tau}}(-1)^{\breve{\tau}-j}\prod_{\nu=j+1}^{\breve{\tau}}\frac{c_\nu}{d_{\nu-1}}Y^r_{0,j} - \frac{b_{\breve{\tau}}}{c_{\breve{\tau}+2}}\sum_{j=\breve{\tau}+2}^{r} g_{\breve{\tau}+2\,j}Y^r_{0,j}. \tag{167_4}$$

In (167_4), we express $Y^r_{0,j}$ via $\widehat{Y}^r_{i,j}$ ($i = 1,2$); to this end, we introduce the constants

$$\begin{aligned}
v_j &= -2\widetilde{\alpha}^{\breve{\tau}-j+1}\frac{2\alpha-2(r-j)+1}{2\alpha-2(r-\breve{\tau})+1}\prod_{\nu=j+1}^{\breve{\tau}}\frac{\alpha(2\nu+1)+2(r-\nu)-1}{2\nu+1},\\
u_j &= \widetilde{\alpha}(\alpha(2j+1)+2(r-j)-1)(-2\alpha+2(r-j)-1)^{-1}v_j \qquad (j=-1,\ldots,\breve{\tau});\\
u_{\breve{\tau}+1} &= -\widetilde{\alpha}(2\breve{\tau}+3); \qquad v_j = 2\widetilde{\alpha}b_{\breve{\tau}}g_{\breve{\tau}+2j}/c_{\breve{\tau}+2} \qquad (j=\breve{\tau}+2,\ldots,r),\\
u_j &= -b_{\breve{\tau}}((\alpha(2j+1)+2(r-j)-1)g_{\breve{\tau}+2j} - \widetilde{\alpha}(2j+1)g_{\breve{\tau}+2j-1})/c_{\breve{\tau}+2}.
\end{aligned} \tag{168_4}$$

Then, by (159^1) and (161_4),

$$(-1)^{\breve\tau-j}\prod_{\nu=j+1}^{\breve\tau}\frac{c_\nu}{d_{\nu-1}}=-\frac{\widetilde\alpha v_j}{2},$$

$$u_j=\widetilde\alpha(-\alpha(2j+1)-2(r-j)+1)v_j/2+(2j+1)v_{j-1}/2\qquad(\tau=-1,\ldots,\breve\tau);$$

$$v_{\breve\tau}=-2\widetilde\alpha,\qquad u_{\breve\tau}=2(\alpha(2\breve\tau+1)+2(r-\breve\tau)-1)/(2\alpha-2(r-\breve\tau)+1).$$

Now in (167_4), we have

$$-\widetilde\alpha\sum_{j=-1}^{\breve\tau}\frac{v_jY^r_{0,j}}{2}=\sum_{j=-1}^{\breve\tau}(\widetilde\alpha(-\alpha(2j+1)-2(r-j)+1)v_j/2+v_{j-1}(2j+1)/2)\widehat Y^r_{1,j})$$
$$+v_{\breve\tau}(2\breve\tau+3)\widehat Y_{1,\breve\tau+1}/2+\sum_{j=-1}^{\breve\tau}v_j\widehat Y^r_{2,j}=\sum_{j=0}^{\breve\tau+1}u_j\widehat Y^r_{1,j}+\sum_{j=-1}^{\breve\tau}v_j\widehat Y^r_{2,j},$$

and

$$\sum_{j=\breve\tau+2}^{r}g_{\breve\tau+2\,j}Y^r_{0,j}=\sum_{j=\breve\tau+2}^{r}g_{\breve\tau+2\,j}((\alpha(2j+1)+2(r-j)-1)\widehat Y^r_{1,j}-2\widetilde\alpha\widehat Y^r_{2,j})$$
$$-\sum_{j=\breve\tau+3}^{r+1}g_{\breve\tau+2\,j-1}\widetilde\alpha(2j+1)\widehat Y^r_{1,j}.$$

As a result, Eq. (167_4) acquires the form

$$v_{-1}\widehat Y^r_{2,-1}+\sum_{j=0}^{\breve\tau}(u_j\widehat Y^r_{1,j}+v_j\widehat Y^r_{2,j})+u_{\breve\tau+1}\widehat Y^r_{1,\breve\tau+1}+\sum_{j=\breve\tau+2}^{r}(u_j\widehat Y^r_{1,j}+v_j\widehat Y^r_{2,j})=0.\qquad(169_4)$$

If $(\alpha,r)\in\{\alpha,r\}^1_4$, then, by Lemma 12 and Corollary 22, $\breve\tau=\tau_1$ and $c_{\tau_1+1}=0$; therefore, relation (166_4) for $\tau=\tau_1+1$ provides the additional relation $0\cdot h^r_{1,\tau_1+1}=\sum_{j=\tau_1+1}^r g_{\tau_1+1\,j}Y^r_{0,j}$, which, by analogy with (169_4), can be represented in the form

$$\sum_{j=\tau_1+1}^{r}(u^1_j\widehat Y^r_{1,j}+v^1_j\widehat Y^r_{2,j})=0,\qquad(\alpha,r)\in\{\alpha,r\}^1_4\qquad(h^r_{1,\tau_1+1}\text{ is arbitrary}),\qquad(170_4)$$

where $u^1_j=(\alpha(2j+1)+2(r-j)-1)g_{\tau_1+1\,j}-\widetilde\alpha(2j+1)g_{\tau_1+1\,j-1}$ and $v^1_j=-2\widetilde\alpha g_{\tau_1+1\,j}$.

Let us rewrite the relations (169_4) and (170_4) in terms of the coefficients of system (3).

By (168_4), $v_j=0$ if $\prod_{\nu=j+1}^{\breve\tau}(\alpha(2\nu+1)+2(r-\nu)-1)=0$, i.e., if $c''_{j+1}\cdots c''_{\breve\tau}=0$ $(-1\le j\le\breve\tau-1)$, and $u_j=0$ if $c''_j\cdots c''_{\breve\tau}=0$ $(-1\le j\le\breve\tau)$. By using Corollary 22, we readily transform relation (169_4) in case (ii) with $\breve\tau=r$.

Let $(\alpha,r)\in\{\alpha,r\}^{3c}_4\subset\{\alpha,r\}^0_4$; then $\tau^c_3=2ln-l-n$. Therefore, $v_{\tau^c_3},\ldots,v_r,u_{\tau^c_3+1},\ldots,u_r\ne0$, since they do not contain $c''_{\tau^c_3}=0$, and Eq. (169_4) acquires the form

$$v_{\tau^c_3}Y_2^{(2\tau^c_3+2,2(r-\tau^c_3))}+\sum_{j=\tau^c_3+1}^{r}(u_jY_1^{(2j+1,2(r-j)+1)}+v_jY_2^{(2j+2,2(r-j))})=\widetilde c.\qquad(169^{3_c}_4)$$

Let $(\alpha,r)\in\{\alpha,r\}^{4c}_4\subset\{\alpha,r\}^0_4$; i.e., $\alpha=1/(2r+1)$. Then $\tau^c_4=r$; therefore, only $v_r\ne0$, and Eq. (169_4) acquires the form

$$-2Y_2^{(2r+2,0)}=\widetilde c\qquad(r\ge2).\qquad(169^{4_c}_4)$$

Now if $(\alpha, r) \in \{\alpha, r\}_4^0 \backslash (\{\alpha, r\}_4^{3c} \cup (\{\alpha, r\}_4^{4c})$, in particular, $(\alpha, r) \in \{\alpha, r\}_4^{1c} \subset \{\alpha, r\}_4^0$, then all $u_j, v_j \neq 0$, and relation (169_4) acquires the form

$$v_{-1} Y_2^{(0,2r+2)} + \sum_{j=0}^{r} (u_j Y_1^{(2j+1, 2(r-j)+1)} + v_j Y_2^{(2j+2, 2(r-j))}) = \widetilde{c}. \qquad (169_4^0)$$

Let

$$(\alpha, r) \in \{\alpha, r\}_4^{2_2} = \{-1/2, m\}_{m \in M_4} \cup \{(2\ell - 1)/2, m\}_{\ell, m \in \mathbb{N},\, m \geq \max\{\ell, 2\}}.$$

If $(\alpha, r) \in \{-1/2, m\}_{m \in M_4}$, then $\tau_2 = r - 1$, all $u_j, v_j \neq 0$, and

$$v_{-1} Y_2^{(0,2r+2)} + \sum_{j=0}^{r-1} (u_j Y_1^{(2j+1, 2(r-j)+1)} + v_j Y_2^{(2j+2, 2(r-j))}) + u_r Y_1^{(2r+1,1)} = \widetilde{c}. \qquad (169_4^{2_1})$$

Now let $(\alpha, r) \in \{(2\ell - 1)/2, m\}_{\ell, m \in \mathbb{N},\, m \geq \max\{\ell, 2\}}$ ($\breve{\tau} = \tau_2 = m - \ell - 1$). We estimate u_j and v_j for $j = \tau_2 + 2, \ldots, r$ in (169_4) with regard of the condition $b_j > 0$ ($j \geq -1$).

If $\ell = 1$, then $j = r = m$, $\alpha = 1/2$, and $c_m = m(m - 1/2) > 0$ in (159^1). Then $v_m = 2b_{m-2} g_{mm}/c_m > 0$, and $u_m = -b_{m-2}(m - 1/2)/c_m < 0$.

Let $\ell \geq 2$. For $\alpha = (2\ell - 1)/2$ and $r = m$ in (159^1), we have

$$c_\tau = ((2\ell - 1)(\tau - 1) + 2m - 2\tau + 1)((\ell - 1/2)(2\tau + 1) + 2m - 2\tau - 1) > 0 \quad \text{for} \quad \tau \geq 0.$$

If $\ell = 2$, then $j = m - 1, m$. Therefore,

$$v_{m-1} = 2b_{m-3} g_{m-1\, m-1}/c_{m-1} > 0, \qquad u_{m-1} = -b_{m-3}(3m - 1/2)/c_{m-1} < 0.$$

By formula (165_1), we have

$$v_m = 2b_{m-3} g_{m-1\, m}/c_{m-1} = -2b_{m-3} a_{m-1}/(c_{m-1} c_m) = 2b_{m-3}(12m^2 - 2m - 4)/(c_{m-1} c_m) > 0.$$

We have $u_m = -b_{m-3}(12m^2 - 2m - 4)(3m + 1/2)/(c_{m-1} c_m) - 2m - 1 < 0$.

Now let $\ell \geq 3$. By (172_1),

$$g_{\tau_2 + 2\, j} = g_{\tau_2 + 2\, j-1} f_{\tau_2 + 2\, j-1}/c_j$$

for $j = \tau_2 + 3, \ldots, r$ such that $f_{\tau_2 + 2\, j-1} \neq 0$.

To estimate $f_{\tau_2 + 2\, j}$ from below, set $\zeta_j = (2j + 4)(2\alpha j + 2(r - j) - 1) > 0$ for $j \geq -1$. By induction, $f_{\tau_2 + 2\, j} > \zeta_j$ for $j = \tau_2 + 2, \ldots, r - 1$. Therefore, $g_{\tau_2 + 2\, \tau_2 + 2} = 1$ and $g_{\tau_2 + 2\, j} > g_{\tau_2 + 2\, j-1} \zeta_{j-1}/c_j > 0$ for $j = \tau_2 + 3, \ldots, r$.

Since $\varkappa_4 = \alpha(2j + 1) + 2(r - j) - 1 > 0$ ($j \geq 0$), we have

$$\varkappa_4 g_{\tau_2 + 2\, j} - (2j + 1) g_{\tau_2 + 2\, j-1} > g_{\tau_2 + 2\, j-1}(\varkappa_4 \zeta_{j-1}/c_j - 2j - 1) = g_{\tau_2 + 2\, j-1} > 0$$

for u_j in (168_4).

Thus if $(\alpha, r) \in \{(2\ell - 1)/2, m\}_{\ell, m \in \mathbb{N},\, m \geq \max\{\ell, 2\}}$, then all $u_j < 0$ and $v_j > 0$, and Eq. (169_4) acquires the form

$$v_{-1} Y_2^{(0,2r+2)} + \sum_{j=0}^{\tau_2} (u_j Y_1^{(2j+1, 2(r-j)+1)} + v_j Y_2^{(2j+2, 2(r-j))})$$
$$+ u_{\tau_2 + 1} Y_1^{(2\tau_2 + 3, 2(r - \tau_2) - 1)} + \sum_{j=\tau_2 + 2}^{r} (u_j Y_1^{(2j+1, 2(r-j)+1)} + v_j Y_2^{(2j+2, 2(r-j))}) = \widetilde{c}. \quad (169_4^{2_2})$$

Finally, let $(\alpha, r) \in \{\alpha, r\}_4^1 = \{\alpha, r\}_4^{2c}$ [$\breve{\tau} = \tau_1 = l(2n - 1)$]. We estimate u_j and v_j for $j = \tau_1 + 2, \ldots, r$ in (169_4) and u_j^1 and v_j^1 for $j = \tau_1 + 1, \ldots, r$ in (170_4).

In (159^1), we have

$$c_\tau = (2(k+l)(2n-1) - 2n - 2\tau + 3 - (2k-1)(\tau-1)/l)(2(k+l)(2n-1) \\ - 2n - 2\tau + 1 - (2k-1)(2\tau+1)/(2l)) > 0$$

for $\tau \ge \tau_1 + 2$. By (172_1), $g_{\tau j} = g_{\tau\, j-1} f_{\tau\, j-1}/c_j$ for $f_{\tau\, j-1} \ne 0$ ($\tau = \tau_1+1, \tau_1+2$).

To estimate $f_{\tau j}$ from below, we introduce

$$\xi_j = -(2j+3)(2\alpha j + 2(r-j) - 1) > 0$$

for $j \ge \tau_1 + 1$. By induction, $f_{\tau j-1} > \xi_{j-1}$ ($\tau = \tau_1+1$, τ_1+2, $j = \tau+1, \dots, r$); therefore, $g_{\tau\tau} = 1$ and $g_{\tau j} > g_{\tau\, j-1}\xi_{j-1}/c_j > 0$ for the same τ and j.

Since $\varkappa_4 < 0$ for $j \ge \tau_1$, it follows that, for u_j in (168_4) and u_j^1 in (170_4), one has

$$\varkappa_4 g_{\tau j} - \widetilde{\alpha}(2j+1) g_{\tau\, j-1} < g_{\tau\, j-1}(\xi_{j-1}(\varkappa_4/c_j + 2j - 1)) = 0,$$

$\tau = \tau_1+1, \tau_1+2$, $j = \tau, \dots, r$.

Therefore, if $(\alpha, r) \in \{\alpha, r\}_4^1$, then all $u_j^1, v_j^1 < 0$; and relation (170_4) has the form

$$\sum_{j=\tau_1+1}^{r} (u_j^1 Y_1^{(2j+1,2(r-j)+1)} + v_j^1 Y_2^{(2j+2,2(r-j))}) = \widetilde{c}; \qquad (170_4^1)$$

$u_j, v_j > 0$ in (169_4) for $j = \tau_1+2, \dots, r$, and relation (169_4) acquires the form

$$v_{-1} Y_2^{(0,2r+2)} + \sum_{j=0}^{\tau_1} (u_j Y_1^{(2j+1,2(r-j)+1)} + v_j Y_2^{(2j+2,2(r-j))}) \\ + u_{\tau_1+1} Y_1^{(2\tau_1+3,2(r-\tau_1)-1)} + \sum_{j=\tau_1+2}^{r} (u_j Y_1^{(2j+1,2(r-j)+1)} + v_j Y_2^{(2j+2,2(r-j))}) = \widetilde{c}. \qquad (169_4^1)$$

5^0. Let us state the obtained results. For $p = 2r + \mu$ ($r \ge 1$, $\mu \in \{0,1\}$), we split the coefficients of the forms $Y_1^{(p+1)}$ and $Y_2^{(p+1)}$ into four disjoint sets $\{Y\}_\lambda^{r,\mu}$ ($\lambda = 1, \dots, 4$):

$$\{Y\}_1^{r,0} = (Y_1^{(1,2r)}, Y_1^{(3,2r-2)}, \dots, Y_1^{(2r+1,0)}, Y_2^{(0,2r+1)}, Y_2^{(2,2r-1)}, \dots, Y_2^{(2r,1)}),$$
$$\{Y\}_2^{r,1} = (Y_1^{(0,2r+2)}, Y_1^{(2,2r)}, \dots, Y_1^{(2r+2,0)}, Y_2^{(1,2r+1)}, Y_2^{(3,2r-1)}, \dots, Y_2^{(2r+1,1)}),$$
$$\{Y\}_3^{r,0} = (Y_1^{(0,2r+1)}, Y_1^{(2,2r-1)}, \dots, Y_1^{(2r,1)}, Y_2^{(1,2r)}, Y_2^{(3,2r-2)}, \dots, Y_2^{(2r+1,0)}),$$
$$\{Y\}_4^{r,1} = (Y_1^{(1,2r+1)}, Y_1^{(3,2r-1)}, \dots, Y_1^{(2r+1,1)}, Y_2^{(0,2r+2)}, Y_2^{(2,2r)}, \dots, Y_2^{(2r+2,0)}).$$

Theorem 20. 1. *System* (151) *is formally equivalent to system* (3) *with*

$$P = (\alpha y_1^2 - \operatorname{sgn}\alpha\, y_2^2, y_1 y_2) \qquad (\alpha \ne 0)$$

if the coefficients of the homogeneous polynomials $Y_i^{(p+1)}$ of system (3) *satisfy the following resonance equations for all $p \ge 2$:*

(a) *Eq.* (153) *for $p = 2$;*

(b) *Eq.* (154) *for $p = 3$;*

(c) *if $p = 2r$ ($r \ge 2$, $\mu = 0$), then, depending on α, the coefficients in $\{Y\}_1^{r,0}$ satisfy one of Eqs.* ($169_1^{2_c}$), (169_1^0), *and* (169_1^1); *and if $(\alpha, r) \in \{\alpha, r\}_1^1 = \{-k/l, (k+l)n+1\}_{k,l,n\in\mathbb{N}}$, then they also satisfy Eq.* (170_1^1) *and have nonzero factors in it except for the coefficient $Y_2^{(2(m-n),2n+1)}$ in* (169_1^0) *if $(\alpha, r) \in \{n, m\}_{m,n\in\mathbb{N},\, m\ge 2}$; depending on α, the coefficients in $\{Y\}_3^{r,0}$ satisfy one of Eqs.* (169_1^1),

$(169_1^{2_2})$, $(169_1^{2_1})$, *and* (169_3^0); *moreover, in the first two equations, the coefficients have nonzero factors, and in the last two equations, the coefficients can be zero for some* (α, r);

(d) *if* $p = 2r + 1$ $(r \geq 2,\ \mu = 1)$, *then, depending on* α, *the coefficients in* $\{Y\}_2^{r,1}$ *satisfy one of Eqs.* (169_2^1), (169_2^2), *and* (169_2^0); *moreover, in the first two equations, the coefficients have nonzero factors, and in the last equation, the factors can be zero for some* (α, r); *depending on* α, *the coefficients in* $\{Y\}_4^{r,1}$ *satisfy one of Eqs.* $(169_4^{3_c})$, $(169_4^{4_c})$, (169_4^0), $(169_4^{2_1})$, $(169_4^{2_2})$, *and* (169_4^1); *if* $(\alpha, r) \in \{\alpha, r\}_4^1 = \{-(2k-1)/(2l)$ *and* $(k+l)(2n-1) - n + 1\}_{k,l,n \in \mathbb{N}}$, *then they also satisfy Eq.* (170_4^1) *and have nonzero factors in these equations.*

2. *For each* $p \geq 2$, *the coefficients of the forms* $Y_i^{(p+1)}$ *that do not occur in the above-listed resonance equations or have only zero factors in them are nonresonance and can take arbitrary values.*

3. *If* $(\alpha, r) \in \{\alpha, r\}_1^1$, *then, in the change of variables* (2) *relating* (151) *and* (3), *the resonance coefficients* $h_2^{(2\tau_1+1, 2(r-\tau_1)-1)}$, *where* $\tau_1 = ln$ $(l, n \in \mathbb{N})$, *are not subjected to any constraints*; *if* $(\alpha, r) \in \{\alpha, r\}_4^1$, *then the resonance coefficients* $h_1^{(2\tau_1+2, 2(r-\tau_1)-1)}$, *where* $\tau_1 = l(2n-1)$, *are not subjected to any constraints.*

$$\text{Let } n_p = \begin{cases} 2 & \text{if} \quad p = 3,\ \alpha \neq 2,\ p = 2r,\ (\alpha, r) \notin \{\alpha, r\}_1^1, \\ & \qquad p = 2r+1,\ (\alpha, r) \notin \{\alpha, r\}_4^1 \\ 3 & \text{if} \quad p = 2,\ \alpha \neq 1/2,\ p = 3,\ \alpha = 2, \\ & \qquad p = 2r,\ (\alpha, r) \in \{\alpha, r\}_1^1,\ p = 2r+1,\ (\alpha, r) \in \{\alpha, r\}_4^1 \\ 4 & \text{if} \quad p = 2,\ \alpha = 1/2. \end{cases}$$

Corollary 23. *In system* (3), n_p *distinct resonance coefficients of the forms* $Y_i^{(p+1)}$ *form a resonance set if these coefficients are the following.*

For $p = 2$: (i) *any of* (153^1); (ii) $Y_1^{(3,0)}$; (iii) *any of* (153^3) *except for* $Y_2^{(3,0)}$ *for* $\alpha = 1/2$; (iv) $Y_2^{(3,0)}$ *if* $\alpha = 1/2$.

For $p = 3$: (i) *any of* (154^1); (ii) *any of* (154^2); (iii) *if* $\alpha = 2$, *then any of* (154^3) *other than the one chosen in* (154^1).

For $p = 2r$ $(r \geq 2)$: (i) *any coefficient in* $\{Y\}_3^{r,0}$ *occurring in the corresponding equation* (169_3^1), $(169_3^{2_1})$, $(169_3^{2_2})$, *or* (169_3^0) *with a nonzero factor*; (ii) *any coefficient in* $\{Y\}_1^{r,0}$ *occurring in the corresponding equation* $(169_1^{2_c})$, (169_1^0), *or* (169_1^1) *with a nonzero factor*; (iii) *if* $(\alpha, r) \in \{\alpha, r\}_1^1$, *then any coefficient in* (170_1^1) *other than the coefficient chosen in* (169_1^1) *but such that these equations are solvable for them.*

For $p = 2r + 1$ $(r \geq 2)$: (i) *any coefficient in* $\{Y\}_2^{r,1}$ *occurring in the corresponding equation* (169_2^1), (169_2^2), *or* (169_2^0) *with a nonzero factor*; (ii) *any coefficient in* $\{Y\}_4^{r,1}$ *occurring in the corresponding equation* $(169_4^{3_c})$, $(169_4^{4_c})$, (169_4^0), $(169_4^{2_1})$, $(169_4^{2_2})$, *or* (169_4^1) *with a nonzero factor*; (iii) *if* $(\alpha, r) \in \{\alpha, r\}_4^1$, *then any coefficient in* (170_4^1) *other than the one chosen in* (169_4^1) *but such that these equations are solvable for them.*

Corollary 24. *System* (3) *with the nonperturbed part* (15_2) *is a generalized normal form if, for each* $p \geq 2$, *the forms* $Y_i^{(p+1)}$ *contain at most* n_p *terms with arbitrary coefficients that form one of the resonance sets described in Corollary* 23 *and the remaining coefficients are zero.*

Theorem 21. *Let us arbitrarily fix the structure of the generalized normal form* (3); *i.e., for each* $p \geq 2$, *one fixes the order of those* n_p *terms of the forms* $Y_i^{(p+1)}$ *whose coefficients occur in the resonance set chosen for a given* p; *in addition, in the change of variables* (2), *one fixes the coefficient* $h_2^{(2\tau_1+1, 2(r-\tau_1)-1)}$ *if* $(\alpha, r) \in \{\alpha, r\}_1^1$ *or* $h_1^{(2\tau_1+2, 2(r-\tau_1)-1)}$ *if* $(\alpha, r) \in \{\alpha, r\}_4^1$. *Then there exists a unique normalizing change of variables* (2) *reducing an arbitrary system* (151) *to the generalized normal form* (3) *with the chosen structure in which, for each* $p \geq 2$, *the coefficients in the chosen resonance set described in Corollary* 23 *are uniquely determined from the resonance equations containing them.*

Example 9. Consider the general case in which α occurs in none of the above-introduced families $\{\alpha, r\}$; therefore, $\alpha \neq -k/l,\ 1/l,\ (2l-1)/2,\ k\ (k,l \in \mathbb{N})$. Then system (151) can be reduced to a generalized normal form (3) that has, say, the following structures:

$$\dot{y}_1 = \alpha y_1^2 - \widetilde{\alpha} y_2^2 + Y_1^{(3,0)} y_1^3 + \sum_{r=1}^{\infty} (Y_1^{(1,2r)} y_1 y_2^{2r} + Y_1^{(1,2r+1)} y_1 y_2^{2r+1} + Y_1^{(0,2r+1)} y_2^{2r+1} + Y_1^{(0,2r+2)} y_2^{2r+2}),$$
$$\dot{y}_2 = y_1 y_2;$$
$$\dot{y}_1 = \alpha y_1^2 - \widetilde{\alpha} y_2^2 + Y_1^{(3,0)} y_1^3 + \sum_{r=1}^{\infty} (Y_1^{(0,2r+1)} y_2^{2r+1} + Y_1^{(0,2r+2)} y_2^{2r+2}),$$
$$\dot{y}_2 = y_1 y_2 + \sum_{r=1}^{\infty} (Y_2^{(0,2r+1)} y_2^{2r+1} + Y_2^{(0,2r+2)} y_2^{2r+2}).$$

In conclusion, we note that various methods for the normalization of systems whose linear part matrix has zero eigenvalues were discussed in [4, 5]. The so-called resonance equation method was suggested in [6] for the practical finding of all possible structures of generalized normal forms, which can be called generalized normal forms of the first order. Systems with linear-quadratic unperturbed part were studied in [5] with the use of this method. Next, systems with eleven distinct canonical forms of the quadratic unperturbed part were studied in [1–3] and in the present paper in accordance with the partition [1] of the set of two-dimensional quadratic systems by linear nondegenerate changes of variables into seventeen disjoint classes according to the minimization principle for the number of nonzero terms in the resulting canonical forms. Finally, systems whose unperturbed parts contain cubic terms were studied in [6, 7] by the resonance equation method.

REFERENCES

1. Basov, V.V. and Skitovich, A.V., A Generalized Normal Form and Formal Equivalence of Two-Dimensional Systems with Quadratic Zero Approximation. I, *Differ. Uravn.*, 2003, vol. 39, no. 8, pp. 1016–1029.
2. Basov, V.V. and Skitovich, A.V., A Generalized Normal Form and Formal Equivalence of Two-Dimensional Systems with Quadratic Zero Approximation. II, *Differ. Uravn.*, 2005, vol. 41, no. 8, pp. 1011–1023.
3. Basov, V.V., A Generalized Normal Form and Formal Equivalence of Two-Dimensional Systems with Quadratic Zero Approximation: III, *Differ. Uravn.*, 2006, vol. 42, no. 3, pp. 308–319.
4. Bryuno, A.D. and Petrovich, V.Yu., Normal Forms of a System of ODE, *Preprint Ist. Prikl. Mat. RAN*, Moscow, 2000, no. 18.
5. Basov, V.V. and Fedotov, A.A., A Generalized Normal Forms for Two-Dimensional Systems of Ordinary Differential Equations with Linear and Quadratic Unperturbed Parts, *Vestnik St. Petersburg. Gos. Univ.*, 2007, no. 1, pp. 13–33.
6. Basov, V.V., A Generalized Normal Form and Formal Equivalence of Systems of Differential Equations with Zero Characteristic Numbers, *Differ. Uravn.*, 2003, vol. 39, no. 2, pp. 154–170.
7. Basov, V.V., A Generalized Normal Form and Formal Equivalence of Systems of Differential Equations with Zero Approximation $(x_2^3, -x_1^3)$, *Differ. Uravn.*, 2004, vol. 40, no. 8, pp. 1011–1022.