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ESTIMATES OF THE CONCENTRATION FUNCTION OF LINEAR COMBINATIONS OF ORDER STATISTICS OF A UNIFORM DISTRIBUTION

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1. Introduction and Formulation of Results. The study of the limit properties of the concentration functions of sums of independent random variables has been given and is being given much attention in the literature; on the contrary, there are almost no publications in which the asymptotic behavior of the concentration functions of sums of variables, connected by dependence, has been investigated. The present note is devoted to the estimation of the Lévy concentration function of linear combinations of order statistics. The results, obtained here, generalize a result of [1].

Let $U_1, \ldots, U_n$ be n independent uniformly (on (0, 1)) distributed random variables and $U_{1n} \leqslant \ldots \leqslant U_{nn}$ be the order statistics corresponding to them. Let us consider random variables of the form

$$S_n = \sum_{k=1}^{n} c_{kn} U_{kn}, \tag{1.1}$$

where c_{kn} are arbitrary constants. Let $Q(S_n; \lambda)$ denote the Lévy concentration function of S_n:

$$Q(S_n; \lambda) = \sup_{x \in \mathbf{R}} \mathsf{P}\{S_n \in [x, x+\lambda]\} \quad (\lambda > 0).$$

Let us set

$$\Delta_n = \sum_{k=1}^{n} \sum_{m=k}^{n} (c_{kn} + \ldots + c_{mn})^2. \tag{1.2}$$

THEOREM 1. There exists an absolute positive constant C such that

$$Q(S_n; \lambda) < C\lambda (1 + 1/n)(n+1)^{3/2} \Delta_n^{-1/2} \tag{1.3}$$

for all $n \in \mathbf{N}$.

COROLLARY 1. Let there exist a positive constant K such that

$$\Delta_n > Kn^4 \tag{1.4}$$

for sufficiently large n. Then

$$Q(S_n; \lambda) \leqslant C\lambda/\sqrt{n}, \tag{1.5}$$

where $C = r/\sqrt{K}$, r being an absolute positive constant.

This result follows immediately from Theorem 1. We give one more corollary, in which stronger, but more easily verifiable, conditions that are sufficient for an estimate of the form (1.5) are given.

COROLLARY 2. Let there exist a positive constant K such that

$$\left|\sum_{k=1}^{n} k(n-k+1) c_{kn}\right| > Kn^3 \tag{1.6}$$

for sufficiently large n. Then

$$Q(S_n; \lambda) \leqslant C\lambda/\sqrt{n},$$

where $C = r/\sqrt{K}$, r being an absolute positive constant.

To prove the second corollary, it is sufficient to observe that

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$$\sum_{k=1}^{n}\sum_{m=k}^{n}(c_{kn}+\ldots+c_{mn})=\sum_{k=1}^{n}k(n-k+1)c_{kn},$$

and, by the Cauchy–Bunyakovskii inequality, (1.4) follows from (1.6).

COROLLARY 3. Let $c_{kn}\geqslant 0$ for all k and $n\,(k\leqslant n)$ and suppose that there exist positive constants c and γ such that for sufficiently large n the number of the elements

$$\{k:\ c_{kn}\geqslant c\}\geqslant \gamma n. \tag{1.7}$$

Then

$$Q\,(S_n;\,\lambda)\leqslant K\lambda/\sqrt{n}\ ,$$

where K is a positive constant depending only on c and γ.

Remark 1. The last result has been obtained in [1].

Remark 2. In each of the above results, it is assumed that λ depends on n.

Remark 3. For large n estimate (1.3) becomes trivial in the case where the number of the nonzero coefficients in the linear combination is small in comparison with the size of the sample and these coefficients are bounded above in absolute value by a constant that does not depend on n.

2. Auxiliary Results. Let $\xi_1,\xi_2,\ldots,\xi_{n+1}$ be independent exponential random variables with the common distribution function $F(x)=1-e^{-x}$ $(x\geqslant 0)$. Let us set $b_k=\sum_{j=k}^{n}c_{jn}$ $(1\leqslant k\leqslant n)$ and $b_{n+1}=0$. We set

$$T_n=\left(\sum_{k=1}^{n+1}\xi_k\right)^{-1}\cdot\sum_{k=1}^{n+1}b_k\xi_k. \tag{2.1}$$

LEMMA 1. a) The distribution of the random variable T_n coincides with the distribution of the random variable S_n.

b) The conditional distribution of the random variable $s^{-1}\sum_{k=1}^{n}b_k\xi_k$ under the condition $\sum_{k=1}^{n+1}\xi_k=s$ $(s>0)$ coincides with the distribution of S_n.

Proof. The statement b) follows from the arguments carried out in [2]. The statement a) follows from b). The lemma is proved.

LEMMA 2.

$$\inf_{x\in\mathbf{R}}\sum_{k=1}^{n+1}(b_k-x)^2=\frac{\Delta_n}{n+1}\,.$$

Proof.

$$\inf_{x\in\mathbf{R}}\sum_{k=1}^{n+1}(b_k-x)^2=\sum_{k=1}^{n+1}\left(b_k-(n+1)^{-1}\sum_{k=1}^{n+1}b_k\right)^2=$$
$$=\frac{1}{n+1}\left[(n+1)\sum_{k=1}^{n+1}b_k^2-\left(\sum_{k=1}^{n+1}b_k\right)^2\right]=\frac{1}{n+1}\sum_{k=1}^{n}\sum_{m=k+1}^{n+1}(b_k-b_m)^2=\frac{\Delta_n}{n+1}.$$

The lemma is proved.

Before formulating the next lemma, we introduce the following notation:

$$M_1(x)=\max_{1\leqslant k\leqslant n+1}|\,b_k-x\,|=|\,b_{k_0}-x\,|,$$

$M_2(x)=\max_{k\neq k_0}|\,b_k-x\,|$ is the second (in magnitude) maximum after $M_1(x)$ and $Y_k=\frac{1}{n+1}(b_k-x)\xi_k$ $(1\leqslant k\leqslant n+1)$ are independent exponential random variables (ξ_k have been defined above).

Let $v_k(t)$ denote the characteristic function of the random variable Y_k:

$$v_k(t)=\frac{1}{1-(b_k-x)(n+1)^{-1}it}\,. \tag{2.2}$$

We fix an arbitrary ε such that $0 < \varepsilon < 1$ and an arbitrary $n \in \mathbf{N}$. We decompose the set of real numbers $\mathbf{R}$ into two subsets:

$$\mathbf{R}_{(n)} = \left\{x\colon \sum{}' (b_k - x)^2 \geqslant \varepsilon \frac{\Delta_n}{n+1}\right\}, \quad \mathbf{R}^c_{(n)} = \mathbf{R} \setminus \mathbf{R}_{(n)}. \tag{2.3}$$

Here, and in the sequel, Σ' denotes summation over $k = 1, 2, \ldots, n+1, k \neq k_0$. One of the subsets in (2.3) may be empty.

LEMMA 3. a) Let $x \in \mathbf{R}_{(n)}$. Then

$$I(x) = \int_{-\infty}^{\infty} \prod_{k=1}^{n+1} |v_k(t)|\, dt \leqslant C_1 (n+1)^{3/2} (\varepsilon\Delta_n)^{-1/2},$$

where C_1 is an absolute constant.

b) Let $x \in \mathbf{R}^c_{(n)}$. Then $M_1(x) > [(1-\varepsilon)(n+1)^{-1}\Delta_n]^{1/2}$.

Proof. We prove a). If $R_{(n)}$ is not empty, we take an arbitrary $x \in \mathbf{R}_{(n)}$. By virtue of (2.2), we have

$$I(x) = 2\int_0^{\infty} \prod_{k=1}^{n+1} \frac{1}{\sqrt{1 + (n+1)^{-2}(b_k - x)^2 t^2}}\, dt. \tag{2.4}$$

Let us observe that if $x \in \mathbf{R}_{(n)}$, then $M_2(x) \neq 0$ and, after a change of variable in (2.4), we get

$$I(x) = \frac{2(n+1)}{M_2(x)} \int_0^{\infty} \prod_{k=1}^{n+1} \frac{dt}{\sqrt{1 + M_2^{-2}(x)(b_k - x)^2 t^2}} = I_1 + I_2, \tag{2.5}$$

where

$$I_1 = \frac{2(n+1)}{M_2(x)} \int_0^1 \ldots dt, \quad I_2 = \frac{2(n+1)}{M_2(x)} \int_1^{\infty} \ldots dt.$$

Let us estimate I_1. We have

$$I_1 \leqslant \frac{2(n+1)}{M_2(x)} \int_0^1 \prod{}' \frac{1}{\sqrt{1 + M_2^{-2}(x)(b_k - x)^2 t^2}}\, dt,$$

where Π' denotes product over all $k = 1, 2, \ldots, n+1, k \neq k_0$, and, applying the inequality

$$\frac{1}{1+z^2} \leqslant e^{-z^2/2} \quad (|z| \leqslant 1) \tag{2.6}$$

to each factor under the sign of integral, we get

$$I_1 \leqslant \frac{2(n+1)}{M_2(x)} \int_0^1 \exp\left[-t^2 (2M_2(x))^{-2} \sum{}' (b_k - x)^2\right] dt \leqslant \frac{4(n+1)}{\left(\sum' (b_k - x)^2\right)^{1/2}} \int_0^{\infty} e^{-t^2}\, dt \leqslant 2\sqrt{\pi}(n+1)^{3/2} (\varepsilon\Delta_n)^{-1/2}. \tag{2.7}$$

Let us estimate I_2. Since $M_2(x)$ is the second (in magnitude) maximum of $|b_k - x|$ with respect to k, the factor $I_2((1+t^2)(1+t^2M_1^2(x)/M_2^2(x)))^{-1/2}$ under the sign of the integral I_2 does not exceed $1/(1+t^2)$. We set $t = 1$ in the remaining factors. Then

$$I_2 \leqslant \frac{2\sqrt{2}(n+1)}{M_2(x)} \prod{}' \frac{1}{\sqrt{1 + M_2^{-2}(x)(b_k - x)^2}} \int_1^{\infty} \frac{1}{1+t^2}\, dt. \tag{2.8}$$

Applying inequality (2.6) to the product in (2.8), we have

$$I_2 \leqslant \frac{\pi\sqrt{2}(n+1)}{2M_2(x)} \exp\left[-(2M_2(x))^{-2} \sum{}' (b_k - x)^2\right]. \tag{2.9}$$

The maximum with respect to $M_2(x)$ (x fixed) in (2.9) is attained for $M_2(x) = \left(\frac{1}{2}\sum' (b_k - x)^2\right)^{1/2}$. Substituting this value in (2.9), we get

$$I_2 \leqslant \frac{\pi(n+1)e^{-1/2}}{\left(\sum' (b_k - x)^2\right)^{1/2}} \leqslant \frac{\pi}{\sqrt{e}} (n+1)^{3/2} (\varepsilon\Delta_n)^{-1/2}. \tag{2.10}$$

Combining (2.10) and (2.7), we have

$$I(x) = I_1 + I_2 \leqslant c_1 (n+1)^{3/2} (\varepsilon\Delta_n)^{-1/2},$$
$$c_1 = 2\sqrt{\pi} + \pi/\sqrt{e}.$$

The statement a) is proved. Let us pass to the proof of b). If $\mathbf{R}^c_{(n)}$ is not empty, we fix $x \in \mathbf{R}^c_{(n)}$. Then

$$M_1^2(x) = \sum_{k=1}^{n+1} (b_k - x)^2 - \sum{}' (b_k - x)^2. \tag{2.11}$$

By Lemma 2 and the definition of the set $\mathbf{R}^c_{(n)}$, the right-hand side of (2.11) is greater than $(1-\varepsilon)\frac{\Delta_n}{n+1}$, which proves b). The lemma is proved.

3. Proof of Theorem 1. We fix an arbitrary $n \in \mathbf{N}$ and let $0 < \varepsilon < 1$, where ε is, at present, arbitrary. Let us set $Q_x(T_n;\ \lambda) = \mathsf{P}\{x \leqslant T_n \leqslant x + \lambda\}$. Then

$$Q(T_n;\ \lambda) = \sup_{x \in \mathbf{R}} Q_x(T_n;\ \lambda).$$

By Lemma 1

$$Q(S_n; \lambda) = Q(T_n; \lambda). \tag{3.1}$$

We will prove that for all $x \in \mathbf{R}$

$$Q_x(T_n; \lambda) \leqslant C\lambda\left(1 + \frac{1}{n}\right)(n+1)^{3/2}\Delta_n^{-1/2}, \tag{3.2}$$

where C is an absolute positive constant. The assertion of the theorem follows immediately from (3.2), with regard for (3.1). We prove (3.2). We have

$$Q_x(T_n; \lambda) = \mathsf{P}\{T_n \leqslant x + \lambda\} - \mathsf{P}\{T_n < x\} =$$
$$= \mathsf{P}\left\{\sum_{k=1}^{n+1} b_k\xi_k \leqslant (x+\lambda)\sum_{k=1}^{n+1}\xi_k\right\} - \mathsf{P}\left\{\sum_{k=1}^{n+1} b_k\xi_k < x\sum_{k=1}^{n+1}\xi_k\right\}.$$

Let us set $\Sigma_1 = \sum_{k=1}^{n+1}(b_k - x)\xi_k$ and $\Sigma_2 = \sum_{k=1}^{n+1}\xi_k$. Then

$$Q_x(T_n;\ \lambda) = \mathsf{P}\{\Sigma_1 \leqslant \lambda\Sigma_2\} - \mathsf{P}\{\Sigma_1 < 0\} = \mathsf{P}\{\Sigma_1 \leqslant \lambda\Sigma_2,\ \Sigma_2 \leqslant 2(n+1)\} +$$
$$+ \mathsf{P}\{\Sigma_1 \leqslant \lambda\Sigma_2,\ \Sigma_2 > 2(n+1)\} - \mathsf{P}\{\Sigma_1 < 0\} \leqslant \mathsf{P}\{\Sigma_1 \leqslant 2\lambda(n+1)\} -$$
$$- \mathsf{P}\{\Sigma_1 \leqslant 2\lambda(n+1),\ \Sigma_2 > 2(n+1)\} + \mathsf{P}\{\Sigma_1 \leqslant \lambda\Sigma_2,\ \Sigma_2 > 2(n+1)\} - \mathsf{P}\{\Sigma_1 < 0\} = P_1 + P_2, \tag{3.3}$$

where

$$P_1 = \mathsf{P}\left\{0 \leqslant \sum_{k=1}^{n+1}(b_k - x)\xi_k \leqslant 2\lambda(n+1)\right\},$$
$$P_2 = \mathsf{P}\left\{2\lambda(n+1) < \sum_{k=1}^{n+1}(b_k - x)\xi_k \leqslant \lambda\sum_{k=1}^{n+1}\xi_k\right\}.$$

Let us estimate P_1:

$$P_1 = \mathsf{P}\left\{0 \leqslant \frac{1}{n+1}\sum_{k=1}^{n+1}(b_k - x)\xi_k \leqslant 2\lambda\right\} \leqslant Q\left(\sum_{k=1}^{n+1} Y_k; 2\lambda\right). \tag{3.4}$$

By Lemma 3 [3],

$$Q\left(\sum_{k=1}^{n+1} Y_k; 2\lambda\right) \leqslant 2\left(\frac{96}{95}\right)^2 \lambda I(x). \tag{3.5}$$

If $x \in \mathbf{R}_{(n)}$, then by Lemma 3, (3.4), and (3.5) we have

$$P_1 \leqslant \frac{C_2}{\sqrt{\varepsilon}}\lambda(n+1)^{3/2}\Delta_n^{-1/2}, \tag{3.6}$$

and we can set

$$C_2 = 2\left(\frac{96}{95}\right)^2\left(2\sqrt{\pi} + \frac{\pi}{\sqrt{e}}\right).$$

If $x \in \mathbf{R}^C_{(n)}$, then we can rewrite P_1 in the form

$$P_1 = \mathsf{P}\left\{0 \leqslant M_1^{-1}(x) \sum_{k=1}^{n+1} (b_k - x)\xi_k \leqslant \frac{2\lambda(n+1)}{M_1(x)}\right\}. \tag{3.7}$$

The coefficient of ξ_{k_0} in (3.7) is equal in absolute value to one, and, by a property of the concentration function for independent random variables (see [3]),

$$P_1 \leqslant Q\left(\xi_{k_0}; \frac{2\lambda(n+1)}{M_1(x)}\right). \tag{3.8}$$

It is easily verified that for the exponential random variable ξ with the distribution density $p(x) = e^{-x}$ $(x \geqslant 0)$ we have $Q(\xi; \lambda) \leqslant \lambda$. Therefore, by virtue of (3.8) and the fact that $x \in \mathbf{R}^C_{(n)}$, we have the estimate

$$P_1 \leqslant 2\lambda(n+1)^{3/2}((1-\varepsilon)\Delta_n)^{-1/2}. \tag{3.9}$$

Combining (3.9) and (3.6), we see that for all $x \in \mathbf{R}$

$$P_1 \leqslant \left(\frac{C_2}{\sqrt{\varepsilon}} + \frac{2}{\sqrt{1-\varepsilon}}\right)\lambda(n+1)^{3/2}\Delta_n^{-1/2}. \tag{3.10}$$

Now, we choose ε. The minimum of the right-hand side of (3.10) is attained for $\varepsilon = C_2^{2/3}/(2^{2/3} + C_2^{2/3})$. Therefore, we finally have

$$P_1 \leqslant C\lambda(n+1)^{3/2}\Delta_n^{-1/2}, \tag{3.11}$$

where $C = (C_2^{2/3} + 2^{2/3})^{3/2}$ and $C_2 = 2\left(\frac{96}{95}\right)^2\left(2\sqrt{\pi} + \frac{\pi}{\sqrt{e}}\right)$.

It remains to estimate P_2:

$$P_2 = \mathsf{P}\{2\lambda(n+1) < \sum_{k=1}^{n+1}(b_k - x)\xi_k \leqslant \lambda \sum_{k=1}^{n+1}\xi_k\} =$$

$$= \int_{2(n+1)}^{\infty} \mathsf{P}\left\{2\lambda(n+1) < \sum_{k=1}^{n+1}(b_k - x)\xi_k \leqslant \lambda y \,\Big|\, \sum_{k=1}^{n+1}\xi_k = y\right\} p'(y)\,dy, \tag{3.12}$$

where $p'(y)$ is the distribution density of the random variable $\sum_{k=1}^{n+1}\xi_k$. Let us consider separately the conditional probability under the sign of integral in (3.12). We have

$$\mathsf{P}\{2\lambda(n+1) < \sum_{k=1}^{n+1}(b_k - x)\xi_k \leqslant \lambda y \,|\, \sum_{k=1}^{n+1}\xi_k = y\} =$$

$$= \mathsf{P}\{2\lambda(n+1) < \sum_{k=1}^{n} b_k\xi_k - xy \leqslant \lambda y \,|\, \sum_{k=1}^{n+1}\xi_k = y\} \leqslant \mathsf{P}\{x \leqslant y^{-1}\sum_{k=1}^{n} b_k\xi_k \leqslant x + \lambda \,|\, \sum_{k=1}^{n+1}\xi_k = y\}. \tag{3.13}$$

By Lemma 1, the right-hand side of (3.13) is equal to $Q_x(T_n; \lambda)$; whence, using (3.12) and the Chebyshev inequality, we get

$$P_2 \leqslant Q_x(T_n; \lambda)(n+1)^{-1}. \tag{3.14}$$

(3.2) follows from (3.3), (3.11), and (3.14). The theorem is proved.

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