are satisfied. They lead to the linear dependence of the equations of system (5). In this case i_q can be expressed in terms of other variables; one can find values of $i_1, \ldots, i_{q-1}$ such that i_q is an integer as well because all the β_i are integers. But then the second condition of the theorem is violated, and, consequently, the distribution of $\xi_t, \ldots, \xi_{t+q-1}$ need not be the distribution of q independent random variables.

The theorem above does not imply that all terms in the sequence are independent. Consider the following situation

Example. Suppose that the process is defined by the formula $\xi_t = \xi_{t-4} \oplus \zeta_t \oplus \zeta_{t-2}$. The generating function of its coefficients can be reduced to the form

$$\beta(s) = (1-s^2)/(1-s^4) = 1/(1+s^2).$$

Each two successive values of the process, ξ_t and ξ_{t+1} are independent and uniformly distributed. Yet, it is easy to see that the values $\xi_t, \xi_{t+2}, \xi_{t+4}, \ldots$ are a random walk on the circle. Thus, the process we consider decomposes into two independent processes.

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ON ANALOGUES OF BERRY–ESSEEN INEQUALITY FOR TRUNCATED LINEAR COMBINATIONS OF ORDER STATISTICS*

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(*Translated by M. V. Khatuntseva*)

1. Let $X_1, X_2, \ldots, X_n$ be independent random variables (r.v'.s) with common distribution function (d.f.) F and let $X_{n,1} \leqq \cdots \leqq X_{n,n}$ be the corresponding order statistics. Consider a linear combination of the order statistics

$$L = n^{-1/2} \sum_{i=1}^{n} c_{n,i} X_{n,i}, \tag{1}$$

where the $c_{n,i}$ are real. Many authors have shown interest in studying the asymptotic properties of L-statistics (i.e., statistics of type (1)) due to their applicability in estimation theory (references and a review of the research in this area may be found in [1]).

This paper assumes that

$$\begin{aligned} &c_{n,i} = 0, \quad \textit{for} \quad i < k \quad \textit{and} \quad i > m \\ &\textit{where} \quad k, m \quad \textit{are integers}, \quad 1 \leqq k < m \leqq n, \\ &\textit{and} \quad \liminf_{n\to\infty} k/n = \alpha, \quad \limsup_{n\to\infty} m/n = \beta, \qquad 0 < \alpha < \beta < 1. \end{aligned} \tag{2}$$

Bjerve [2] studied the rate of convergence of the distributions of r.v.'s of type (1) to the normal law for the truncated linear combination at the level of the central order statistics.

*Received by the editors May 10, 1990.

He was the first to obtain an orderwise optimum estimate for L-statistics of the form

$$\sup_{x\in R}\left|F_n(x)-\Phi(x)\right| \leqq C n^{-1/2}, \tag{3}$$

where $F_n(x)$ is the d.f. of normalized r.v. L, $\Phi(x)$ is the standard normal distribution function and C is a constant independent of n. Inequality (3) was proved in [2] under the following assumptions:

($\mathcal{F}1$) *the second derivative* $[F^{-1}(u)]''$ *of the inverse of the d.f.* F *satisfies a Lipschitz condition of order* 1 *in an open interval containing closed interval* $[\alpha, \beta]$,

($\mathcal{C}1$) $n^{-1}\sum_{i=1}^{n}|c_{n,i}|$ *is bounded above uniformly in* n;

($\mathcal{D}1$) $S_n/\sqrt{n}$ *is defined by*

$$S_n^2=\sum_{i=1}^{n}\alpha_{n,i}^2, \qquad \alpha_{n,i}=(n-i+1)^{-1}\sum_{j=i}^{n}c_{n,j}H'\left(\sum_{k=n-i+1}^{n}k^{-1}\right),$$

where H' *is the derivative of* $H=F^{-1}(1-\exp(-x))$, $(x\geqq 0)$, *and acts as a variation for the normed* L. H' *is bounded away from zero.*

In [3] an estimate of type (3) was derived for a truncated linear combination with $k=n\alpha+o(n)$, $m=n\beta+o(n)$, $n\to\infty$, under the following assumptions:

($\mathcal{F}2$) $F'(u)=\inf\{x\colon F(x)\geqq u\}, 0<u<1,$ *satisfies a Lipschitz condition of order* 1 *in given neighborhoods of* α *and* β.

($\mathcal{C}2$) $a_n=\max_{1\leqq i\leqq n}|c_{n,i}|$ *and* $b_n=n\max_{k\leqq i\leqq m-1}|c_{n,i+1}-c_{n,i}|$ *are bounded above uniformly in* n;

($\mathcal{D}2$) *the constant* $\bar\sigma$ *(to be defined below in* § 3*) normalizing* L, *is bounded away from zero.*

This paper proposes one more variant of assumptions which are sufficient for the estimate of type (3):

($\mathcal{F}3$) $F^{-1}(u)$ *satisfies a Lipschitz condition of order* 1 *with a factor* l *in an open interval* I *containing the* $[\alpha, \beta]$;

($\mathcal{C}3$)

$$a_n=\max_{1\leqq i\leqq n}|c_{n,i}| \quad \textit{and} \quad v_n=\sum_{i=1}^{n-1}|c_{n,i+1}-c_{n,i}|$$

are bounded above uniformly in n;

($\mathcal{D}3$) = ($\mathcal{D}2$).

The potentialities of the method applied here are investigated in § 5.

2. The main result of this paper (as of [3]) is obtained with the help of Van Zvet's theorem on symmetric statistics. Let $T=\tau(X_1,\ldots,X_n)$, where $\tau\colon R^n\to R$ is a symmetric function of n variables and $ET=0$, $ET^2=1$.

THEOREM A ([4]). *Suppose that there exist positive constants* A *and* B *such that*

$$E\left|E(T\,|\,X_1)\right|^3\leqq An^{-3/2}, \tag{4}$$

$$1+E\left\{E(T\,|\,X_1,\ldots,X_{n-2})\right\}^2-2E\left\{E(T\,|\,X_1,\ldots,X_{n-1})\right\}^2\leqq Bn^{-3}. \tag{5}$$

Then

$$\sup_{x\in R}\left|P(T\leqq x)-\Phi(x)\right|\leqq C(A+B)n^{-1/2},$$

where C *is an absolute positive constant.*

Section 5 of this paper is devoted to a method of obtaining estimates connected with Theorem A. Note that condition ($\mathcal{C}3$) is weaker than ($\mathcal{C}2$) although ($\mathcal{F}3$) is stronger than the respective condition ($\mathcal{F}2$). This brings up the question: Is it possible to weaken the condition on the coefficients to ($\mathcal{C}1$) (i.e., not assume any smoothness on $c_{n,i}$), by strengthening the smoothness conditions of the original distribution function (possibly to ($\mathcal{F}1$)) while remaining in the framework of the method of [4]? It would be interesting because the proof of Bjerve's result [2] is quite complicated.

It turns out that Theorem A does not cover Bjerve's result. It will be proved in § 5 that obtaining an estimate of optimal order of type (3) is closely related to v_n being uniformly bounded above in n (condition ($\mathcal{C}3$)) and the further strengthening (in comparison with ($\mathcal{F}3$)) of the smoothness conditions of original d.f. will not lead to new results. Therefore, the method in (4) as applied to L-statistics is not sensitive to more than first order smoothness of $F^{-1}(u)$.

The proof of these results requires the following lemma.

LEMMA. *Let $k = n\alpha + o(n), n \to \infty$, where $k \in N, 0 < \alpha < 1$. If $F(\Delta) < \alpha$, then $P\{X_{n,k} < \Delta\} \leqq e^{-\delta n}$, where $\delta > 0$ is a constant independent of n.*

This Lemma, which is a consequence of Bernstein's inequality, is proved in [2] (see [2, p. 35, Lemma 1.1] and [3, Lemma]).

3. Put

$$\bar{c}_n = n^{-1}\sum_{i=1}^{n}|c_{n,i}|, \qquad v_n = \sum_{i=1}^{n-1}|c_{n,i+1} - c_{n,i}|$$

and suppose that conditions (2) and ($\mathcal{F}3$) hold. To normalize r.v. L without using any moment assumptions, we choose some $\varepsilon > 0$ such that $[\alpha - \varepsilon, \beta + \varepsilon] \subset I$ (see ($\mathcal{F}3$)) and introduce truncated r.v.'s $\overline{X}_i = \min(\max[F^{-1}(\alpha - \varepsilon), X_i], F^{-1}(\beta + \varepsilon))$, $i = 1, \ldots, n$. Let $\overline{X}_{n,1} \leqq \cdots \leqq \overline{X}_{n,n}$ be the order statistics for the sample $\overline{X}_1, \ldots, \overline{X}_n$. Consider a linear combination

$$\overline{L} = n^{-1/2}\sum_{i=1}^{n} c_{n,i}\overline{X}_{n,i},$$

where the $c_{n,i}$ are the same as the coefficients of L.

Suppose, that $\bar{\sigma} = (E\overline{L}^2 - E^2\overline{L})^{1/2} > 0$. Introduce the normalized r.v.'s $L^* = (L - E\overline{L})/\bar{\sigma}$, $\overline{L}^* = (\overline{L} - E\overline{L})/\bar{\sigma}$.

Set

$$\Delta_n = \sup_{x \in R}\left|P(L^* \leqq x) - \Phi(x)\right|.$$

THEOREM 1. *If conditions* (2) *and* ($\mathcal{F}3$) *hold, then for all n*

$$\Delta_n \leqq \exp(-\delta n) + C\left\{(l\,\bar{c}_n/\bar{\sigma})^3 + (l\,v_n/\bar{\sigma})^2\right\}n^{-1/2},$$

where $C > 0$ is an absolute constant and δ is a positive constant independent of n.

Theorems 2 and 3 follow from Theorem 1 immediately.

THEOREM 2. *If conditions* (2) *and* ($\mathcal{F}3$) *hold and $\bar{c}_n + v_n = O(\bar{\sigma})$ as $n \to \infty$, then $\Delta_n = O(n^{-1/2})$ as $n \to \infty$.*

THEOREM 3. *If conditions* (2), ($\mathcal{F}3$), ($\mathcal{C}3$), *and* ($\mathcal{D}3$) *hold, then $\Delta_n = O(n^{-1/2})$, as $n \to \infty$.*

4. *Proof of Theorem* 1. First note that due to (2) and the definition of $\overline{X}_{n,i}$ (see § 3) the following inclusions hold:

$$\{L \leqq x\} \subset \{\overline{L} \leqq x\} \cup \left\{X_{n,k} < F^{-1}(\alpha - \varepsilon)\right\} \cup \left\{X_{n,m} > F^{-1}(\beta + \varepsilon)\right\},$$
$$\{\overline{L} \leqq x\} \subset \{L \leqq x\} \cup \left\{X_{n,k} < F^{-1}(\alpha - \varepsilon)\right\} \cup \left\{X_{n,m} > F^{-1}(\beta + \varepsilon)\right\}.$$

Thus

$$\left|P\{L \leqq x\} - P\{\overline{L} \leqq x\}\right| \leqq P\left\{X_{n,k} < F^{-1}(\alpha - \varepsilon)\right\} + P\left\{X_{n,m} > F^{-1}(\beta + \varepsilon)\right\} \tag{6}$$

for all $x \in R$. (Similar arguments were used in generalizing Esseen's inequality [5, p. 160].) By the definition of Δ_n, the lemma of § 2, and inequality (6), we deduce that there exists a positive constant δ independent of n such that

$$\Delta_n \leqq \exp(-\delta n) + \sup_{x \in R} \left| P(\overline{L}^* \leqq x) - \Phi(x) \right|. \tag{7}$$

Let us verify that conditions (4) and (5) of Theorem A are obeyed for $\overline{L}^*$ a normalized symmetric function of the independent and identically distributed r.v's $\overline{X}_1, \ldots, \overline{X}_n$. Set $\overline{X}_{n,0} = F^{-1}(\alpha - \varepsilon)$ and $\overline{X}_{n,n+1} = F^{-1}(\beta + \varepsilon)$. Then, for all $i = 1, \ldots, n$

$$\overline{X}_{n,i} = \min(\overline{X}_{n-1,i}, \overline{X}_n) - \min(\overline{X}_{n-1,i-1}, \overline{X}_n) + \overline{X}_{n-i,i-1},$$

where $\overline{X}_{n-1,i-1}$, $i = 1, \ldots, n-1$, are the order statistics corresponding to the sample $\overline{X}_1, \ldots, \overline{X}_{n-1}$ (see [6, p. 679]). To verify (4), we note that since the distribution of $\overline{X}_{n,i}$, $i = 1, \ldots, n-1$, does not depend on $\overline{X}_n$

$$\begin{aligned} &\bar{\sigma}^3 E \left| E(\overline{L}^* \mid \overline{X}_1) \right|^3 \\ &= n^{-3/2} E \left| E\left\{ \sum_{i=1}^n c_{n,i} \left[\min(\overline{X}_{n-1,i}, \overline{X}_n) - \min(\overline{X}_{n-1,i-1}, \overline{X}_n) \right] \middle| \overline{X}_n \right\} \right. \\ &\quad \left. - E \sum_{i=1}^n c_{n,i} \left[\min(\overline{X}_{n-1,i}, \overline{X}_n) - \min(\overline{X}_{n-1,i-1}, \overline{X}_n) \right] \right|^3 \\ &\leqq 8 n^{-3/2} \left[E \sum_{i=1}^n |c_{n,i}| (\overline{X}_{n-1,i} - \overline{X}_{n-1,i-1}) \right]^3. \end{aligned} \tag{8}$$

Consider $G(u) = \inf\{x \colon \overline{F}(x) \geqq u\}$ the inverse of d.f. $\overline{F}(x) = P\{\overline{X}_1 \leqq x\}$. It is well known (see, for example, [7]) that the joint distribution of the order statistics $\overline{X}_{n-1,i}$, $i = 1, \ldots, n-1$, coincides with the distribution of r.v's the $G(U_{n-1,i})$, $i = 1, \ldots, n-1$, where the $U_{n-1,i}$ are the order statistics of a sample of size $n-1$ from the uniform distribution on (0,1). Thus the right-hand side of (8) is equal to

$$8n^{-3/2} \left[E \sum_{i=1}^n |c_{n,i}| \big(G(U_{n-1,i}) - G(U_{n-1,i-1}) \big) \right]^3,$$

where $U_{n-1,0} = 0$, $U_{n-1,n} = 1$. By the Lipschitz condition, this last expression does not exceed

$$8 n^{-3/2} l^3 \left[E \sum_{i=1}^n |c_{n,i}| (U_{n-1,i} - U_{n-1,i-1}) \right]^3 = 8 n^{-3/2} l^3 \left[n^{-1} \sum_{i=1}^n |c_{n,i}| \right]^3 = 8 l^3 \bar{c}_n^3 n^{-3/2}.$$

So, condition (4) of Theorem A is true with the constant $A = (2l\bar{c}_n/\bar{\sigma})^3$. Now we verify condition (5). Let R_{n-1} and R_n be the respective ranks of $\overline{X}_{n-1}$ and $\overline{X}_n$ in the sample $\overline{X}_1, \ldots, \overline{X}_n$. Let $k_1 = \min(R_{n-1}, R_n)$ and $k_2 = \max(R_{n-1}, R_n)$. As in [4], define a r.v.

$$\begin{aligned} Z = \overline{L} - E(\overline{L} \mid \overline{X}_1, \ldots, \overline{X}_{n-1}) - E(\overline{L} \mid \overline{X}_1, \ldots, \overline{X}_{n-2}, \overline{X}_n) \\ + E(\overline{L} \mid \overline{X}_1, \ldots, \overline{X}_{n-2}). \end{aligned}$$

Since

$$E Z^2/\bar{\sigma}^2 = 1 + E\left\{ E(\overline{L}^* \mid \overline{X}_1, \ldots, \overline{X}_{n-2}) \right\}^2 - 2E\left\{ E(\overline{L}^* \mid \overline{X}_1, \ldots, \overline{X}_{n-1}) \right\}^2,$$

it remains to estimate EZ^2. Consider the functions

$$P(x)=\int_{-\infty}^{x}\left(\overline{F}(y)\right)^2dy,\qquad M(x)=\int_{-\infty}^{x}\overline{F}(y)\left(1-\overline{F}(y)\right)dy,$$
$$Q(x)=\int_{-\infty}^{x}\left(1-\overline{F}(y)\right)^2dy.$$

It is clear that $E|\overline{L}|<\infty$ and so

$$\begin{aligned} n^{1/2}Z=&-\sum_{i=1}^{k_1}(c_{n,i+1}-c_{n,i})\left(P(\overline{X}_{n,i})-P(\overline{X}_{n,i-1})\right)\\ &+\sum_{i=k_1}^{k_2-1}(c_{n,i+1}-c_{n,i})\left(M(\overline{X}_{n,i+1})-M(\overline{X}_{n,i})\right)\\ &-\sum_{i=k_2}^{n}(c_{n,i}-c_{n,i-1})\left(Q(\overline{X}_{n,i+1})-Q(\overline{X}_{n,i})\right) \end{aligned}\tag{9}$$

((9) follows from (4.21) [4, p. 437]; a detailed derivation of these formulas was done in [8]). Note that the difference $(c_{n,k_1+1}-c_{n,k_1})$ occurs twice on the right-hand side of (9): once with a non-negative factor and once with nonpositive factor. Thus, one can reject one of these summands without decreasing the absolute value of the right-hand side of (9). The same is true for the difference $(c_{n,k_2}-c_{n,k_2-1})$.

Now using the inverse transform $G(u)$, the Lipschitz condition, and the known representation

$$U_{n,i}\overset{d}{=}(\xi_1+\cdots+\xi_i)/(\xi_1+\cdots+\xi_{n+1}),\qquad i=1,\ \ldots,\ n,$$

where ξ_i, $i=1,\ \ldots,\ n+1$, are independent r.v'.s with the standard exponential distribution (see, for example, [9, pp. 73–75], we obtain

$$\begin{aligned} nEZ^2&\leqq l^2E\left\{\sum_{i=1}^{n-1}|c_{n,i+1}-c_{n,i}|\,\xi_i/(\xi_1+\cdots+\xi_{n+1})\right\}^2\\ &\leqq l^2\max\left\{E\left(\xi_1/(\xi_1+\cdots+\xi_{n+1})\right)^2,\ E\left(\xi_1\xi_2/(\xi_1+\cdots+\xi_{n+1})^2\right)\right\}\\ &\quad\times\left(\sum_{i=1}^{n-1}|c_{n,i+1}-c_{n,i}|\right)^2\leqq l^2EU_{n,1}^2v_n^2=2l^2v_n^2/(n+1)(n+2)<2l^2v_n^2n^{-2}. \end{aligned}$$

Therefore, condition (5) holds with $B=2(lv_n/\bar{\sigma})^2$. Now the theorem follows from Theorem A.

5. Let condition $(\mathcal{F}1)$ hold. We construct an example of a linear combination of order statistics for which conditions $(\mathcal{C}1)$ and $(\mathcal{D}1)$ hold (see § 1) with v_n of order n^t, $0<t<\frac{1}{4}$, as $n\to\infty$. We prove that $EZ^2/\bar{\sigma}^2\geqq Kv_nn^{-3}$ for sufficiently large n, where K is a positive constant independent of n. Now $EZ^2/\bar{\sigma}^2$ coincides with the left-hand side of (5) for $T=\overline{L}^*$. Thus the estimate of order $n^{-1/2}$ for the rate of convergence of the distribution of L^* to the normal law, which is true by the theorem in [2], can not be derived by immediately applying the results in [4] (which only allow one to obtain an estimate of order $n^{-1/2+t}$).

Suppose that condition $(\mathcal{C}1)$ holds. First, let us show that $\bar{\sigma}=O(\bar{c}_n)=O(1)$ as $n\to\infty$. Clearly, $EU_{n,i}=i/(n+1)\subset(\alpha-\varepsilon,\beta+\varepsilon)$ for $k\leqq i\leqq m$ for sufficiently large n. Let $\widehat{M}$ be the greatest value of the derivative G' on $[\alpha-\varepsilon,\ \beta+\varepsilon]$ where $G(u)=\inf\{x\colon \overline{F}(x)\geqq u\}$ (we shall use the notation of § 1–4). Then

$$\bar{\sigma}^2=n^{-1}E\left(\sum_{i=1}^{n}c_{n,i}\left(\overline{X}_{n,i}-E\overline{X}_{n,i}\right)\right)^2=n^{-1}E\left(\sum_{i=1}^{n}c_{n,i}\left(G(U_{n,i})-EG(U_{n,i})\right)\right)^2$$

$$\leqq 4\widehat{M}^2 n^{-1} E\left(\sum_{i=1}^{n}|c_{n,i}|\left|U_{n,i}-\frac{i}{n+1}\right|\right)^2$$

$$\leqq 4\widehat{M}^2 n^{-1}\max_{k\leqq i\leqq m} E\left(U_{n,i}-\frac{i}{n+1}\right)^2\left(\sum_{i=1}^{n}|c_{n,i}|\right)^2 = O(\bar{c}_n^2) \tag{10}$$

as $n\to\infty$. Now we obtain a lower bound for EZ^2.

$$\begin{aligned} EZ^2 &\geqq E\{Z^2\,|\,k_1<k-1,\,k_2>m+1\}\,P\{k_1<k-1,\,k_2>m+1\} \\ &= E\{Z^2\,|\,k_1<k-1,\,k_2>m+1\}\,2(k-2),\,(n-m-1)/n^2. \end{aligned} \tag{11}$$

Relations (11), (9), and condition (2) imply

$$EZ^2 \geqq 2\alpha(1-\beta)\,n^{-1}E\left\{\sum_{i=1}^{n-1}(c_{n,i+1}-c_{n,i})\left(M(\overline{X}_{n,i+1})-M(\overline{X}_{n,i})\right)\right\}^2 \tag{12}$$

for sufficiently large n. By virtue of the lemma of § 2, the probability that $U_{n,k}$ and $U_{n,m}$ are outside of $[\alpha-\varepsilon,\,\beta+\varepsilon]$ is exponentially small in n and all further calculations are carried out up to an exponentially small term. If $V(u)=M(G(u))$, then the right-hand side of (12) becomes

$$\begin{aligned} &2\alpha(1-\beta)n^{-1}E\left\{\sum_{i=1}^{n-1}(c_{n,i+1}-c_{n,i})\left[V(U_{n,i+1})-V(U_{n,i})\right]\right\}^2 \\ &= 2\alpha(1-\beta)\,n^{-1}E\{\mathcal{L}_n+\mathcal{Q}_n\}^2 \\ &= 2\alpha(1-\beta)\,n^{-1}\left\{E\mathcal{L}_n^2+2E(\mathcal{L}_n\mathcal{Q}_n)+E\mathcal{Q}_n^2\right\}, \end{aligned} \tag{13}$$

where

$$\mathcal{L}_n=\sum_{i=1}^{n-1}(c_{n,i+1}-c_{n,i})\,V'\left(\frac{i}{n+1}\right)(U_{n,i+1}-U_{n,i}),$$

$$\begin{aligned} \mathcal{Q}_n=\sum_{i=1}^{n-1}(c_{n,i+1}-c_{n,i})&\left[V''\left(\frac{i}{n+1}+\theta_{i+1}\left(U_{n,i+1}-\frac{i}{n+1}\right)\right)\left(U_{n,i+1}-\frac{i}{n+1}\right)^2\right. \\ &\left.-V''\left(\frac{i}{n+1}+\theta_i\left(U_{n,i}-\frac{i}{n+1}\right)\right)\left(U_{n,i}-\frac{i}{n+1}\right)^2\right], \qquad |\theta_i|\leqq 1. \end{aligned}$$

By the Lipschitz condition and the known estimates for moments of the uniform order statistics

$$\begin{aligned} E\mathcal{Q}_n^2 = O\Bigg(E\Bigg\{\sum_{i=1}^{n-1}&(c_{n,i+1}-c_{n,i}) \\ &\times\left[\left[\theta_{i+1}\left(U_{n,i+1}-\frac{i}{n+1}\right)-\theta_i\left(U_{n,i}-\frac{i}{n+1}\right)\right]\left(U_{n,i+1}-\frac{i}{n+1}\right)^2\right. \\ &+V''\left(\frac{i}{n+1}+\theta_i\left(U_{n,i}-\frac{i}{n+1}\right)\right)(U_{n,i+1}-U_{n,i}) \\ &\left.\times\left(U_{n,i+1}+U_{n,i}-\frac{2i}{n+1}\right)\right]\Bigg\}^2\Bigg) = O(v_n^2/n^3). \end{aligned}$$

Therefore, the quantity $E\mathcal{Q}_n^2$ may be neglected; moreover,

$$E(\mathcal{L}_n\mathcal{Q}_n)\leqq(E\mathcal{L}_n^2\,E\mathcal{Q}_n^2)^{1/2}=O(n^{-5/2}v_n^2)$$

as $n \to \infty$.

Combining (12), (13), and the last estimates, we arrive at

$$E\,Z^2 \geqq 2\alpha\,(1-\beta)\,n^{-1}\,E\,\mathcal{L}_n^2 + O(n^{-7/2}v_n^2), \tag{14}$$

and

$$\begin{aligned} n^{-1}\,E\,\mathcal{L}_n^2 &= n^{-1}\,E\left\{\sum_{i=1}^{n-1}(c_{n,i+1}-c_{n,i})\,V'\Big(\frac{i}{n+1}\Big)\,(U_{n,i+1}-U_{n,i})\right\}^2 \\ &= 2\big[n\,(n+1)\,(n+2)\big]^{-1}\left\{\sum_{i=1}^{n-1}\sum_{j=1}^{i}(c_{n,i+1}-c_{n,i})\,(c_{n,j+1}-c_{n,j})\right. \\ &\qquad \left.\times V'\Big(\frac{i}{n+1}\Big)V'\Big(\frac{j}{n+1}\Big)\right\}. \end{aligned} \tag{15}$$

gives the main contribution to $E\,Z^2$. The sum in the braces on the right-hand side of (15) is equal to

$$\sum_{i=1}^{n-1}(c_{n,i+1}-c_{n,i})\,V'\Big(\frac{i}{n+1}\Big)\Sigma_i, \tag{16}$$

where

$$\Sigma_i = \sum_{j=1}^{i} c_{n,j}\left(V'\Big(\frac{j-1}{n+1}\Big)-V'\Big(\frac{j}{n+1}\Big)\right) + c_{n,i+1}V'\Big(\frac{i}{n+1}\Big).$$

For n sufficiently large, we have $c_{n,1}=c_{n,n}=0$ and as a result of Abel's transform, we find that (16) is equal to

$$\sum_{i=1}^{n} c_{n,i}\left[V''\Big(\frac{i-1}{n+1}+\frac{\theta_i}{n+1}\Big)\frac{1}{n+1}\Sigma_{i-1} + V'\Big(\frac{i}{n+1}\Big)\,\big(\Sigma_{i-1}-\Sigma_i\big)\right] = S_1 + S_2,$$

where $0 \leqq \theta_i \leqq 1$ and

$$S_1 = \sum_{i=1}^{n} c_{n,i}V''\Big(\frac{i-1}{n+1}+\frac{\theta_i}{n+1}\Big)\frac{1}{n+1}\Sigma_{i-1},$$

$$S_2 = \sum_{i=1}^{n} c_{n,i}V'\Big(\frac{i}{n+1}\Big)\,(\Sigma_{i-1}-\Sigma_i) = \sum_{i=1}^{n} c_{n,i}(c_{n,i}-c_{n,i+1})\left(V'\Big(\frac{i}{n+1}\Big)\right)^2. \tag{17}$$

Now we construct an example. Suppose that $V'(u) \geqq \gamma > 0$ for all $u \in [\alpha-\varepsilon, \beta+\varepsilon]$. Let $c_{n,i} = c_0 > 0$, for all $i \in (\{k, k+1, \dots, m\}\backslash\{i_1, i_2, \dots, i_s\})$ where $s \asymp n^t$ as $n \to \infty$ for some $0 < t < \frac{1}{4}$ with $i_j - i_{j-1} > 1$, $j = 2, \dots, s$, and let $c_{n,i} = 0$ for $i = i_j$, $j = 1, \dots, s$. Condition $(\mathcal{F}1)$ holds by assumption. It is clear that $(\mathcal{C}1)$ and $(\mathcal{D}1)$ also hold in this example. So, the estimate of the rate of convergence to the normal law of order $n^{-1/2}$ of type (3) is true due to the theorem in [2]. On the other hand, we can note that $v_n \asymp n^t$, $v_n^2 = o(n^{1/2})$, and $n^{-7/2}v_n^2 = o(n^{-3})$ as $n \to \infty$. In view of the definitions of S_1, Σ_i, and conditions $(\mathcal{C}1)$ and $(\mathcal{F}1)$, we have

$$\frac{2|S_1|}{(n+1)\,(n+2)} = O(n^{-2})$$

as $n \to \infty$ and (17) implies that $2S_2/\{(n+1)(n+2)\} \geqq \gamma c_0 v_n/\{(n+1)(n+2)\} \asymp n^{t-2}$ as $n \to \infty$. Relations (10), (14)–(17) together with the last estimates imply the existence of a constant $K > 0$ independent of n such that $E\,Z^2/\bar{\sigma}^2 \geqq Kn^{t-3}$. So, the best order of the estimate which may be achieved with the help of Theorem A is in our example $n^{-1/2+t}$.

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TRANSIENT PHENOMENA FOR REAL-VALUED MARKOV CHAINS*

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This paper considers transient phenomena arising in investigations of stationary real-valued ergodic Markov chains. They are similar in a sense to nonergodic chains having paths tending to infinity. This approach enables one to construct approximations for the stationary distributions of the chains.

Let $\{X_n^{(\varepsilon)}\}_{n=0}^{\infty}$ be a sequence (in ε) of homogeneous real-valued Markov chains (in n) with transition function $P^{(\varepsilon)}(x, B)$, $x \in R$, $B \in \mathfrak{B}(R)$, where $\mathfrak{B}(R)$ is the σ-algebra of the Borel sets in R. An invariant measure $\pi^{(\varepsilon)}$ corresponding to the chain $\{X_n^{(\varepsilon)}\}$, i.e., a measure satisfying the equation

$$\pi^{(\varepsilon)}(B) = \int_R P^{(\varepsilon)}(x, B)\, \pi^{(\varepsilon)}(dx), \qquad \pi^{(\varepsilon)}(R) = 1, \tag{1}$$

is our main subject of study. If the chains $\{X_n^{(\varepsilon)}\}$ are ergodic for $\varepsilon > 0$, then the asymptotic behavior of their stationary distribution (as $\varepsilon \to 0$) will be discussed. In the sequel, it is supposed that equation (1) has a unique solution when $\varepsilon > 0$. This is the case if conditions hold for the chains $\{X_n^{(\varepsilon)}\}$ to be ergodic involving a "mean drift" of the chain towards some compact set (see Theorem A) and a "mixing" condition of Doob-Doeblin type (see [2]). In this situation the distribution $P^{(\varepsilon)}(x, n, \cdot)$ converges in variation to $\pi^{(\varepsilon)}(\cdot)$ with the measure $\pi^{(\varepsilon)}(\cdot)$ unique.

Consider a family of random variables (r.v.'s) $\xi^{(\varepsilon)}(x)$ whose distribution coincides with the distribution of the step of the chain $\{X_n^{(\varepsilon)}\}$ from the state x: $P\{x + \xi^{(\varepsilon)}(x) \in B\} = P^{(\varepsilon)}(x, B)$. Below we shall use some regularity conditions. The first one concerns the assumption of "loadability" of the Markov chains $\{X_n^{(\varepsilon)}\}$ meaning that the "average drift" tends to zero: $E\xi^{(\varepsilon)}(x) \to 0$ as $x \to \infty$ and $\varepsilon \downarrow 0$. We then assume that the transient kernel is "weakly continuous" (we shall omit the index (0) for the parameters of the limiting chain $X_n \equiv X_n^{(0)}$): $P^{(\varepsilon)}(x, \cdot) \Longrightarrow P(y, \cdot)$ as $x \to y$, $\varepsilon \downarrow 0$ for any $y \in R$, and that the limiting kernel

*Received by the editors November 20, 1990.