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ORDINARY DIFFERENTIAL EQUATIONS

Invariant Surfaces of Standard Two-Dimensional Systems with Conservative First Approximation of the Third Order

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Abstract—We simultaneously study two classes of two-dimensional time-periodic systems of differential equations with a small positive parameter, namely, systems with "slow" or "fast" time whose first-approximation systems are autonomous and conservative and do not contain terms of order higher than three. Thus, the corresponding unperturbed systems have one, two, or three rest points.

For the perturbations, we indicate explicit conditions, independent of the small parameter, under which every original system of either class with coefficients three times continuously differentiable with respect to the phase variables and the parameter in a neighborhood of zero has finitely many two-dimensional invariant surfaces homeomorphic to tori for all sufficiently small parameter values. We also give formulas for these surfaces.

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1. INTRODUCTION

1.1. Statement of the Problem

Consider the standard two-dimensional system

$$\begin{aligned} \dot{x}_1 &= (-x_2 + X_1(t, x_1, x_2, \varepsilon)\,\varepsilon)\,\varepsilon^\nu, \\ \dot{x}_2 &= \left(x_1^3 - 2\sigma x_1^2 + \eta x_1 + X_2(t, x_1, x_2, \varepsilon)\,\varepsilon\right)\varepsilon^\nu \quad (\nu = 0, 1) \end{aligned} \tag{1^ν}$$

with a small parameter ε, where $\sigma = 0, 1$, $\eta = -1, 0, 1$ for $\sigma = 0$, $\eta \in \mathbf{R}$ for $\sigma = 1$, and X_1 and X_2 are continuous functions defined for $t \in \mathbf{R}$, $|x_1| < x_0$, $|x_2| < x_0$, and $0 \le \varepsilon < \varepsilon_0$, T-periodic with respect to t, and belonging to the class C^3 with respect to x_1, x_2, and ε in the above-mentioned domain.

Essentially, formula (1) describes two different systems, (1^0) and (1^1); comparing them, one can say that system (1^1) has "fast" time, since when reducing it to (1^0) one obtains a system with period $T\varepsilon$.

The first approximation system, or the unperturbed system, corresponding to system (1^ν) is naturally defined as the autonomous system

$$\dot{x}_1 = -x_2\varepsilon^\nu, \qquad \dot{x}_2 = \left(x_1^3 - 2\sigma x_1^2 + \eta x_1\right)\varepsilon^\nu \quad (\nu = 0, 1). \tag{2^ν}$$

Obviously, the phase portraits of systems (2^0) and (2^1) coincide.

Let us introduce notation for the roots of the polynomial $x_1^3 - 2\sigma x_1^2 + \eta x_1$. We set

$$\gamma_0 = 0; \qquad \gamma_1 = \sigma - \sqrt{\sigma^2 - \eta}, \qquad \gamma_2 = \sigma + \sqrt{\sigma^2 - \eta} \quad \text{if} \quad \eta \le \sigma^2. \tag{3}$$

If $\sigma = 1$ and $\eta \le 1$, then $\gamma_1 = 1 - \sqrt{1-\eta}$, $\gamma_2 = 1 + \sqrt{1-\eta}$, and $\eta = \gamma_j(2 - \gamma_j)$; if $\sigma = 0$ and $\eta = -1$, then $\gamma_j = (-1)^j$; if $\eta = 0$, then $\gamma_j = 0$ $(j = 1, 2)$.

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System (2^ν) is conservative. The phase plane of system (2^ν) is filled with closed trajectories and separatrices, which are given by the integrals

$$x_1^4 - 8x_1^3\sigma/3 + 2\eta x_1^2 + 2x_2^2 = a$$

and surround one, two, or three rest points $(0,0)$, $(\gamma_1, 0)$, and $(\gamma_2, 0)$; here the last two points can be absent or coincide with the first one.

The aim of the present paper is as follows. For any sufficiently small $\varepsilon > 0$, we find several two-dimensional cylindrical invariant surfaces of system (1^ν) which are homeomorphic to tori if time is taken modulo the period and whose phase projections lie in a small neighborhood of some closed trajectories of the unperturbed system (2^ν) and thereby surround one or all rest points.

As a result, we write out closed-form conditions on the functions $X_i(t, x_1, x_2, 0)$ under which the perturbed system (1^ν) has the above-mentioned invariant surfaces. These conditions depend on σ and η. We also write out estimates for the number of such surfaces and present the asymptotic expansion of each of these surfaces in powers of the small parameter.

Systems (1^0) and (1^1) are studied simultaneously, since the invariant tori are found with the use of the same method developed in [1, 2] and substantially modified in [3]. However, when finding the bifurcation equations, there are important distinctions between the cases $\nu = 0$ and $\nu = 1$, and we discuss these distinctions in detail.

1.2. Approaches to the Solution

The general method for solving such problems is quite standard and includes three stages. One should

1. Rewrite system (1^ν) in special polar coordinates r and φ (called the "action–angle variables" in [4, Sec. 50]) in a neighborhood of an arbitrary closed trajectory of the unperturbed system (2^ν).

2. Find and analyze conditions on the functions $X_i(t, x_1, x_2, 0)$ in system (1^ν) sufficient for the bifurcation equation to have at least one admissible solution satisfying the nondegeneracy condition, which guarantees the existence of a nonzero focal quantity generated by the above-mentioned functions.

3. Make a number of averaging and scaling changes of variables reducing system (1^ν) to a system that, by the Hale lemma [5], has a two-dimensional invariant surface homeomorphic to a torus.

Therefore, nowadays the main problems that necessitate the development of new approaches and methods arise at the first stage of the analysis when constructing special polar coordinates for increasingly complicated first approximation systems with several rest points.

Additional interest is also attracted by the second stage, since the bifurcation equation, which is quadratic for systems with simpler first approximations, in this case has a very complicated integral structure.

The third stage is standard but should be realized with care, since it permits one to obtain closed formulas for each invariant surface corresponding to a root of the bifurcation equation.

1.3. Versions of System (1^1)

1. System (1^1) can be obtained from the more general T-periodic system

$$\begin{aligned} \dot{y}_1 &= -y_2\varepsilon + Y_1(t, y_1, y_2, \varepsilon)\varepsilon^2, \\ \dot{y}_2 &= \left(\upsilon_0(t)y_1^3 + \upsilon_1(t)y_1^2 + \upsilon_2(t)y_1 + \upsilon_3(t)\right)\varepsilon + Y_2(t, y_1, y_2, \varepsilon)\varepsilon^2 \end{aligned} \qquad (1_1^1)$$

with a small positive parameter ε, where $\int_0^T \upsilon_0(t)dt > 0$ and the Y_i are functions similar to X_i.

Indeed, after the change of variables $y_2 = w_2 + \sum_{k=0}^3 \tilde{\upsilon}_k^*(t) y_1^{3-k}\varepsilon$ in (1_1^1), the first approximation $\left(-y_2, \sum_{k=0}^3 \upsilon_k y_1^{3-k}\right)\varepsilon$ acquires the form $\left(-w_2, \sum_{k=0}^3 \bar{\upsilon}_k y_1^{3-k}\right)\varepsilon$. Here $\upsilon(t) = \bar{\upsilon} + \tilde{\upsilon}(t)$, where $\bar{\upsilon}$ is the mean value of $\upsilon(t)$, and $\tilde{\upsilon}^*(t) = \int_{t^*}^t \tilde{\upsilon}(\zeta)d\zeta$ is a T-periodic function with zero mean for an appropriate choice of t^*.

After the change of variables $y_1 = w_1 + \zeta$, we obtain $(-w_2, \bar{\upsilon}_0 w_1^3 + \check{\upsilon}_1 w_1^2 + \check{\upsilon}_2 w_1)\varepsilon$, where $\bar{\upsilon}_0 > 0$, $\check{\upsilon}_1 = \bar{\upsilon}_1 + 3\zeta\bar{\upsilon}_0$, and $\check{\upsilon}_2 = \bar{\upsilon}_2 + 2\zeta\bar{\upsilon}_1 + 3\zeta^2\bar{\upsilon}_0$ provided that ζ is a real root of the cubic equation $\sum_{k=0}^3 \bar{\upsilon}_k \zeta^{3-k} = 0$.

Let $\check{v}_1 = 0$. If $\check{v}_2 = 0$, then we make the scaling change of variables $w_1 = x_1$, $w_2 = \bar{v}_0^{1/2} x_2$, $\varepsilon = \bar{v}_0^{-1/2}\varepsilon$, but if $\check{v}_2 \neq 0$, then we make the change of variables $w_1 = \bar{v}_0^{-1/2}|\check{v}_2|^{1/2} x_1$, $w_2 = \bar{v}_0^{-1/2}|\check{v}_2| x_2$, $\varepsilon = |\check{v}_2|^{-1/2}\varepsilon$. We obtain the first approximation $(-x_2, x_1^3)\varepsilon$ in the first case and $(-x_2, x_1^3 + \operatorname{sgn}\check{v}_2 x_1)\varepsilon$ in the second case. But if $\check{v}_1 \neq 0$, then we make the change of variables $w_1 = -2^{-1}\bar{v}_0^{-1}\check{v}_1 x_1$, $w_2 = -2^{-2}\bar{v}_0^{-3/2}\check{v}_1|\check{v}_1| x_2$, $\varepsilon = 2\bar{v}_0^{1/2}|\check{v}_1|^{-1}\varepsilon$ and obtain the first approximation

$$\left(-x_2, x_1^3 - 2x_1^2 + 2\bar{v}_0^{1/2}|\check{v}_1|^{-1}\check{v}_2 x_1\right)\varepsilon.$$

In particular, if $\sigma \neq 0$ in a system of the form (1^1), then it can be scaled to unity; if $\sigma = 0$, then a nonzero η can be scaled to ± 1.

2. The system

$$\dot{u}_1 = u_2 + U_1(t, u_1, u_2, \varepsilon), \qquad \dot{u}_2 = -u_1^3 + 2u_1^2\varepsilon - \eta u_1\varepsilon^2 + U_2(t, u_1, u_2, \varepsilon) \tag{1_2^1}$$

was considered in [3], where $U_i = U_i^{[2+i]} + U_i^{>[2+i]}$; $U_i^{[k]}$ is a form of order k in u_1, u_2, and ε with coefficients 2π-periodic in t assuming that u_1 and ε have the first order of smallness and u_2 has the second order of smallness; $U_i^{>[k]}$ is a function 2π-periodic in t and sufficiently smooth in a neighborhood of the point $u_1 = u_2 = \varepsilon = 0$ whose expansion starts from some order $> k$. This system can be reduced by the change of variables $u_1 = x_1\varepsilon$, $u_2 = -x_2\varepsilon^2$ to system (1^1) with $\sigma = 1$ and

$$X_i(t, x_1, x_2, 0) = \sum_{p_1+2p_2=0}^{2+i} U_i^{(p_1, 2p_2, 2+i-p_1-2p_2)}(t) x_1^{p_1}(-x_2)^{p_2};$$

i.e., if $\varepsilon = 0$, then X_1 has six terms at most quadratic in x_2 and X_2 has nine terms at most cubic in x_2.

3. Another special case of system (1^1) is given by the Duffing equation

$$\ddot{y} + y^3 - 2\sigma y^2\varepsilon + \eta y\varepsilon^2 = Y(t, y, \dot{y}, \varepsilon), \tag{1_3^1}$$

which describes small oscillations of a periodic oscillator whose restoring force has the third order of smallness and whose quadratic and linear terms depend on the small parameter.

Equation (1_3^1) can be reduced by the change of variables $y = u_1$, $\dot{y} = u_2$ to system (1_2^1) with

$$U_1(t, x_1, x_2, \varepsilon) \equiv 0, \qquad U_2(t, x_1, x_2, \varepsilon) = -Y(t, x_1, x_2, \varepsilon).$$

2. PARAMETRIZATION OF CLOSED TRAJECTORIES AND SEPARATRICES OF A UNPERTURBED SYSTEM

2.1. Admissible Initial Values

By using the structure of system (2^ν), we introduce special polar coordinates by analogy with the coordinates that were introduced in [6, p. 290], in particular, for the unperturbed system $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1^3$ and were then generalized in [2] with the normalization preserved to the unperturbed system $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1^3 - \eta^2 x_1\varepsilon^2$ with a single rest point.

The main difference between the special trigonometric functions $C(\varphi)$ and $S(\varphi)$ introduced in [3] and in the present paper is that the function $C(\varphi)$ is not only nonsymmetric owing to the presence of a term containing x_1^2 in system (2^ν) but is also chosen to be not normalized at zero, which permits directly parametrizing any closed trajectory and separatrix of the unperturbed system by the above-mentioned trigonometric functions.

Thus, consider the real autonomous system

$$C'(\varphi) = -S(\varphi), \qquad S'(\varphi) = C^3(\varphi) - 2\sigma C^2(\varphi) + \eta C(\varphi). \tag{4}$$

Obviously, its trajectories coincide with those of system (2^ν).

The closed trajectories of (4) that are generators of invariant cylindrical surfaces in the (φ, C, S)-space meet the abscissa axis at certain points. To find out what these points are for various values of σ and η, we introduce the function

$$f(\zeta) = \zeta^4 - 8\sigma\zeta^3/3 + 2\eta\zeta^2.$$

Then $f(0) = 0$, $f(\zeta) \to +\infty$ as $\zeta \to \pm\infty$; by (3), if $\sigma = 1$ and $\eta \le 1$, then $f(\gamma_j) = \gamma_j^3 (4/3 - \gamma_j)$, and if $\sigma = 0$ and $\eta = -1$, then $f(\gamma_j) = -1$. In addition, the derivative $f' = 4\zeta(\zeta^2 - 2\sigma\zeta + \eta)$ is zero at the points γ_0, γ_1, and γ_2, and $f'(C(\varphi)) \equiv 4S'(\varphi)$ for the trajectories of system (4).

For certain values of the real parameter c, the relation

$$f(C) + 2S^2 = f(c), \quad \text{or} \quad C^4 - 8\sigma C^3/3 + 2\eta C^2 + 2S^2 = f(c), \tag{5}$$

is obviously an integral of system (4).

Definition 1. A real parameter c is said to be *admissible* if $c \ne \gamma_l$, $l = 0, 1, 2$, [see (3)] and there exists a number $b = b(c)$ such that

$$b < c, \quad b \ne \gamma_l, \quad f(b) = f(c), \quad f(\zeta) < f(c) \quad \text{for} \quad \zeta \in (b, c).$$

For any admissible parameter c, (b, c) is referred to as an *admissible interval.* It is a solution of the inequality $C^2(C^2 - 8\sigma C/3 + 2\eta) < f(c)$, or $f(C) < f(c)$, and formula (5) describes a closed curve passing through the points $(b, 0)$ and $(c, 0)$ in the coordinates C and S. Since $b, c \ne \gamma_l$, it follows that this curve is not a separatrix loop of system (4).

Thus if c is an admissible parameter, then the solution of the Cauchy problem for system (4) with the initial data $C(0) = c$, $S(0) = 0$ is given by real-analytic $\omega = \omega(c)$-periodic functions $C(\varphi) = C(\varphi, c)$ and $S(\varphi) = S(\varphi, c)$. In addition, $C(\omega/2) = b$ and $S(\omega/2) = 0$.

2.2. Classification of Closed Trajectories of System (4)

For any values of the parameters σ and η occurring in the unperturbed part of system (1^ν), we split the set of closed trajectories of system (4), or the set of admissible values of the parameter c, into three disjoint classes 0, 1, and 2 depending on which of the rest points $(\gamma_0, 0)$ of system (4) [the origin, $(\gamma_1, 0)$, or $(\gamma_2, 0)$] is surrounded by the closed trajectory $C(\varphi)$, $S(\varphi)$ passing through the point $(c, 0)$ for $\varphi = 0$. The trajectory can also surround other rest points, only together with the adjacent separatrices.

For each of these classes of closed trajectories, we establish the boundary of the set of admissible values of the parameter c, the corresponding boundary for the parameter b, and the range of the function $f(c)$ depending on the values of the parameters σ and η.

Lemma 1. *The parameter c is admissible in the following cases:*

a. $\sigma = 1$ *and* $\eta \le 0$ $(\eta \le \gamma_1 \le 0,\ 2 \le \gamma_2 \le 2 - \eta)$; *then*

0a. $b \in (-\infty, b_a^*)$, $c \in (c_a^*, +\infty)$, *and* $f(c) > 0$.

1a. $b \in (b_a^*, \gamma_1)$, $c \in (\gamma_1, 0)$, *and* $f(\gamma_1) < f(c) < 0$ *for* $\eta < 0$ $(\gamma_1 < 0,\ \gamma_2 > 2)$.

2a. $b \in (0, \gamma_2)$, $c \in (\gamma_2, c_a^*)$, *and* $f(\gamma_2) < f(c) < 0$, *where* $b_a^* = 4\left(1 - \sqrt{1 - 9\eta/8}\right)/3 < \gamma_1$ *for* $\eta < 0$ *and* $b_a^* = 0$ *for* $\eta = 0$, $c_a^* = 4\left(1 + \sqrt{1 - 9\eta/8}\right)/3 > \gamma_2$; *i.e.,* b_a^* *and* c_a^* *are roots of the equation* $f(C) = f(0)$.

b. $\sigma = 1$ *and* $0 < \eta \le 1$ $(0 < \gamma_1 \le \eta,\ 2 - \eta \le \gamma_2 < 2)$; *then*

0b. $b \in (b_b^*, 0)$, $c \in (0, \gamma_1)$, *and* $0 < f(c) < f(\gamma_1)$.

1b. $b \in (-\infty, b_b^*)$, $c \in (c_b^*, +\infty)$, *and* $f(c) > f(\gamma_1)$.

2b. $b \in (\gamma_1, \gamma_2)$, $c \in (\gamma_2, c_b^*)$, *and* $f(\gamma_2) < f(c) < f(\gamma_1)$ *for* $0 < \eta < 1$ $(\gamma_1 < \eta,\ \gamma_2 > 2 - \eta)$, *where* $b_b^* < 0 : f(b_b^*) = f(\gamma_1)$, $c_b^* = 1$ *for* $\eta = 1$ *and* $c_b^* > \gamma_2$; $f(c_b^*) = f(\gamma_1)$ *for* $0 < \eta < 1$; *moreover,* $f(\gamma_1) = \gamma_1^3(4/3 - \gamma_1) > 0$.

c. $\sigma = 0$, $\eta = -1$ $[\gamma_1 = -1,\ \gamma_2 = 1$, *and* $f(\zeta) = \zeta^4 - 2\zeta^2]$, *then*

0c. $b \in (-\infty, b^*_{\rm c})$, $c \in (c^*_{\rm c}, +\infty)$, *and* $f(c) > 0$.

1c. $b \in (b^*_{\rm c}, \gamma_1)$, $c \in (\gamma_1, 0)$, *and* $-1 < f(c) < 0$.

2c. $b \in (0, \gamma_2)$, $c \in (\gamma_2, c^*_{\rm c})$, *and* $-1 < f(c) < 0$, *where* $b^*_{\rm c} = -\sqrt{2}$ *and* $c^*_{\rm c} = \sqrt{2}$; *i.e.*, $b^*_{\rm c}$ *and* $c^*_{\rm c}$ *are roots of the equation* $f(C) = f(0)$.

d. *either* $\sigma = 0$ *and* $\eta = 0, 1$ *or* $\sigma = 1$ *and* $\eta > 1$; *then*

0d. $b \in (-\infty, 0)$, $c \in (0, +\infty)$, *and* $f(c) > 0$.

Proof. The properties of the function $f(\zeta)$ permit one to conclude that the boundary of admissible values of c coincides with the boundary of the intervals on which $f'(C) > 0$ with "separatrix" points deleted.

In case a, we have $f'(C) > 0$ for $C \in (\gamma_1, 0) \cup (\gamma_2, +\infty)$, and f attains the zero maximum for $C = 0$ and minima at the points γ_1 and γ_2. (If $\gamma_1 = 0$, then it is an inflection point.) Therefore, for $c = 0$ Eq. (5) defines one ($\eta = 0$) or two ($\eta < 0$) separatrices entering the point $(0, 0)$, and the equation $f(C) = 0$ has the roots $c^*_{\rm a} = 4/3 + \sqrt{16/9 - 2\eta} > \gamma_2$ and $b^*_{\rm a} = 4/3 - \sqrt{16/9 - 2\eta} \le \gamma_1$.

In case b, we have $f'(C) > 0$ for $C \in (0, \gamma_1) \cup (\gamma_2, +\infty)$, and f attains the maximum at the point γ_1 and the minimum at the point γ_2. (If $\gamma_1, \gamma_2 = 1$, then it is an inflection point.) Moreover, $f(\gamma_1) > 0$. Therefore, for $c = \gamma_1$ Eq. (5) defines one ($\eta = 1$) or two ($0 < \eta < 1$) separatrices at the point $(\gamma_1, 0)$, and the equation $f(C) = f(\gamma_1)$ always has a root $b^*_{\rm b} < 0$ and a root $c^*_{\rm b} > \gamma_2$ if $0 < \eta < 1$.

Case c is similar to the two preceding cases.

In case d, we have $f'(C) > 0$ for $C > 0$; therefore, every $c > 0$ is admissible.

By M we denote the set of admissible values of the parameter c. Let $M^0_{\rm a} = (c^*_{\rm a}, +\infty)$, $M^0_{\rm b} = (0, \gamma_1)$, $M^0_{\rm c} = (c^*_{\rm c}, +\infty)$, $M^0_{\rm d} = (0, +\infty)$, $M^1_{\rm a} = (\gamma_1, 0)$, $M^1_{\rm b} = (c^*_{\rm b}, +\infty)$, $M^1_{\rm c} = (\gamma_1, 0)$, $M^2_{\rm a} = (\gamma_2, c^*_{\rm a})$, $M^2_{\rm b} = (\gamma_2, c^*_{\rm b})$, and $M^2_{\rm c} = (\gamma_2, c^*_{\rm c})$. Then , by the classification in Lemma 1, the set M has the form $M = M_{\rm a} = \bigcup_{l=0}^2 M^l_{\rm a}$ in case a, in which $\sigma = 1$ and $\eta \le 0$, $M = M_{\rm b} = \bigcup_{l=0}^2 M^l_{\rm b}$ in case b, in which $\sigma = 1$ and $0 < \eta \le 1$, $M = M_{\rm c} = \bigcup_{l=0}^2 M^l_{\rm c}$ in case c, in which $\sigma = 0$ and $\eta = -1$, and $M = M_{\rm d} = M^0_{\rm d}$ in case d, in which either $\sigma = 0$ and $\eta = 0, 1$ or $\sigma = 1$ and $\eta > 1$.

In turn, class l ($l = 0, 1, 2$) consists of admissible values of c in the set $M^l = \{M^l_{\rm a}$ in case a, $M^l_{\rm b}$ in case b, $M^l_{\rm c}$ in case c, and $M^l_{\rm d}$ in case d$\}$; i.e., $M = \bigcup_{l=0}^2 M^l$.

2.3. Separatrices of System (4)

The partition into classes 0, 1, and 2 introduced for closed trajectories determines the singular point $(0, 0)$, $(\gamma_1, 0)$, or $(\gamma_2, 0)$, respectively, of system (4) for which the admissible intervals (b, c) and separatrices are constructed in Lemma 1 for various values of the parameters σ and η distinguished, in turn, between cases a–d.

Corollary 1. *In case* a, *formula* (5) *with* $c = c^*_{\rm a}$ *describes the equations of two separatrices, left and right, forming a separatrix figure-eight entering the point* $(0, 0)$ *and denoted by* $S^b_{\rm a} \cup S^c_{\rm a}$, *where* $S^b_{\rm a}$ *is the trajectory of system* (4) *passing through the point* $(b^*_{\rm a}, 0)$ *and* $S^c_{\rm a}$ *is the trajectory passing through the point* $(c^*_{\rm a}, 0)$. *If* $\eta \nearrow 0$, *then* $b^*_{\rm a} \searrow \gamma_1 \searrow 0$ *and* $c^*_{\rm a} \nearrow \gamma_2 \nearrow 2$; *therefore,* $S^b_{\rm a} = \varnothing$ *for* $\eta = 0$.

In cases b, *formula* (5) *with* $c = c^*_{\rm b}$ *describes the equations of the separatrix figure-eight entering the point* $(\gamma_1, 0)$ *and denoted by* $S^b_{\rm b} \cup S^c_{\rm b}$, *where* $S^b_{\rm b}$ *is the trajectory passing through the point* $(b^*_{\rm b}, 0)$ *and* $S^c_{\rm b}$ *is the trajectory passing through the point* $(c^*_{\rm b}, 0)$. *If* $\eta \nearrow 0$, *then* $b^*_{\rm b} \searrow b_1$, *where* b_1 *is a root of the equation* $f(C) = f(1)$, $\gamma_1 \nearrow 1$, *and* $c^*_{\rm b} \searrow \gamma_2 \searrow 1$; *therefore,* $S^c_{\rm b} = \varnothing$ *for* $\eta = 1$.

In case c, *formula* (5) *with* $c = c^*_{\rm c}$ *describes the equations of the separatrix figure-eight again entering the point* $(0, 0)$ *and denoted by* $S^b_{\rm c} \cup S^c_{\rm c}$, *where* $S^b_{\rm c}$ *is the trajectory of the solution with the initial data* $C(0) = -\sqrt{2}$, $S(0) = 0$, *and* $S^c_{\rm c}$ *is the trajectory with the initial data* $C(0) = \sqrt{2}$, $S(0) = 0$.

In cases d, *the separatrices are absent.*

Note that for the parametrization of S^b the initial data on the abscissa axis have to be posed at the left point b^* rather than at the right point (which is a rest point).

Thus, the above-introduced partition into classes 0, 1, and 2 implies the following: if the parameter c ranges over the set M^l $(l = 0, 1, 2)$ of admissible values, then the closed trajectories $C(\varphi)$, $S(\varphi)$ of system (4) passing through the points $(c, 0)$ for $\varphi = 0$ fill either a neighborhood of the singular point $(\gamma_l, 0)$ inside a separatrix surrounding that point or (for classes 0 and 1) the entire phase plane outside a separatrix figure-eight entering the points $(\gamma_0, 0)$ and $(\gamma_1, 0)$, respectively.

This remark completes the construction of the phase portrait of the unperturbed system (2^ν), which coincides with (4) after the change of time $t = \varphi/\varepsilon^\nu$.

3. PASSAGE INTO A NEIGHBORHOOD OF A CLOSED TRAJECTORY OF CLASS 0

3.1. Special Polar Change of Variables

Let c range over the set of admissible values in class 0. Then the closed trajectories $C(\varphi)$, $S(\varphi)$ of system (4) passing through the point $(c, 0)$ for $\varphi = 0$ fill the following neighborhoods of the origin: outside $S_{\mathrm{a}}^b \cup S_{\mathrm{a}}^c$ in cases 0a and 0c, inside S_{b}^b in case 0b, and any neighborhood in case 0d.

We fix an arbitrary c and find the ω-periodic solution $C(\varphi)$, $S(\varphi)$ of system (4) with the initial data $C(0) = c$, $S(0) = 0$.

We investigate the perturbed system (1^ν) in a small neighborhood of the closed cylindrical surface $f(x_1) + 2x_2^2 = f(c)$ generated by the chosen trajectory $x_1 = C(\varphi)$, $x_2 = S(\varphi)$ of system (2^ν). To this end, in system (1^ν) we perform a special affine polar change of variables with a nonsymmetric nonnormalized generalized cosine:

$$x_1 = C(\varphi)(1 + r), \qquad x_2 = S(\varphi)(1 + r)^2 \quad (|r| < 1), \tag{6}$$

which transforms the cylindrical surface to the point $r = 0$.

For the realization of the substitution (6), it is necessary that system (1^ν) satisfies the inequality $|x_1|, |x_2| < x_0$ for all $r < 1$. Since the interval $[b, c]$ is the range of the function $C(\varphi)$, the function $f(C)$ attains its minimum at one of the points γ_l $(l = 0, 1, 2)$, and $2S^2 = f(c) - f(C)$, from (5), we have $|C(\varphi)| \le \max\{-b, c\}$ and $|S(\varphi)| \le \max \sqrt{(f(c) - f(\gamma_l))/2}$.

We have $|x_1| < 2|C(\varphi)|$ and $|x_2| < 4|S(\varphi)|$ in the change of variables (6); consequently, the desired inequality is valid if

$$\max\left\{-b, c, \sqrt{2f(c) - 2f(\gamma_l)}\right\} < x_0/2. \tag{7}$$

If c grows inside some range of admissible values, then the left-hand side of inequality (7) increases; therefore, condition (7) can substantially restrict the set of closed trajectories in class 0.

The function

$$p(C) = f(c) + 2\sigma C^3/3 - \eta C^2 \quad \left(p'(C) = 2\sigma C^2 - 2\eta C\right),$$

depending on c, plays an important role when making the change of variables (6). By (5),

$$p(C) = C^4 - 2\sigma C^3 + \eta C^2 + 2S^2 = CS' + 2S^2.$$

By differentiating the change of variables (6) according to system (1^ν) and by solving the resulting equations for $\dot r$ and $\dot\varphi$, we obtain the system

$$\begin{aligned} p(C)\varrho\dot r &= (S\varrho X_0(C, \varrho) + (S'\varrho X_1 + SX_2)\,\varepsilon)\,\varepsilon^\nu, \\ p(C)\varrho^2\dot\varphi &= \left(p(C)\varrho^3 + C\varrho X_0(C, \varrho) + (CX_2 - 2S\varrho X_1)\,\varepsilon\right)\varepsilon^\nu, \end{aligned} \tag{8^ν}$$

where

$$\varrho = 1 + r, \qquad X_0 = 2\sigma C^2\varrho(\varrho - 1) - \eta C\left(\varrho^2 - 1\right), \qquad X_i = X_i\left(t, C\varrho, S\varrho^2, \varepsilon\right).$$

3.2. Verification of the Monotonicity of the Angular Variable

In general, the function $p(C)$ occurring on the left-hand side in system (8^ν) can change sign.

Lemma 2. *Let c be an admissible parameter in class 0. Then $p(C) \ge p_0 > 0$ in system (8^ν) for $C \in [b, c]$; moreover, $b < 0 < c$. But if c belongs to class* 1b *and $\eta < 1$, then $p(C)$ changes sign on $[b, c]$.*

Proof. We have $p(0) = f(c) > 0$ in all cases considered in the lemma; therefore, $p(C) > f(c)/2$ for $|C| \le \tau$. In addition, $p(C) \ge q(C) = C^2 (C^2 - 2C + \eta)$ for $C \in [b, c]$, since $f(C) \le f(c)$.

The inequality $\eta \le 0$ is valid in case 0a; then $p(C) \ge f(c)$ for $C \ge 0$ and $C \in [\eta, 0)$, since $p'(C) < 0$ for $C \in (\eta, 0)$. Since $q(C)$ has the roots γ_1, 0, 0, and γ_2, we have $q(C) > 0$ for $C < \gamma_1$. Since $\eta \le \gamma_1$, we find that $p(C) \ge p_* > 0$ for $C \in [b, \eta]$.

In cases 0b and 1b, we have $\eta > 0$; therefore, $q(C) \ge q_* > 0$ for $C \in [b, -\tau]$.

If $C \in [\tau, c]$, then in case 0b, we have $0 < c < \gamma_1$ and $q(C) > 0$ for $C \in (0, \gamma_1)$.

In case 1b, we have $p'(C) < 0$ for $C \in (0, \eta)$ and $p(\eta) = f(c) - \eta^3/3$ at the point of minimum. But $\eta \le \gamma_2 < c$, and so $p(C) \ge p^* > 0$ for $C \ge 0$ if $f(c) > \eta^3/3$, which is valid only for $\eta = 1$. If, on the other hand, $0 < \gamma_1 < \eta < 1$, then this condition fails for c close to c_{b}^*, since $\eta^3/3 > f(\gamma_1) = \gamma_1^3 (4/3 - \gamma_1)$.

Case 0c is obvious, since $p(C) = f(c) + C^2$ in it.

Case 0d is also obvious, since $q(C) > 0$ for any $C \ne 0$ if $\eta > 1$.

Remark 1. It follows from geometric considerations that $p(C)$ is zero in cases 1a, 1c, and 2, provided that the interval $[b, c]$ does not contain zero for any admissible value of c, since the derivative $\dot\varphi$ in system (8^ν) should change sign when going around a closed trajectory not surrounding the origin if the motion is observed from the point $(0, 0)$. In case b, $b < 0 < \gamma_1 < \gamma_2 < c$. Nevertheless, if from the point $(0, 0)$ we observe going around a closed trajectory lying outside a separatrix figure-eight and sufficiently close to it, the polar angle is oscillating in a neighborhood of the adjacency of the figure-eight to the singular point $(\gamma_1, 0)$, which again implies a change of sign of $p(C)$. Therefore, in cases 1 and 2, special polar coordinates should be introduced in neighborhoods of the points $(\gamma_1, 0)$ and $(\gamma_2, 0)$ after the preliminary shift of the origin to these points.

3.3. Completion of the Polar Change of Variables

For a smooth function $v(\zeta, \varepsilon)$, we set $v^\zeta = \partial v/\partial\zeta$ and $v^\varepsilon = \partial v/\partial\varepsilon$. We denote the arguments $(t, C(1 + r), S(1 + r)^2, \varepsilon) = (\varkappa)$ and $(t, C, S, 0) = (\varkappa_0)$. Then in system (8^ν), we have

$$X_i(\varkappa) = X_i(\varkappa_0) + \left(X_i^{x_1}(\varkappa_0) C + X_i^{x_2}(\varkappa_0) \cdot 2S\right) r + X_i^\varepsilon(\varkappa_0)\varepsilon + O\left((|r| + \varepsilon)^2\right)$$

and, by (1^ν), (6), and (7), $O(\cdots) \in C^3_{r,\varepsilon}$ for $|r| < 1$, $0 \le \varepsilon < \varepsilon_0$.

Let $R(t, \varphi, r, \varepsilon) = X_1(\varkappa)S' + X_2(\varkappa)(1+r)^{-1}S$ and $R_0(t, \varphi) = R(t, \varphi, 0, 0)$. By taking into account Lemma 2, one can rewrite system (8^ν) in the form

$$\begin{aligned} \dot r &= \left(\left(R_1 r + R_2 r^2\right) S + \left(R_0 + R_0^r r + R_0^\varepsilon \varepsilon + O\left((|r| + \varepsilon)^2\right)\right) p^{-1}\varepsilon\right)\varepsilon^\nu, \\ \dot\varphi &= \left(1 + p\Phi_1 r + O\left(|r|^2\right) + \left(\Phi_0 + O(|r| + \varepsilon)\right)\varepsilon\right)\varepsilon^\nu, \end{aligned} \tag{9^ν}$$

where

$$\begin{aligned} R_1 &= \left(2\sigma C^2 - 2\eta C\right)p^{-1}, \qquad R_2 = \left(2\sigma C^2 - \eta C\right)p^{-1}, \\ \Phi_1 &= \left(1 + CR_1(C)\right)p^{-1}, \qquad \Phi_0(t, \varphi) = \left(X_2(\varkappa_0)C - X_1(\varkappa_0 r) \cdot 2S\right)p^{-1} \end{aligned}$$

in view of the fact that $C = C(\varphi)$, $S = S(\varphi)$, and $O(\cdots) \in C^3_{r,\varepsilon}$ for $|r| < 1$, $0 \le \varepsilon < \varepsilon_0$.

Therefore,

$$\begin{aligned} R_1(C) &= p'p^{-1}, \qquad R_2(C) = (p' + \eta C)p^{-1}, \qquad \Phi_1(C) = p^{-1} + Cp'p^{-2}, \\ R_0(t, \varphi) &= X_1(\varkappa_0)S' + X_2(\varkappa_0)S, \qquad R_0^\varepsilon(t, \varphi) = X_1^\varepsilon(\varkappa_0)S' + X_2^\varepsilon(\varkappa_0)S, \\ R_0^r(t, \varphi) &= \left(X_1^{x_1}(\varkappa_0)C + X_1^{x_2}(\varkappa_0) \cdot 2S\right)S' + \left(X_2^{x_1}(\varkappa_0)C + X_2^{x_2}(\varkappa_0) \cdot 2S\right)S \end{aligned}$$

in system (9^ν); moreover,

$$S' = C^3 - 2\sigma C^2 + \eta C, \qquad p(C) = f(c) + 2\sigma C^3/3 - \eta C^2 = CS' + 2S^2.$$

3.4. Preliminary Averaging

First, we introduce the ω-periodic real-analytic function

$$p^*(C(\varphi)) = \int\limits_c^{C(\varphi)} p^{-3}(\theta)\theta \left(p'^2(\theta) - \eta p(\theta)\right) d\theta$$

satisfying the equation $p^{*\prime} = p^{-1}\left(p'\Phi_1 - R_2\right)$ and show that the averaging change of variables

$$r = p^{-1}(C(\varphi))\left(z + p^*(C(\varphi))z^2\right) \tag{10}$$

reduces system (9^ν) to the system

$$\begin{aligned} \dot z &= \left(O\left(|z|^3\right) + \left(R_0 + Zz + R_0^\varepsilon\varepsilon + O\left((|z|+\varepsilon)^2\right)\right)\varepsilon\right)\varepsilon^\nu, \\ \dot\varphi &= \left(1 + \Phi_1 z + O\left(|z|^2\right) + \left(\Phi_0 + O(|z|+\varepsilon)\right)\varepsilon\right)\varepsilon^\nu, \end{aligned} \tag{11^ν}$$

where $Z(t,\varphi) = p^{-1}R_0^r - p^{-1}p'S\Phi_0 - 2p^*R_0$.

To this end, we differentiate (10) according to systems (9^ν) and (11^ν) and cancel ε^ν; then

$$\begin{aligned} &\left(R_1p^{-1}\left(z + p^*z^2\right) + R_2p^{-2}z^2\right)S + \left(R_0 + R_0^r p^{-1}z + R_0^\varepsilon\varepsilon\right)p^{-1}\varepsilon + O\left((|z|+\varepsilon)^2\right)\varepsilon + O\left(|z|^3\right) \\ &\quad = p^{-1}\left(1 + 2p^*z\right)\left(R_0 + Zz + R_0^\varepsilon\varepsilon\right)\varepsilon + \left(-p^{-2}p'\left(z + p^*z^2\right) + p^{-1}p^{*\prime}z^2\right)(-S)\left(1 + \Phi_1 z + \Phi_0\varepsilon\right). \end{aligned}$$

By matching the coefficients of z, z^2, ε, $z\varepsilon$, and ε^2, we obtain obvious identities for the above-mentioned $p^*(C)$ and $Z(t,\varphi)$.

Remark 2. If desired, in the first equation in system (9^ν), one can nullify all terms containing ε^ν, i.e., nullify $O\left(|z|^3\right)$ in system (11^ν). This was carried out in [2].

Now in systems (11) one should average the functions $R_0(t,\varphi)$ and $Z(t,\varphi)$, but since $\dot\varphi = 1 + \cdots$ in (11^0) and $\dot\varphi = \varepsilon + \cdots$ in (11^1), it follows that the averaging changes of variables and conditions for their existence are different.

For functions $\upsilon(t,\varphi)$ T-periodic in t and ω-periodic in φ in the case $\nu = 1$, we use the expansion

$$\upsilon = \bar\upsilon + \hat\upsilon(\varphi) + \tilde\upsilon(t,\varphi), \qquad \bar\upsilon = \frac{1}{\omega T}\int\limits_0^\omega\int\limits_0^T \upsilon(t,\varphi)\,dt\,d\varphi, \qquad \hat\upsilon = \frac{1}{T}\int\limits_0^T \upsilon(t,\varphi)\,dt - \bar\upsilon.$$

Then the functions $\int \hat\upsilon(\varphi)d\varphi$ and $\int \tilde\upsilon(t,\varphi)dt$ are also periodic and are uniquely determined by the condition that their mean value is zero.

If $\nu = 1$, then it is unnecessary to single out the term $\hat\upsilon$; therefore, we simply write $\upsilon = \bar\upsilon + \tilde\upsilon(t,\varphi)$.

Let us show that, by using the change of variables

$$z = y + G^\nu(t,\varphi,y,\varepsilon)\varepsilon \quad \left(\overline{G^\nu}(y,\varepsilon) = 0\right), \tag{12^ν}$$

where $G^0 = \widetilde{g_0^0}(t,\varphi) + \widetilde{g_1^0}(t,\varphi)y$ and $G^1 = \widehat{g_0^1}(\varphi) + \widehat{g_1^1}(\varphi)y + \widetilde{g_0^1}(t,\varphi)\varepsilon + \widetilde{g_1^1}(t,\varphi)y\varepsilon$, one can reduce system (11^ν) to the system

$$\begin{aligned} \dot y &= \left(\left(\overline{R_0} + \overline{L^\nu}y + Y_0^\nu\varepsilon\right)\varepsilon + O\left((|y|+\varepsilon)^3\right)\right)\varepsilon^\nu, \\ \dot\varphi &= \left(1 + \Phi^\nu\varepsilon + \Phi_1 y + O\left((|y|+\varepsilon)^2\right)\right)\varepsilon^\nu. \end{aligned} \tag{13^ν}$$

Here $\Phi^0(t,\varphi) = \Phi_1\widetilde{g_0^0} + \Phi_0$; $\Phi^1(t,\varphi) = \Phi_1\widehat{g_0^1} + \Phi_0$; $\overline{L^\nu} = \bar{Z} - \overline{Y^\nu}$, where $Y^0(\varphi) = \widetilde{g_0^0}'\Phi_1$ and $Y^1(\varphi) = \widehat{R_0}\Phi_1$; $Y_0^0(t,\varphi) = \widetilde{g_0^0}Z + R_0^\varepsilon - \widetilde{g_0^0}'\Phi^0 - \widetilde{g_1^0}\overline{R_0}$ and $Y_0^1(t,\varphi) = \widehat{g_0^1}Z + R_0^\varepsilon - \widehat{g_0^1}'\Phi^1 - \widehat{g_1^1}\overline{R_0} - \widetilde{g_0^1}'$; R_0, R_0^ε, Φ_0, and Φ_1 are the functions introduced in (9^ν), and Z is defined in (11^ν).

Here and throughout the following, the dot stands for the partial derivative with respect to t, and the prime stands for the partial derivative with respect to φ.

By differentiating the change of variables (12^ν) according to systems (11^ν) and $(13)^\nu)$ and by cancelling ε^ν, we obtain

$$\begin{aligned}&(R_0 + (y + G^\nu\varepsilon)\,Z + R_0^e\varepsilon)\,\varepsilon + O\left((|y| + \varepsilon)^3\right)\\ &\quad= (1 + \varepsilon\partial G^\nu/\partial y)\left(\bar{R}_0 + \bar{L}^0 y + Y_0^0\varepsilon\right)\varepsilon + G^{\nu'}\varepsilon\,(1 + \Phi^\nu\varepsilon + \Phi_1 y) + \dot{G}^\nu\varepsilon^{1-\nu}.\end{aligned}$$

Let $\nu = 0$. The terms multiplying ε and $y\varepsilon$ form the equations

$$\widetilde{g_0^0}' + \dot{\widetilde{g_0^0}} = R_0 - \overline{R_0}, \qquad \widetilde{g_1^0}' + \dot{\widetilde{g_1^0}} = Z - g_0^{0\prime}\Phi_1 - \overline{L^0},$$

whose right-hand sides have zero means.

Suppose that the periods T and $\omega = \omega(c)$ satisfy the condition

$$|q\omega - pT| > K(p+q)^{-\tau}, \quad K > 0, \quad \tau \ge 1, \quad p \text{ and } q \text{ are positive integers.} \tag{14^0}$$

Then, by Lemma B.5 in [7, p. 17], the equations have solutions $\widetilde{g_0^0}(t,\varphi)$ and $\widetilde{g_1^0}(t,\varphi)$ of the same smoothness as that of the right-hand sides; i.e., the solutions are continuous, real-analytic, ω-periodic in φ, and T-periodic in t. They are uniquely determined by the conditions $\overline{g_0^0} = 0$ and $\overline{g_1^0} = 0$. Now the terms multiplying ε^2 form an obvious identity by virtue of the choice of Y_0^0.

Let $\nu = 1$. For ε^2 and $y\varepsilon^2$, we obtain the equations

$$R_0 = \bar{R}_0 + \widehat{g_0^1}' + \dot{\widetilde{g_0^1}}, \qquad Z = \bar{L}^1 + \widehat{g_0^1}'\Phi_1 + \widehat{g_1^1}' + \dot{\widetilde{g_1^1}},$$

whence it follows that $\widehat{g_0^1}' = \widehat{R}_0$, $\dot{\widetilde{g_0^1}} = \tilde{R}_0$; $\widehat{g_1^1}' = \hat{Z} - \widehat{Y^1}$, $\dot{\widetilde{g_1^1}} = \tilde{Z}$, and the desired functions g^1 can be found successively and uniquely without additional conditions like (14). Now from the resulting identity for the terms with ε^3, we find $Y^1(t,\varphi)$, which completes the construction of the change of variables (12^ν).

4. PASSAGE TO A NEIGHBORHOOD OF A CLOSED TRAJECTORY OF CLASSES 1 AND 2

4.1. Shift to the Singular Points $(\gamma_1, 0)$ and $(\gamma_2, 0)$ of the unperturbed System

Now let the parameter c be chosen from class 1 or 2; therefore, $\eta \le 1$ for $\sigma = 1$ and $\eta = -1$ for $\sigma = 0$. In this case, the closed trajectories of system (4) passing through the points $(c, 0)$, where c ranges over all admissible values, fill the following neighborhoods of the singular points $(\gamma_1, 0)$ and $(\gamma_2, 0)$ of the unperturbed system (2^ν): 1, inside S_{a}^b; 1b, outside S_{b}; 1c, inside S_{c}^b; 2a, inside S_{a}^c; 2b, inside S_{b}^c; 2c, inside S_{c}^c.

As was mentioned in Remark 1, one cannot directly make the change of variables (6) in system (1^ν) in these domains, since it would require dividing by the alternating function $p(C)$ in system (8^ν).

The difficulty thus arising is removed by the introduction of polar coordinates directly in a neighborhood of the point $(\gamma_1, 0)$ or $(\gamma_2, 0)$; to this end, the origin in systems (1^0) and (4) should be shifted to these points.

We introduce a common notation for classes 1 and 2. Set

$$\begin{aligned}&\breve{\gamma} = \gamma_j \quad (j = 1, 2); \qquad \breve{\sigma} = \sigma - 3\breve{\gamma}/2, \qquad \breve{\eta} = 3\breve{\gamma}^2 - 4\sigma\breve{\gamma} + \eta;\\ &\breve{b} = b - \breve{\gamma}, \qquad \breve{c} = c - \breve{\gamma}.\end{aligned}$$

By (3), if $\sigma = 1$ and $\eta \le 1$, then $\breve{\sigma} = 1 - 3\gamma_j/2$ and $\breve{\eta} = 2\gamma_j(\gamma_j - 1)$; and if $\sigma = 0$ and $\eta = -1$, then $\breve{\sigma} = 3(-1)^{j-1}/2$ and $\breve{\eta} = 2$.

We perform two shifting changes of variables

$$x_1 = \breve{\gamma} + \breve{x}_1, \quad x_2 = \breve{x}_2; \qquad C = \breve{\gamma} + \breve{C}, \quad S = \breve{S}. \tag{15}$$

The first change of variables in (15) reduces system (1^ν) to the form

$$\begin{aligned} \dot{\breve{x}}_1 &= \left(-\breve{x}_2 + \breve{X}_1(t, \breve{x}_1, \breve{x}_2, \varepsilon)\,\varepsilon\right)\varepsilon^\nu, \\ \dot{\breve{x}}_2 &= \left(\breve{x}_1^3 - 2\breve{\sigma}\breve{x}_1^2 - \breve{\eta}\breve{x}_1 + \breve{X}_2(t, \breve{x}_1, \breve{x}_2, \varepsilon)\,\varepsilon\right)\varepsilon^\nu \quad (\nu = 0, 1), \end{aligned} \tag{$\breve{1}^\nu$}$$

where $\breve{X}_i(t, \breve{x}_1, \breve{x}_2, \varepsilon) = X_i(t, \breve{\gamma} + \breve{x}_1, x_2, \varepsilon)$.

The second change of variables in (15) reduces system (4) to the system

$$\breve{C}'(\varphi) = -\breve{S}(\varphi), \qquad \breve{S}'(\varphi) = \breve{C}^3(\varphi) - 2\breve{\sigma}\breve{C}^2(\varphi) + \breve{\eta}\breve{C}(\varphi), \tag{$\breve{4}$}$$

which has the singular points $(0,0)$ and either $(-\gamma_j, 0)$, $(2 - 2\gamma_j, 0)$ if $\sigma = 1$ and $\eta \le 1$ (cases a and b) or $((-1)^{j-1}, 0)$ and $(2(-1)^{j-1}, 0)$ if $\sigma = 0$ and $\eta = -1$ (case c). System ($\breve{4}$) parametrizes the same phase portrait as (4).

By analogy with $f(C)$, we introduce the function

$$\breve{f}(\breve{C}) = \breve{C}^4 - 8\breve{\sigma}\breve{C}^3/3 + 2\breve{\eta}\breve{C}^2.$$

Then $\breve{f}(0) = 0$, $f(\breve{\gamma} + \breve{C}) = f(\breve{\gamma}) + \breve{f}(\breve{C})$, and $f'(\breve{\gamma} + \breve{C}) = \breve{f}'(\breve{C}) = 4\breve{S}'$. For any $\breve{c} = c - \breve{\gamma}$, the relation

$$\breve{f}(\breve{C}) + 2\breve{S}^2 = \breve{f}(\breve{c}) \quad \text{or} \quad \breve{C}^4 - 8\breve{\sigma}\breve{C}^3/3 + 2\breve{\eta}\breve{C}^2 + 2\breve{S}^2 = \breve{f}(\breve{c}) \tag{$\breve{5}$}$$

is an integral of system ($\breve{4}$).

By Lemma 1, if c is an admissible parameter, then we have the following for $\breve{\gamma} = \gamma_1$: 1a, $0 < \breve{f}(\breve{c}) < -f(\gamma_1)$; 1b, $\breve{f}(\breve{c}) > 0$; 1c, $0 < \breve{f}(\breve{c}) < -f(\gamma_1) = 1$. We also have the following for $\breve{\gamma} = \gamma_2$: 2a, $0 < \breve{f}(\breve{c}) < -f(\gamma_2)$; 2b, $0 < \breve{f}(\breve{c}) = f(\gamma_1) - f(\gamma_2) = 16(\gamma_2 - 1)^3/3$; 2c, $0 < \breve{f}(\breve{c}) < -f(\gamma_2) = 1$. In this case, one always has $\breve{b} < 0$ and $\breve{c} > 0$.

4.2. Polar Change of Variables in Classes 1 and 2

By analogy with the changes of variables without the symbol $\breve{}$, we subject system ($\breve{1}^\nu$) to the successive changes of variables ($\breve{6}$), ($\breve{10}$), and ($\breve{12}^\nu$) leading to systems ($\breve{8}^\nu$) and ($\breve{9}^\nu$), ($\breve{11}^\nu$), ($\breve{13}^\nu$).

The only fact to be justified is the positivity of the function

$$\breve{p}(\breve{C}) = \breve{f}(\breve{c}) + 2\breve{\sigma}\breve{C}^3/3 - \breve{\eta}\breve{C}^2 \quad \text{for} \quad \breve{C} \in [\breve{b}, \breve{c}];$$

i.e., one should prove an analog of Lemma 2, which permits one to derive systems ($\breve{9}$).

By ($\breve{4}$) and ($\breve{5}$), we have $\breve{p} = \breve{C}^4 - 2\breve{\sigma}\breve{C}^3 + \breve{\eta}\breve{C}^2 + 2\breve{S}^2 = \breve{C}\breve{S}' + 2\breve{S}'^2$, whence we obtain

$$\breve{p}(\breve{C}) \ge \breve{q}(\breve{C}) = \breve{C}^4 - 2\breve{\sigma}\breve{C}^3 + \breve{\eta}\breve{C}^2$$

for $\breve{C} \in [\breve{b}, \breve{c}]$, and $\breve{q}(\breve{C})$ has the roots 0, 0 and $-\gamma_j$, $2 - 2\gamma_j$ in cases a and b or $(-1)^{j-1}$, $2(-1)^{j-1}$ in cases c.

Since $\breve{p}(0) = \breve{f}(\breve{c}) > 0$, we have $\breve{p}(\breve{C}) \ge \breve{f}(\breve{c})/2$ for $|\breve{C}| \le \tau$ ($\tau > 0$) and $\breve{q}(\breve{C}) > 0$ if $\breve{C}$ is nonzero and does not lie between nonzero roots of $\breve{q}$.

Consider the following cases: 1a, $\breve{c} < -\gamma_1 < 2 - 2\gamma_1$; 2a, $2 - 2\gamma_2 \le -\gamma_2 < \breve{b}$; 2b, $-\gamma_2 < 2 - 2\gamma_2 < \breve{b}$; 1c, $\breve{c} < -\gamma_1 = 1$ (the roots 1 and 2); 2c, $\breve{b} > -\gamma_2 = -1$ (the roots -1 and -2). Therefore, $\breve{q}(\breve{C}) \ge q_0 > 0$ for $\breve{C} \in [\breve{b}, -\tau] \cup [\tau, \breve{c}]$.

It remains to consider case 2b, in which $0 < \breve{\gamma} = \gamma_1 \le 1$. If $\gamma_1 = 2/3$, then $\breve{\sigma} = 0$ and $\breve{p}\left(\breve{C}\right) = \breve{f}\left(\breve{c}\right) + 4\breve{C}^2/9 \ge p_0 > 0$. We set $\gamma_* = 4\gamma_1\left(\gamma_1 - 1\right)/\left(2 - 3\gamma_1\right)$; then $\breve{p}'\left(\breve{C}\right) = 2\breve{\sigma}\breve{C}^2 - 2\breve{\eta}\breve{C} = \left(2 - 3\breve{\gamma}\right)\breve{C}\left(\breve{C} - \gamma_*\right)$. If $\gamma_1 > 2/3$, then $-\gamma_1 < 0 < 2 - 2\gamma_1 < \gamma_*$ and $\breve{p}'\left(\breve{C}\right) > 0$ for $\breve{C} \in (0, \gamma_*)$; and if $\gamma_1 < 2/3$, then $\gamma_* < -\gamma_1 < 0 < 2 - 2\gamma_1$ and $\breve{p}'\left(\breve{C}\right) < 0$ for $\breve{C} \in (\gamma_*, 0)$. Therefore, we always have $\breve{p}\left(\breve{C}\right) \ge \breve{f}\left(\breve{c}\right) > 0$ for $\breve{C} \in [-\gamma_1, 2 - 2\gamma_1]$ and $\breve{q}\left(\breve{C}\right) > 0$ outside this interval.

Thus, in cases 1 and 2, in system $(\breve{7})$, we have $\breve{p}\left(\breve{C}\right) \ge p_0 > 0$ for $\breve{C} \in [\breve{b}, \breve{c}]$. In addition, $\breve{p}\left(\breve{C}\right) = p(C) - \breve{\gamma}S' = p\left(\breve{\gamma} + \breve{C}\right) - \breve{\gamma}\breve{S}'$.

Remark 3. It is reasonable to shift the origin to the point $(\gamma_j, 0)$ only if the existence and smoothness domain of the functions X_i of system (1^ν) specified by the constant x_0 is sufficiently large; otherwise, the set of admissible values c satisfying inequality (7) would be empty. Therefore, we assume that $x_0 > -\gamma_1$ in case 1a, $x_0 > c_{\rm b}$ in case 1b, and $x_0 > \gamma_2$ in case 2.

Remark 4. The notation introduced for classes 1 and 2 can be used for class 0 as well. To this end, it suffices to assume that $\breve{\gamma}$ (in addition to γ_1 and γ_2) can be equal to $\gamma_0 = 0$. Then, for class 0, the change of variables (15_1) reducing system (1^ν) to $(\breve{1}^\nu)$ proves to be the the identity mapping, $\breve{\sigma} = \sigma$, $\breve{\eta} = \eta$, and $\breve{C} = C$; i.e., all formulas marked by $\breve{}$ coincide with the corresponding formulas without this symbol.

5. EXISTENCE CONDITIONS FOR INVARIANT SURFACES

5.1. Invariance of the Function $\overline{R_0}(c)$

In forthcoming considerations, the function $\overline{R_0} = \overline{R_0}(c)$ in system (13^ν) and the function $\overline{\breve{R}_0} = \overline{\breve{R}_0}\left(\breve{c}\right)$ in system $(\breve{13}^\nu)$ $(\breve{c} = c - \breve{\gamma})$ are of interest.

We extend the domain of the function $R_0(c)$ introduced in system (9^ν) for $c \in M^0$ to admissible values c in classes 1 and 2.

For any admissible parameter $c \in M$, we set

$$\overline{R_0} = \frac{1}{\omega T} \int_0^\omega \int_0^T \left(S'(\varphi) X_1(t, C(\varphi), S(\varphi), 0) + S(\varphi) X_2(t, C(\varphi), S(\varphi), 0)\right) dt\, d\varphi, \tag{16}$$

where X_1 and X_2 are given in system (1^ν) and $C(\varphi)$, $S(\varphi)$ is a real-analytic ω-periodic solution of system (4) with the initial data $C(0) = c$, $S(0) = 0$. In class 0, formula (16) is a formula for the mean value R_0 in (13^ν).

Lemma 3. *Let c be an admissible operator, and let $\breve{c} = c - \breve{\gamma}$; then, in systems* (4) *and* $(\breve{4})$, *the periods $\omega = \omega(c)$ and $\breve{\omega} = \breve{\omega}\left(\breve{c}\right)$ of solutions with the initial data $(c, 0)$ and $(\breve{c}, 0)$ are given by the formula*

$$\omega(c) = \breve{\omega}\left(\breve{c}\right) = 2 \int_b^c \left(\frac{f(c) - f(\zeta)}{2}\right)^{-1/2} d\zeta \qquad (f(b) = f(c), \quad b < c), \tag{17}$$

and $\overline{\breve{R}_0}\left(\breve{c}\right) = \overline{R_0}(c)$ in system $(\breve{13}^\nu)$ obtained for classes 1 *and* 2.

Proof. We have $S^2 = C'^2$ in system (4); therefore, $2C'^2(\varphi) + f(C(\varphi)) \equiv f(c)$ in (5), whence it follows that $d\varphi = \pm((f(c) - f(C))/2)^{-1/2} dC$. By integrating from b to c (over a half-period), we obtain (17) for $\omega(c)$. By proceeding in a similar way in system $(\breve{4})$, we obtain

$$\breve{\omega}\left(\breve{c}\right) = 2 \int_{\breve{b}}^{\breve{c}} \left(\left(\breve{f}\left(\breve{c}\right) - \breve{f}\left(\breve{C}\right)\right)\Big/ 2\right)^{-1/2} d\breve{C}.$$

By performing the change of variables $\breve{C} = C - \breve{\gamma}$ in (15_2) and by taking into account the relation $\breve{f}(\zeta) = f(\breve{\gamma} + \zeta) - f(\breve{\gamma})$, we again obtain (17).

By $(\breve{1})$, (15), and (17), we have $\breve{X}_i\left(t, \breve{C}, \breve{S}, 0\right) = X_i\left(t, \breve{\gamma} + \breve{C}, \breve{S}, 0\right) = X_i(t, C, S, 0)$ and $\breve{S}' = S'$; i.e., $\breve{C}^3 - 2\breve{\sigma}\breve{C}^2 + \breve{\eta}\breve{C} = C^3 - 2\sigma C^2 + \eta C$, and $\breve{\omega} = \omega$; therefore,

$$\overline{\breve{R}_0}(\breve{c}) = (\breve{\omega}T)^{-1} \int_0^{\breve{\omega}}\int_0^{T} \left(\breve{S}'\breve{X}_1\left(t, \breve{C}, \breve{S}, 0\right) + \breve{S}\breve{X}_2\left(t, \breve{C}, \breve{S}, 0\right)\right) dt\, d\varphi = \overline{R_0}(c).$$

5.2. The Structure and Analysis of $\overline{R_0}(c)$ in the Analytic Case

Let us analyze the function $\overline{R_0}(c)$ for the case in which $X_i(t, x_1, x_2, 0)$ are analytic functions.

Suppose that the functions $X_i(t, x_1, x_2, 0)$ in system (1^ν) are continuous, T-periodic in t, and analytic with respect to x_1 and x_2 in the domain $G = \{(t, x_1, x_2):\ t \in \mathbf{R},\ |x_1|, |x_2| < x_0\}$; i.e., $X_i(t, x_1, x_2, 0) = \sum_{p,q=0}^{\infty} X_i^{(p,q)}(t) x_1^p x_2^q$ are absolutely and uniformly convergent power series in H with real continuous T-periodic coefficients. Then under condition (7) in (9^ν), the series

$$R_0(c) = \sum_{p,q=0}^{\infty} \left(X_1^{(p,q)}(t) S'(\varphi) C^p(\varphi) S^q(\varphi) + X_2^{(p,q)}(t) C^p(\varphi) S^{q+1}(\varphi)\right)$$

is absolutely convergent for arbitrary t and φ.

Since the integral of the product of an even and an odd function and that of S' over the period are zero, it follows that formula (16) acquires the form

$$\omega(c)\overline{R_0}(c) = \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \overline{X_1^{(p,2q)}} \int_0^{\omega} S' C^p S^{2q} d\varphi + \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \overline{X_2^{(p,2q-1)}} \int_0^{\omega} C^p S^{2q} d\varphi.$$

By integrating the relation $(2q+1)C^p S^{2q} S' = (C^p S^{2q+1})' + pC^{p-1} S^{2q+2}$, we obtain

$$\omega(c)\overline{R_0}(c) = \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \left(\frac{p+1}{2q-1}\overline{X_1^{(p+1,2q-2)}} + \overline{X_2^{(p,2q-1)}}\right) \int_0^{\omega} C^p(\varphi) S^{2q}(\varphi) d\varphi.$$

By (5), we have $S^{2q}(\varphi) \equiv ((f(c) - f(C(\varphi)))/2)^q$; therefore,

$$\int_0^{\omega} C^p(\varphi) S^{2q}(\varphi) d\varphi = 2\int_b^c C^p \left(\frac{f(c) - f(C)}{2}\right)^{q-1/2} dC.$$

As a result,

$$\begin{aligned}
\overline{R_0}(c) &= \frac{2}{\omega(c)} \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} a^{(p,q)} d_{p,q}(c), \\
a^{(p,q)} &= \frac{p+1}{2q-1}\overline{X_1^{(p+1,2q-2)}} + \overline{X_2^{(p,2q-1)}}, \\
d_{p,q}(c) &= \int_b^c \zeta^p \left(\frac{f(c) - f(\zeta)}{2}\right)^{q-1/2} d\zeta
\end{aligned} \qquad (16_{\mathrm{a}})$$

for any admissible parameter c under condition (7), where the $X_i^{(p,q)}(t)$ are the coefficients of the series $X_i(t, x_1, x_2, 0)$ in system (1^ν), $\omega(c)$ is the period given by (17), and $f(\zeta) = \zeta^4 - 8\sigma\zeta^3/3 + 2\eta\zeta^2 < f(c)$ for $\zeta \in (b(c), c)$.

For example, $\overline{R_0} = 2\omega^{-1}(c)\sum_{p=0}^{2} a^{(p,1)} d_{p,1}(c)$ for system (1_2^1) in [3]; i.e.., $\overline{R_0}$ is a linear combination of three integral functions $d_{p,1}(c)$.

Let us analyze the functions $d_{p,q}(c)$ and the number of zeros of $\overline{R}_0(c)$. Note that

$$f(c) - f(\zeta) = (c-\zeta)(f(c)/c + f(\zeta)/\zeta + (c+\zeta - 8/3)c\zeta).$$

But $f(b) = f(c)$; therefore, $f(c)/c + f(b)/b + (c+b-8/3)bc = 0$. By substituting $f(c)/c$ into the first equation, we obtain the expansion

$$f(c) - f(\zeta) = (c-\zeta)(\zeta - b)\left(f(b+\zeta+c)(b+\zeta+c)^{-2} - (bc + \zeta b + \zeta c)\right).$$

By Lemma 1, if $c \in M$, then $f'(c) = 4\left(c^3 - 2c^2 + \gamma c\right) > 0$ and $f'(b) < 0$. The differentiation of the identity $f(c) - f(b(c)) \equiv 0$ implies that $b'(c) = f'(c)/f'(b) < 0$.

Consider the integral functions $d_{p,q}(c)$. It follows from (16_{a}) that

$$d'_{p,q}(c) = \zeta^p \left(\frac{f(c)-f(\zeta)}{2}\right)^{q-1/2}\Bigg|_{\zeta=c} - \zeta^p \left(\frac{f(c)-f(\zeta)}{2}\right)^{q-1/2}\Bigg|_{\zeta=b} b'(c)$$
$$+ \int_b^c \zeta^p \left(q - \frac{1}{2}\right)\left(\frac{f(c)-f(\zeta)}{2}\right)^{q-3/2} \cdot 2^{-1} f'(c)\, d\zeta,$$

whence we obtain

$$d'_{p,q}(c) = \frac{2q-1}{4} f'(c) \int_b^c \zeta^p \left(\frac{f(c)-f(\zeta)}{2}\right)^{q-3/2} d\zeta \quad (p \geq 0, \quad q \geq 1).$$

Consequently, $d'_{2k,q}(c) > 0$ for $c \in M$, $d'_{2k+1,q}(c) > 0$ for $c \in M^2$ [since $b(c) > 0$], and $d'_{2k+1,q}(c) < 0$ for $c \in M_{\mathrm{a}}^1$ or $c \in M_{\mathrm{b}}^1$ (since $c < 0$) ($k = 0, 1, \ldots$). In this case, the functions $d_{p,q}(c)$ have the same signs as their derivatives for the above-mentioned admissible values of the parameter c.

Owing to an appropriate choice of the factors $a^{(p,1)}$ ($p = 0, 1, 2$) fixing some coefficients of the forms $U_i^{[2+i]}(t, u_1, u_2, \varepsilon)$ in (1_2^1), it was shown in [3] that one can find two zeros of the function $\overline{R_0}(c)$, one of which lies in an arbitrarily small given neighborhood of an arbitrary admissible value c, and $\overline{R_0}(c)$ with three zeros was constructed.

5.3. Main Condition

Definition 2. The equation $\overline{R_0}(c) = 0$, where $\overline{R_0}(c)$ is given by (16) and $c \in M$, is referred to as the *bifurcation equation* of system (1^ν).

For system (1^ν), we assume that

$$\exists c^* \in M: \quad \overline{R_0}\,(c^*) = 0, \qquad L_*^\nu \neq 0, \tag{18^ν}$$

where $L_*^\nu = \overline{L^\nu}\,(c^*)$ is given by (13^ν) if $c^* \in M^1$; $L_*^\nu = \overline{\breve{L}^\nu}\,(\breve{c}^*)$ is given by $(\breve{13}^\nu)$ if c^* belongs to class 1 or 2 (and then $\breve{c}^* = c^* - \breve{\gamma}$).

If it turns out that some root of the equation $\bar{R}(c) = 0$, for example, c_*^*, is not admissible, i.e., $c_*^* \notin M$, then it cannot be used, since it is impossible to make the polar change of variables (6) with $C(0) = c_*^*$. From the geometric viewpoint, this implies that the closed trajectory of the unperturbed system (2^ν) with the maximum abscissa equal to c_*^* does not pass through the point $(c_*^*, 0)$.

The condition $L_*^\nu \neq 0$, that is, the presence of a term linear in y in the first equation in system (13^ν) or $(\breve{13}^\nu)$ implies the existence of dissipative terms inducing a nonzero focal quantity due to terms $X_i\,(t, x_1, x_2, 0)$ in system (1^ν).

5.4. Choice of Polar Coordinates

Let the parameter c^* satisfy the main condition (18^ν). We fix the initial data $(c^*, 0)$ for an $\omega^* = \omega(c^*)$-periodic solution $C(\varphi)$, $S(\varphi)$ system (4); this determines a closed trajectory whose small neighborhood, as is shown below, contains the projection of a T-periodic two-dimensional cylindrical surface for any small $\varepsilon > 0$.

If the chosen parameter is $c^* \in M^0$, then the composition of the polar change of variables (6) with $c = c^*$ and the averaging changes of variables (10) and (12^ν) is a polynomial in y and ε with ω^*-periodic coefficients T-periodic in t and real-analytic in φ,

$$x_1 = C\left(1 + p^{-1}(C)\left(y + G^\nu\varepsilon + p^*\left(y + G^\nu\varepsilon\right)^2\right)\right),$$
$$x_2 = S\left(1 + p^{-1}(C)\left(y + G^\nu\varepsilon + p^*\left(y + G^\nu\varepsilon\right)^2\right)\right)^2,$$

and can be represented in the form

$$\begin{aligned} x_1 &= C(\varphi)\left(1 + F_1^\nu(t, \varphi, y, \varepsilon)/p(C(\varphi))\right), \\ x_2 &= S(\varphi)\left(1 + 2F_2^\nu(t, \varphi, y, \varepsilon)/p(C(\varphi))\right), \end{aligned} \qquad (19^\nu)$$

where $F_i^\nu = F_i^{\nu[1]} + F_i^{\nu[2]} + F_i^{\nu[>2]}$; moreover,

$$F_i^{0[1]} = y + \widetilde{g_0^0}\varepsilon, \qquad F_1^{0[2]} = p^*y^2 + \left(2\widetilde{g_0^0}p^* + \widetilde{g_1^0}\right)y\varepsilon + \left(\widetilde{g_0^0}\right)^2 p^*\varepsilon^2,$$
$$F_2^{0[2]} = F_1^{0[2]} + \left(y^2 + 2\widetilde{g_0^0}y\varepsilon + \left(\widetilde{g_0^0}\right)^2\varepsilon^2\right)/(2p), \qquad F_i^{1[1]} = y + \widehat{g_0^1}\varepsilon,$$
$$F_1^{1[2]} = p^*y^2 + \left(2\widehat{g_0^1}p^* + \widehat{g_1^1}\right)y\varepsilon + \left(\left(\widehat{g_0^1}\right)^2 p^* + \widetilde{g_0^1}\right)\varepsilon^2,$$
$$F_2^{1[2]} = F_1^{1[2]} + \left(y^2 + 2\widehat{g_0^1}y\varepsilon + \left(\widehat{g_0^1}\right)^2\varepsilon^2\right)/(2p);$$

the $F_i^{\nu[>2]}$ are polynomials of degree ≥ 3 in y and ε, and, by (17), the period is

$$\omega^* = 2\int_{b^*}^{c^*} \left(f(c^*) - f(\zeta)\right)^{-1/2} d\zeta.$$

If either $c^* \in M^1$ or $c^* \in M^2$, then the composition of the changes of variables (15_1), $(\breve{6})$ with $\breve{c} = \breve{c}^* = c^* - \breve{\gamma}$, $(\breve{9})$, and $(\breve{13})$ can be represented in the form

$$\begin{aligned} x_1 &= \breve{\gamma} + \breve{C}(\varphi)\left(1 + \breve{F}_1^\nu(t, \varphi, y, \varepsilon)/\breve{p}\left(\breve{C}(\varphi)\right)\right), \\ x_2 &= \breve{S}(\varphi)\left(1 + \breve{F}_2^\nu(t, \varphi, y, \varepsilon)/\breve{p}\left(\breve{C}(\varphi)\right)\right), \end{aligned} \qquad (\breve{19}^\nu)$$

where the $\breve{F}_i^\nu$ are polynomials similar to F_i^ν and, by Lemma 3, $\breve{\omega}^* = \omega^*$.

We have thereby proved the following assertion.

Lemma 4. *Let c^* satisfy the main condition (18^ν) and inequality (7), and let condition (14^0) with $\nu = 0$ be valid for the period $\omega = \omega^*$. Then the change of variables (19^ν) with $c^* \in M^0$ or the change of variables $(\breve{19}^\nu)$ with $c^* \in M^1$ or $c^* \in M^2$ reduces system (1^ν) to the system*

$$\dot{y} = \left(L_*^\nu y\varepsilon + Y_*^\nu\varepsilon^2 + O\left((|y| + \varepsilon)^3\right)\right)\varepsilon^\nu, \qquad \dot{\varphi} = \left(1 + \Phi_*^\nu\varepsilon + \Phi_1^* y + O\left((|y| + \varepsilon)^2\right)\right)\varepsilon^\nu, \qquad (20^\nu)$$

which coincides with system (13^ν), where $c = c^*$, or with system $(\breve{13}^\nu)$, where $\breve{c} = c^* - \breve{\gamma}$. In this case, $L_*^\nu \ne 0$ and $Y_*^\nu = Y_0^\nu(t, \varphi, c^*)$ for class 0, and $Y_*^\nu = \breve{Y}_0^\nu(t, \varphi, \breve{c}^*)$ for classes 1 and 2. In a similar way, one can define Φ_*^ν and Φ_1^*.

6. CONSTRUCTION ALGORITHM FOR A TWO-DIMENSIONAL INVARIANT SURFACE

6.1. Final Averaging

The forthcoming considerations are quite standard. They guarantee that a two-periodic invariant surface of system (20^ν) is found for all sufficiently small $\varepsilon > 0$.

First, we nullify $Y_*^\nu(t,\varphi)$ in system (20^ν) by performing the change of variables

$$y = u + H^\nu(t,\varphi,\varepsilon)\varepsilon, \tag{21^ν}$$

where $H^0 = \overline{h^0} + \widetilde{h^0}(t,\varphi)\varepsilon$, $H^1 = \overline{h^1} + \widehat{h^1}(\varphi)\varepsilon + \widetilde{h^1}(t,\varphi)\varepsilon^2$, which reduces ($20^\nu$) to the system

$$\begin{aligned} \dot u &= \left(L_*^\nu u\varepsilon + O\left((|u|+\varepsilon)^3\right)\right)\varepsilon^\nu, \\ \dot\varphi &= \left(1 + \left(\Phi_*^\nu + \overline{h^\nu}\Phi_1^*\right)\varepsilon + \Phi_1^* u + O\left((|u|+\varepsilon)^2\right)\right)\varepsilon^\nu. \end{aligned} \tag{22^ν}$$

The function H^ν occurring in the change of variables (21^ν) satisfies the equation $L_*^0\overline{h^0} + Y_*^0 = \widetilde{h^0}' + \dot{\widetilde{h^0}}$ for $\nu = 0$ and the equation $L_*^1\overline{h^1} + Y_*^1 = \widehat{h^1}' + \dot{\widetilde{h^1}}$ for $\nu = 1$. This implies that $\overline{h^\nu} = -\overline{Y_*^\nu}/L_*^\nu$, the first equation is uniquely solvable by virtue of condition (14^0), and the other implies the uniquely solvable equations $\widehat{h^1}' = \widehat{Y_*^1}$ and $\dot{\widetilde{h^1}} = \widetilde{Y_*^1}$.

Now we average $\Psi_*^\nu(t,\varphi) = \Phi_*^\nu(t,\varphi) + \overline{h^\nu}\Phi_1^*(\varphi)$ in system (22^ν) by making a T- and ω^*-periodic invertible change of the angular variable

$$\varphi = \psi + \Xi^\nu(t,\psi,\varepsilon)\varepsilon, \tag{23^ν}$$

where $\Xi^0 = \widetilde{\xi^0}(t,\psi)$ and $\Xi^1 = \widehat{\xi^1}(\psi) + \widetilde{\xi^1}(t,\psi)\varepsilon$. This change of variables reduces (22^ν) to the system

$$\dot u = \left(L_*^\nu u\varepsilon + O\left((|u|+\varepsilon)^3\right)\right)\varepsilon^\nu, \qquad \dot\psi = \left(1 + \overline{\Psi_*^\nu}\varepsilon + \Phi_1^*(\psi)u + O\left((|u|+\varepsilon)^2\right)\right)\varepsilon^\nu. \tag{24^ν}$$

Obviously, under the change of variables (23^ν), Ξ^ν is uniquely found from the equation $\Psi_*^0 = \overline{\Psi_*^0} + \widetilde{\xi^0}' + \dot{\widetilde{\xi^0}}$ or the equations $\widehat{\xi^1}' = \widehat{\Psi_*^1}$ and $\dot{\widetilde{\xi^1}} = \widetilde{\Psi_*^1}$.

The inverse change of variables for (23^ν) has the form

$$\psi = \varphi + \Upsilon^\nu(t,\varphi,\varepsilon)\varepsilon, \tag{23_0^ν}$$

where

$$\begin{aligned} \Upsilon^0 &= -\widetilde{\xi^0}(t,\varphi) + \widetilde{\xi^0}(t,\varphi)\widetilde{\xi^0}'(t,\varphi)\varepsilon + O\left(\varepsilon^2\right), \\ \Upsilon^1 &= -\widehat{\xi^1}(\varphi) + \left(\widehat{\xi^1}(\varphi)\widehat{\xi^1}'(\varphi) - \widetilde{\xi^1}(t,\varphi)\right)\varepsilon + O\left(\varepsilon^2\right) \end{aligned}$$

are functions T-periodic in t and real-analytic ω^*-periodic in φ.

6.2. Use of the Hale Lemma

To reduce the original system (1^ν) to a form that permits one to establish the existence of a two-periodic invariant surface in it for all sufficiently small parameter values, it remains to perform the scaling change of variables

$$u = v\varepsilon^{3/2} \tag{25}$$

reducing system (24^ν) to the form

$$\dot v = \left(L_*^\nu v\varepsilon + V^\nu(t,\psi,v,\varepsilon)\varepsilon^{3/2}\right)\varepsilon^\nu, \qquad \dot\psi = \left(1 + \overline{\Psi_*^\nu}\varepsilon + \Psi^\nu(t,\psi,v,\varepsilon)\varepsilon^{3/2}\right)\varepsilon^\nu, \tag{26^ν}$$

where V^ν and Ψ^ν are continuous functions of their arguments in a small neighborhood of v and ε, continuously differentiable in v and ψ, T-periodic in t, and ω^*-periodic in ψ.

Indeed,

$$V^\nu(t,\psi,v,\varepsilon) = O\left(\left(|v|\varepsilon^{3/2}+\varepsilon\right)^3\right)\varepsilon^{-3},$$
$$\Psi^\nu(t,\psi,v,\varepsilon) = \Phi_1^* v + O\left(\left(|v|\varepsilon^{3/2}+\varepsilon\right)^2\right)\varepsilon^{-3/2},$$

and the functions $O(\cdots)$ are real-analytic for all ψ and three-times continuously differentiable in a small neighborhood of the point $v=\varepsilon=0$. Therefore, in particular, V_v^ν and Ψ_v^ν are continuous at this point.

System (26^ν) satisfies the assumptions of Lemmas 2.1 and 2.2 [5], which imply that it has an invariant surface $v=\Gamma(t,\psi,\varepsilon)\varepsilon^{1/2}$ for all sufficiently small $\varepsilon>0$, where Γ is a continuous continuously differentiable function T-periodic in t and ω^*-periodic in ψ.

We have thereby proved the following assertion.

Lemma 5. *System* (20^ν) *with any sufficiently small* $\varepsilon>0$ *has a continuous surface*

$$y = H^\nu(t,\varphi,\varepsilon)\varepsilon + \Gamma^\nu\left(t,\varphi+\Upsilon^\nu\varepsilon,\varepsilon\right)\varepsilon^2 \qquad (27^\nu)$$

T-periodic in t and continuously differentiable and ω^*-periodic in φ, which is obtained by the substitution of the invariant surface $v=\Gamma(t,\psi,\varepsilon)\varepsilon^{1/2}$ into the composition of the changes of variables (21^ν), (23_0^ν), *and* (25).

7. RESULTS OF ANALYSIS

7.1. Main Theorem

Lemmas 1–5 imply the following assertion.

Theorem 1. *For any parameter c^* satisfying the test condition (18^ν) and condition (14^0) if $\nu=0$, for any sufficiently small $\varepsilon>0$, system (1^ν) has a continuous two-dimensional invariant surface $T_{c^*}^\nu = T_{c^*}^\nu(t,\varphi,\varepsilon)$ that is T-periodic in t and continuously differentiable and $\omega^* = 2^{3/2}\int_{b^*}^{c^*}(f(c)-f(C))^{-1/2}dC$-periodic in φ.*

If $c^*\in M^0$, i.e., $c^*\in(c_{\rm a}^*,+\infty)$ in case a, *$c^*\in(0,\gamma_1)$ in case* b, *$c^*\in(c_{\rm c}^*,+\infty)$ in case* c, *and $c^*\in(0,+\infty)$ in case* d, *then $T_{c^*}^\nu$ is obtained by the substitution of the surface (27^ν) into the change of variables (19^ν) and has the form*

$$\begin{aligned} x_1 &= C(\varphi) + F_1^\nu\left(t,\varphi,H^\nu(t,\varphi,\varepsilon)\varepsilon+\Gamma^\nu\left(t,\varphi+\Upsilon^\nu\varepsilon,\varepsilon\right)\varepsilon^2,\varepsilon\right)C/p(C),\\ x_2 &= S(\varphi) + 2F_2^\nu\left(t,\varphi,H^\nu(t,\varphi,\varepsilon)\varepsilon+\Gamma^\nu\left(t,\varphi+\Upsilon^\nu\varepsilon,\varepsilon\right)\varepsilon^2,\varepsilon\right)S/p(C). \end{aligned} \qquad (28^\nu)$$

If either $c^*\in M^1$ [i.e., $c^*\in(\gamma_1,0)$ in cases a *and* c *and $c^*\in(c_{\rm b}^*,+\infty)$ in case* b] *or $c^*\in M^2$ (i.e., $c^*\in(\gamma_2,c_{\rm a}^*)$ in case* a, *$c^*\in(\gamma_2,c_{\rm b}^*)$ in case* b, *and $c^*\in(\gamma_2,c_{\rm c}^*)$ in case* c, *then $T_{c^*}^\nu$ is given by formula $(\breve{28}^\nu)$, which is similar to (28^ν) and is obtained by the substitution of the surface (27^ν) into the change of variables $(\breve{19}^\nu)$.*

Corollary 2. *For any sufficiently small $\varepsilon>0$, the invariant surface $T_{c^*}^\nu(t,\varphi,\varepsilon)$ of system (1^ν) is a two-dimensional cylindrical surface whose embedding in the three-dimensional space of the variables x_1, x_2, t is given by formula (28^ν) or $(\breve{28}^\nu)$. The surface $T_{c^*}^\nu$ is homeomorphic to a two-dimensional torus if the time t is taken modulo the period T.*

Corollary 3. *If $c^*\in M^0$, then the invariant torus $T_{c^*}^\nu(t,\varphi,\varepsilon)$ occurring in Theorem 1 admits the asymptotic expansion*

$$x_1 = C + \left(\tau_1^\nu\varepsilon+\delta_1^\nu\varepsilon^2\right)Cp^{-1} + O\left(\varepsilon^3\right), \qquad x_2 = S + 2\left(\tau_2^\nu\varepsilon+\delta_2^\nu\varepsilon^2 S\right)p^{-1} + O\left(\varepsilon^3\right), \qquad (29^\nu)$$

where $\tau_i^0 = \widetilde{g_0^0} + \overline{h^0}$,

$$\delta_1^0(t,\varphi) = \Gamma^0(t,\varphi,0) + \widetilde{h^0} + p^*\left(\overline{h^0}\right)^2 + \left(2\widetilde{g_0^0}p^* + \widetilde{g_1^0}\right)\overline{h^0} + \left(\widetilde{g_0^0}\right)^2 p^*,$$

$$\delta_2^0(t,\varphi) = \delta_1^0(t,\varphi) + \left(\left(\overline{h^0}\right)^2 + 2\widetilde{g_0^0}\overline{h^0} + \left(\widetilde{g_0^0}\right)^2\right)/(2p),$$

and τ_i^1 *and* δ_i^1 *are given by similar formulas with the superscript* 0 *replaced by* 1, *with the coefficient* $\widetilde{g_i^0}(t,\varphi)$ *replaced by* $\widehat{g_i^1}(\varphi)$, *and with the new term* $\widetilde{g_0^1}(t,\varphi)\varepsilon^2$ *added to* δ_1^1.

If $c^* \in M^1$ *or* $c^* \in M^2$, *then* $T_{c^*}^\nu(t,\varphi,\varepsilon)$ *admits an expansion* $(\breve{29}^\nu)$ *that differs from* (29^ν) *by the presence of the sign* $\breve{}$ *over all functions and the addition of the term* γ_1 *or* γ_2 *to the equation for* x_1.

7.2. Results in the Phase Space

For any $t \in [0,T]$, by $P_{c^*}^{t\nu} = P_{c^*}^{t\nu}(\varphi,\varepsilon)$ we denote the projection of the invariant surface $T_{c^*}^\nu$ onto the phase space of system (1^ν). By Theorem 1, the projection $P_{c^*}^{t\nu}$ is a closed curve diffeomorphic to a circle.

Definition 3. The *trace* of system (1^ν) induced by the parameter c^* in condition (18^ν) is defined as the closed set $P_{c^*}^\nu = \bigcup_{t\in[0,T]} P_{c^*}^{t\nu}$.

Obviously, the trace $P_{c^*}^\nu$ lies in an arbitrarily small neighborhood of the smooth closed curve $x_1 = C(\varphi)$, $x_2 = S(\varphi)$ for any sufficiently small $\varepsilon > 0$, where $C(0) = c^*$ and $S(0) = 0$.

Corollary 4. *The following assertions are valid for any sufficiently small* $\varepsilon > 0$.

1. *The trace* $P_{c^*}^\nu$ *is an annular domain whose boundaries* $P_{c^*\text{in}}^\nu$ *and* $P_{c^*\text{out}}^\nu$ *are homeomorphic to a circle and can stick together completely or partially, for example, if system* (1^ν) *is autonomous.*

2. *By* (29^ν), *the width of the trace* $P_{c^*}^1$ *is of the order of at least* ε^2, *and the width of* $P_{c^*}^0$ *can be of the order of* ε *owing to* $\widetilde{g_0^0}(t,\varphi)$ *occurring in* τ_i^0.

3. *In accordance with the classification in Lemma* 1, *the trace* $P_{c^*}^\nu$ *can have the following position and form*:

$c^* \in M_{\rm a}^0 = (c_{\rm a}^*, +\infty) \Rightarrow \operatorname{Int} P_{c^*\rm in}^\nu \supset \left(S_{\rm a}^b \cup S_{\rm a}^c\right)$, $S_{\rm a}^b = \varnothing$ *if* $\eta = 0$;
$c^* \in M_{\rm a}^1 = (\gamma_1, 0) \Rightarrow \eta < 0$, $(\gamma_1, 0) \subset \operatorname{Int} P_{c^*\rm in}^\nu$, $P_{c^*\rm out}^\nu \subset \operatorname{Int} S_{\rm a}^b$;
$c^* \in M_{\rm a}^2 = (\gamma_2, c_{\rm a}^*) \Rightarrow (\gamma_2, 0) \subset \operatorname{Int} P_{c^*\rm in}^\nu$, $P_{c^*\rm out}^\nu \subset \operatorname{Int} S_{\rm a}^c$;
$c^* \in M_{\rm b}^0 = (0, \gamma_1) \Rightarrow (0,0) \subset \operatorname{Int} P_{c^*\rm in}^\nu$, $P_{c^*\rm out}^\nu \subset \operatorname{Int} S_{\rm b}^b$;
$c^* \in M_{\rm b}^1 = (c_{\rm b}^*, +\infty) \Rightarrow \operatorname{Int} P_{c^*\rm in}^\nu \supset \left(S_{\rm b}^b \cup S_{\rm b}^c\right)$, $S_{\rm a}^c = \varnothing$, *if* $\eta = 1$;
$c^* \in M_{\rm b}^2 = (\gamma_2, c_{\rm b}^*)$, $\Rightarrow 0 < \eta < 1$, $(\gamma_2, 0) \subset \operatorname{Int} P_{c^*\rm in}^\nu$, $P_{c^*\rm out}^\nu \subset \operatorname{Int} S_{\rm b}^c$;
$c^* \in M_{\rm c}^0 = (c_{\rm c}^*, +\infty) \Rightarrow \operatorname{Int} P_{c^* in}^\nu \supset \left(S_{\rm a}^b \cup S_{\rm a}^c\right)$;
$c^* \in M_{\rm c}^1 = (\gamma_1, 0) \Rightarrow (\gamma_1, 0) \subset \operatorname{Int} P_{c^*\rm in}^\nu$, $P_{c^*\rm out}^\nu \subset \operatorname{Int} S_{\rm a}^b$;
$c^* \in M_{\rm c}^2 = (\gamma_2, c_{\rm c}^*) \Rightarrow (\gamma_2, 0) \subset \operatorname{Int} P_{c^*\rm in}^\nu$, $P_{c^*\rm out}^\nu \subset \operatorname{Int} S_{\rm a}^c$;
$c^* \in M_{\rm d}^0 = (0, +\infty) \Rightarrow (0,0) \subset \operatorname{Int} P_{c^*\rm in}^\nu$, *where* $\operatorname{Int}\gamma$ *stands for the set of points on the plane lying inside the closed curve* γ.

4. *The trajectory of any solution lying on the torus* $T_{c^*}^\nu$ *never leaves its trace* $P_{c^*}^\nu$ *and performs an infinite rotation with frequency* $\tau_{c^*}^\nu = (1/\omega^* + O(\varepsilon))\,\varepsilon^\nu$ *in accordance with the equation for* $\dot\varphi$ *in system* (20^ν).

5. *Two arbitrary traces of system* (1^ν) *induced by different parameters* c_1^* *and* c_2^* *do not meet each other for all sufficiently small* ε.

Remark 5. By generalizing system (1^ν), one can assume that X_1 and X_2 are quasiperiodic functions of t with m basic frequencies $\omega_1, \ldots, \omega_m$.

The function $\Theta(t)$ is quasiperiodic if $\Theta(t) = \check\Theta(\omega_1 t, \ldots, \omega_m)$, where $\check\Theta(\theta_1, \ldots, \theta_m)$ is a function T-periodic in the variables $\theta_1, \ldots, \theta_m$.

In other words, instead of (1^ν), one can consider system ($\check{1}^\nu$) with quasiperiodic perturbations $\check{X}_i(\theta_1,\ldots,\theta_m,x_1,x_2,\varepsilon)$ by virtually supplementing it with the equations $\dot{\theta}_j=\omega_j$ $(j=1,\ldots,m)$ instead of the equation $\dot{t}=1$, which permits one to apply the Hale lemma to the resulting system ($\check{26}^\nu$).

For the useful realization of all averaging changes of variables, one need only to require that the basic frequencies satisfy the Siegel condition for small exponents similar to (14^0) :

$$|q_1\omega_1+\cdots+q_m\omega_m|>K|q|^{-\tau},$$

where $|q|\neq 0$, $K>0$, and $\tau>1$. (Here $|q|=|q_1|+\cdots+|q_m|$; $q_1,\ldots,q_m$ are integers.)

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