Eikonal algebra on a graph of simple structure

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Abstract

An eikonal algebra $\mathfrak{E}(\Omega)$ is a C*-algebra related to a metric graph Ω. It is determined by trajectories and reachable sets of a dynamical system associated with the graph. The system describes the waves, which are initiated by boundary sources (controls) and propagate into the graph with finite velocity. Motivation and interest to eikonal algebras comes from the inverse problem of reconstruction of the graph via its dynamical and/or spectral boundary data. Algebra $\mathfrak{E}(\Omega)$ is determined by these data. In the mean time, its structure and algebraic invariants (irreducible representations) are connected with topology of Ω. We demonstrate such connections and study $\mathfrak{E}(\Omega)$ by the example of Ω of a simple structure. Hopefully, in future, these connections will provide an approach to reconstruction.

0 Introduction

About the paper

Eikonal algebras appear in the framework of algebraic version of the boundary control method (BC-method), which is an approach to inverse problems based on their relations to control and system theory [4]. These algebras are used for reconstruction of Riemannian manifolds via dynamical and/or spectral boundary inverse data. Namely, these data determine the relevant

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eikonal algebra, which is a *commutative* C*-algebra, whereas its spectrum (a set of irreducible representations) provides an isometric copy of the manifold under reconstruction and, thus, solves the problem [5, 6, 7].

Applications of the BC-method to inverse problems on graphs are represented in [2, 8, 9]. An eikonal algebra associated with a metric graph is introduced in [10]. It is a straightforward analog the algebras, which are successfully used for solving the above-mentioned reconstruction problems. As was hoped, such an analog should reconstruct graphs. However, this analog turns out to be much more difficult for the use and study. The main reason is that the eikonal algebra on a graph is *noncommutative*. In [10] some general features of its structure are revealed: it is represented as a sum of the so-called 'block-algebras'. This structure is connected with geometry of the graph but the connection is of unclear and implicit character. Moreover, the block-algebras are also of rather complicated subtle structure.

However, we hope for availability of the eikonal algebra on graphs and its future role in reconstruction problem. Moreover, it is of certain independent interest as a C*-algebra associated with a concrete important inverse problem of mathematical physics. By this paper we start its systematic study and begin with a simple example. Our goal is to analyze this example in detail.

Contents

• In section 1 a hyperbolic dynamical system associated with a metric graph is introduced. The system describes the waves, which are initiated by the sources (controls) acting from the boundary vertices and propagate into the graph with the unit velocity. The system is endowed with the control theory attributes: outer and inner spaces, and operators. The waves constitute the reachable sets (subspaces) and determine the corresponding projections on them. An *eikonal* is defined as an operator integral composed of these projections. It acts in the inner space and is determined by a single boundary vertex. The eikonals corresponding to a set of boundary vertices generate an *eikonal algebra* $\mathfrak{E}^T_\Sigma$. It is an operator algebra, which is a key object of the paper.

• Section 2 provides the instruments for analyzing the structure of $\mathfrak{E}^T_\Sigma$. The main role is played by a parametrization, which represents the waves as elements of the spaces $L_2([a,b];\mathbb{R}^m)$ and eikonals as operators multiplying elements by the matrix-valued functions of the class $C([a,b];\mathbb{M}^m)$.

• Section 3 contains some general facts on C*-algebras and, in particular, the matrix algebras being in the use. We introduce the so-called *block-algebras*, which play the role of the building blocks constituting $\mathfrak{E}^T_\Sigma$.

• Section 4 is devoted to analysis of the eikonal algebra of a simple graph. The graph is a 3-star: it consists of three edges emanating from a single inner vertex, contains three boundary vertices and is controlled from two of them. The edges are of the different lengths. The corresponding dynamical system, which describes the wave propagation, is considered at the finite time interval $[0, T]$. The waves propagate from the boundary vertices with the speed 1 and gradually fill the graph. Respectively, the algebra $\mathfrak{E}^T_\Sigma$ changes as time T goes on. The evolution of its structure is of our main interest: we analyze it in detail.

Comments

• The bulk of the subject matter of sections 1 and 2 is the same as in [10]. We just repeat the basic notions and facts from [10] to make the paper appropriate for independent reading. A new object appears in subsection 'More hydras', where we introduce the so-called *efficient hydra*. The novelty provides more natural and convenient partition of the graph. As a result, one gets more transparent description of the structure of $\mathfrak{E}^T_\Sigma$.

• The literature on inverse problems on graphs is hardly observable. We refer the reader to the papers by S.Avdonin, P.Kuchment, P.Kurasov, V.Yurko and others. Eikonal algebras are dedicated to solving inverse problems [5, 6, 7]. However, in the given paper we do not solve inverse problems but study an algebra closely related to them. A prospective goal is to recover a graph via its boundary inverse data by the use of $\mathfrak{E}^T_\Sigma$.

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1 Waves on graph

A graph

Basically, our results are meaningful and valid for arbitrary metric graphs [9, 10]. However, for the sake of simplicity, we deal with the following specific case.

A graph $\Omega = E \cup W$ is a connected set in $\mathbb{R}^3$ [1], which consists of the *edges* $E = \{e_i\}_{i=1}^L$ and *vertices* $W = \{w_j\}_{j=1}^M$. Each edge is a finite open interval of the straight line: $e_i = \{x_i + s\omega_i \mid a_i < s < b_i\}$, where $x_i, \omega_i \in \mathbb{R}^3$, $|\omega_i| = 1$; $a_i, b_i \in \mathbb{R}$ provided $e_i \cap e_{i'} = \emptyset$. The vertices $w_j \in \mathbb{R}^3$ are the endpoints of the edges. We say $w \in W$ to be incident to $e \in E$ (and vice versa) and write $w \prec e$ if $w \in \overline{e}$ (the closure in $\mathbb{R}^3$). The number $\mu(w) \geqslant 1$ of the edges incident to the given $w \in W$ is called a valency of w.

In what follows we deal with the graphs obeying $\mu(w) \neq 2$, so that $W = V \cup \Gamma$, where $\Gamma = \{w \in W \mid \mu(w) = 1\}$ and $V = \{w \in W \mid \mu(w) \geqslant 3\}$ are the sets of the *boundary* vertices and *inner* vertices respectively. Also, we assume $\Gamma \neq \emptyset$, rename the boundary vertices by $\{\gamma_1, \dots, \gamma_N\}$ and say Γ to be the boundary of Ω. The inner points are denoted by $\{v_1, \dots, v_{M-N}\} = V$.

Graph Ω is endowed with an intrinsic metric $\tau(x, x')$ induced by the $\mathbb{R}^3$-metric and defined as the length of the shortest path in Ω, which connects x and y. In particular, for $x, x' \in \overline{e}$ one has $\tau(x, x') = |x - x'|$. For a subset $A \subset \Omega$, the set

$$\Omega^r[A] := \{x \in \Omega \mid \tau(x, A) < r\} \;\; (r > 0)$$

is its metric neighborhood of radius r. Thus, Ω is a compact connected metric space.

Derivatives, spaces, operators

In the paper all the functions, function classes and spaces are real.

- Let $e = \{x(s) = \tilde{x} + s\omega \mid a < s < b\}$ be a (parametrized) edge, y a function on Ω. For a point $x = x(s) \in e$ one defines a derivative along the edge by

$$\partial_e y(x) \;:=\; \lim_{\delta \to 0} \frac{y(x(s+\delta)) - y(x(s))}{\delta}\,.$$

Such a derivative depends on the parametrization up to the sign. In the mean time, the second derivative $\partial_e^2 y(x)$ is invariant.

For a vertex w and edge e provided $w \prec e$, one defines an *outward* derivative

$$\partial_e^+ y(w) \;:=\; \lim_{e \ni x \to w} \frac{y(x) - y(w)}{\tau(x, w)}\,.$$

[1] So, we don't need the graph to be planar.

For an interior vertex $v \in V$, an *outward flow*

$$\Pi_v[y] \ := \ \sum_{e \succ v} \partial_e^+ y(v)$$

is introduced.

• By $C(\Omega)$ we denote the Banach space of continuous functions with the norm $\|y\| = \sup_{\Omega} |y(\cdot)|$.

Introduce a Hilbert space $\mathscr{H} = L_2(\Omega)$ of functions with the inner product

$$(y, u)_{\mathscr{H}} := \int_{\Omega} yu \, d\tau = \sum_{e \in E} \int_e yu \, d\tau \,,$$

where $d\tau$ is the length element on Ω.

A function $y \in C(\Omega)$ is assigned to a class $H^2(\Omega)$ if $y(x(\cdot))$ belongs to the Sobolev class $H^2(a, b)$ for each edge $e = \{x(s) = \tilde{x} + s\omega \mid a < s < b\}$. Also, we define the *Kirchhoff class*

$$\mathscr{K} \ := \ \{y \in H^2(\Omega) \mid \ \Pi_v[y] := 0, \ \ v \in V\} \,. \tag{1}$$

• The *Laplace operator* $\Delta : \mathscr{H} \to \mathscr{H}, \ \operatorname{Dom} \Delta = \mathscr{K}$,

$$(\Delta y) \Big|_e \ := \ \partial_e^2 y \,, \qquad e \in E \tag{2}$$

is densely defined and closed.

Dynamical system

• An initial boundary value problem

$$\begin{aligned}
& u_{tt} - \Delta u = 0 && \text{in } \mathscr{H}, \ \ 0 < t < T && (3)\\
& u \in \mathscr{K} && \text{for all } t \in [0, T] && (4)\\
& u|_{t=0} = u_t|_{t=0} = 0 && \text{in } \Omega && (5)\\
& u = f && \text{on } \Gamma \times [0, T] && (6)
\end{aligned}$$

is referred to as a *dynamical system* associated with the graph Ω. Here $T > 0$ is a final moment; $f = f(\gamma, t)$ is a *boundary control*; a solution $u = u^f(x, t)$ describes a *wave* initiated at Γ and propagating into Ω. As is well known, for a C^2-smooth (with respect to t) control f vanishing near $t = 0$ the problem has a unique classical solution u^f. Later on, the (generalized) solutions for $f \in L_2(\Gamma \times [0,T])$ will be defined.

By definition (1), the condition (4) yields the *Kirchhoff laws*:

$$u(\,\cdot\,,t) \in C(\Omega), \quad \Pi_v[u(\,\cdot\,,t)] = 0 \qquad \text{for all } \ t \geqslant 0 \ \text{ and } v \in V\,.$$

By (2), on each (parametrized) edge $e \in E$ the function $\tilde{u} = u^f(x(s),t)$ satisfies the homogeneous string equation

$$\tilde{u}_{tt} - \tilde{u}_{ss} = 0 \qquad \text{in } \ (a,b) \times (0,T). \tag{7}$$

Hence, the waves propagate in Ω with the unit speed.

• A space of controls $\mathscr{F}^T := L_2\left(\Gamma \times [0,T]\right)$ with the inner product

$$(f,g)_{\mathscr{F}^T} := \sum_{\gamma \in \Gamma} \int_0^T f(\gamma,t)\, g(\gamma,t)\, dt$$

is called an *outer space* of system (3)–(6). It contains the subspaces

$$\mathscr{F}^T_\gamma := \left\{ f \in \mathscr{F}^T \mid \operatorname{supp} f \subset \{\gamma\} \times [0,T] \right\}$$

of controls, which act from single boundary vertices $\gamma \in \Gamma$. Each $f \in \mathscr{F}^T_\gamma$ is of the form $f(\gamma',t) = \delta_\gamma(\gamma')\varphi(t)$, where For

$$\delta_\gamma(\gamma') := \begin{cases} 0, & \gamma' \neq \gamma \\ 1, & \gamma' = \gamma \end{cases}$$

and $\varphi \in L_2[0,T]$.

For a subset $\Sigma \subseteq \Gamma$ we put

$$\mathscr{F}^T_\Sigma := \oplus \sum_{\gamma \in \Sigma} \mathscr{F}^T_\gamma \tag{8}$$

and have $\mathscr{F}^T = \oplus \sum_{\gamma \in \Gamma} \mathscr{F}^T_\gamma$.

• The space $\mathscr{H}$ is an *inner space*; the waves $u^f(\,\cdot\,,t)$ are time–dependent elements of $\mathscr{H}$. The linear set of waves

$$\mathscr{U}^s_\gamma := \left\{ u^f(\cdot,s) \mid f \in \mathscr{F}^T_\gamma \right\} \subset \mathscr{H}, \qquad 0 \leqslant s \leqslant T$$

is called *reachable* (from the boundary vertex γ, at the moment $t = s$). We say the set

$$\mathscr{U}^s_\Sigma := \left\{ u^f(\cdot,s) \mid f \in \mathscr{F}^T_\Sigma \right\} = \operatorname{span}\{\mathscr{U}^s_\gamma \mid \gamma \in \Sigma\}$$

(algebraic sum of $\mathscr{U}^s_\gamma$) to be reachable from Σ. As will be noticed later in Remark 1, $\mathscr{U}^s_\gamma$ and $\mathscr{U}^s_\Sigma$ are the (closed) subspaces in $\mathscr{H}$. They are increasing as s grows: $\mathscr{U}^s_\Sigma \subset \mathscr{U}^{s'}_\Sigma$ for $s < s'$.

Eikonals

Here we introduce the algebra which is the main subject of the paper. By $\mathfrak{B}(\mathscr{H})$ we denote the algebra of bounded operators acting in the inner space.

Let P^s_γ be the (orthogonal) projection in $\mathscr{H}$ onto $\mathscr{U}^s_\gamma$. The operator $E^T_\gamma \in \mathfrak{B}(\mathscr{H})$ of the form

$$E^T_\gamma := \int_0^T s\, dP^s_\gamma$$

is called an *eikonal* corresponding to the vertex γ.

For a Banach algebra $\mathfrak{B}$ and a subset $A \subset \mathfrak{B}$ by $\vee A$ we denote the minimal closed (sub)algebra in $\mathfrak{B}$ which contains A. The algebra

$$\mathfrak{E}^T_\Sigma := \vee\{E^T_\gamma \mid \gamma \in \Sigma\} \tag{9}$$

generated by eikonals is called the *eikonal algebra* [10]. Our general goal is to study its structure.

2 Representation of waves and eikonals

Here we derive a relevant representation for elements of $\mathfrak{E}^T_\Sigma$.

Generalized solutions

• Consider the system (3)–(6) with $T = \infty$. Let $\delta(t)$ be the Dirac delta-function.

Fix a boundary vertex γ. Taking the control $f(\gamma', t) = \delta_\gamma(\gamma')\delta(t)$, one can define the (generalized) solution $u^{\delta_\gamma\delta}$ to (3)–(6). A possible way is to use a smooth regularizations $\delta^\varepsilon(t) \underset{\varepsilon\to 0}{\to} \delta(t)$ and then understand $u^{\delta_\gamma\delta}$ as a relevant limit of the classical solutions $u^{\delta_\gamma\delta^\varepsilon}$ as $\varepsilon \to 0$. Such a limit turns out to be a space-time distribution on $\Omega \times [0, T]$ of the class $C\left((0, T); H^{-1}(\Omega)\right)$: see, e.g., [3]. The distribution $u^{\delta_\gamma\delta}$ is called a *fundamental* solution to (3)–(6) corresponding to the given γ. It describes the wave initiated by instantaneous source supported at γ. Let us consider its properties in more detail. It is convenient to use the formal rule, which may be specified as 'dynamics of particles' [10]. By a *measure* is meant a linear continuous functional on the space $C(\Omega)$. A Dirac measure δ_x acts by $\langle\delta_{x_0}, y\rangle = y(x_0)$. The measures $a\delta_{x(t)}$ are said to be the *particles*, the factors $a \in \mathbb{R}$ being called *amplitudes.*

The rule is the following.

1. Each particle $a\delta_{x(t)}$ moves along an edge with velocity 1 in one of two possible directions, so that $|\dot{x}(t)|=1$ holds as $x(t)\in e$.

2. Particles move independently, they do not interact. If by the moment t there are a few particles $a_1\delta_{x(t)},\dots,a_p\delta_{x(t)}$ supported at the point $x(t)\in\Omega\backslash\Gamma$, they are identified with the single particle $[a_1+\dots+a_p]\delta_{x(t)}$.

3. The boundary of the graph reflects particles. As soon as a particle $a\delta_{x(t)}$ reaches a $\gamma\in\Gamma$, it instantly reverses its direction and changes the amplitude from a for $-a$.

4. Moving along the edge e and passing through an inner vertex $v\prec e$, the particle $a\delta_{x(t)}$ splits into $\mu(v)$ particles: one reflected and $\mu(v)-1$ transmitted. The reflected particle moves along e in the opposite direction and is of the amplitude $\frac{2-\mu(v)}{\mu(v)}a$. Each of the transmitted particles moves along the single (incident to v) edge away from v and has the amplitude $\frac{2}{\mu(v)}a$ [2].

Accepting such a convention, we can describe the solution $u^{\delta_\gamma\delta}$ as follows. Recall that τ is the distance in Ω.

A. For $0\leqslant t\leqslant\tau(\gamma,V)$, one has $u^{\delta_\gamma\delta}=\delta_{x(t)}$, where $x(t)$ is the point of the edge $e\succ\gamma$ provided $\tau(x(t),\gamma)=t$. Thus, for the 'small' times, $u^{\delta_\gamma\delta}$ is a single particle injected from γ into the graph and moving along e with the unit velocity.

B. Further evolution for the times $t>\tau(\gamma,V)$ is governed by the rule $1.-4.$

As is easy to recognize, such a description is quite deterministic. So, at any moment $t\geqslant 0$ the solution $u^{\delta_\gamma\delta}$ is a collection of finite number of particles moving into Ω.

• The fundamental solution is a space-time distribution on $\Omega\times\{t\geqslant 0\}$. Owing to its above described specific structure, the function

$$u^f(x,t):=\left[u^{\delta_\gamma\delta}(x,\cdot)*f\right](t),\qquad x\in\Omega,\ 0\leqslant t\leqslant T \tag{10}$$

(the convolution w.r.t. t) is well defined for any boundary control $f\in\mathscr{F}^T_\gamma$ of the form $f(\gamma',t)=\delta_\gamma(\gamma')\varphi(t)$ with $\varphi\in L_2[0,T]$. Moreover, one can show that $u^f\in C([0,T];\mathscr{H})$ and, if φ is C^2-smooth and vanishes near $t=0$ then u^f provides the classical solution to (3)–(6).

[2] Thus, the total amplitude is $\frac{2-\mu(v)}{\mu(v)}\,a+[\mu(v)-1]\frac{2}{\mu(v)}\,a=a$ that corresponds to the Kirchhoff conservation lows.

By the aforesaid, we regard u^f defined by (10) as a *generalized* solution to (3)–(6) for $f \in \mathscr{F}^T_\gamma$. Also, in accordance with (8), for $f \in \mathscr{F}^T : \ f = \sum_{\gamma\in\Gamma} f_\gamma$ with $f_\gamma \in \mathscr{F}^T_\gamma$, we put

$$u^f(x,t) \,=\, \sum_{\gamma\in\Gamma} u^{f_\gamma}(x,t), \qquad x \in \Omega,\ \ 0 \leqslant t \leqslant T\,. \tag{11}$$

Later on an efficient representation of the waves u^f will be provided.

• As was mentioned above (see (7)), the waves propagate with the unit speed. As a consequence, for $f \in \mathscr{F}^T_\gamma$ one has

$$\operatorname{supp} u^f(\cdot,t) \,\subset\, \overline{\Omega^t[\gamma]}, \qquad t > 0\,. \tag{12}$$

By the latter and (11), for a subset $\Sigma \subseteq \Gamma$ we have

$$\operatorname{supp} u^f(\cdot,t) \,\subset\, \overline{\Omega^t[\Sigma]} \qquad \text{for} \quad f \in \oplus\sum_{\gamma\in\Sigma} \mathscr{F}^T_\gamma, \ \ t > 0\,.$$

Thus, $\Omega^t[\Sigma]$ is the part of the graph filled by waves, which move from Σ, at the moment t.

Hydra

• Fix a boundary vertex γ. Considering the fundamental solution as a space-time distribution, we introduce the set

$$H_\gamma \,:=\, \operatorname{supp} \mathrm{u}^{\delta_\gamma\delta} \subset \Omega \times \overline{\mathbb{R}}_+$$

and call it a *hydra* [10]. Thus, the hydra is a space-time graph formed by trajectories of particles: see Fig. 1.

Let

$$\begin{aligned} &\pi : H_\gamma \ni h = (x,t) \mapsto x \in \Omega, \quad \pi^{-1}(x) := \{h \in H_\gamma \mid \pi(h) = x\};\\ &\rho : H_\gamma \ni h = (x,t) \mapsto t \in \overline{\mathbb{R}}_+, \quad \rho^{-1}(t) := \{h \in H_\gamma \mid \rho(h) = t\} \end{aligned} \tag{13}$$

be the space- and time-projections. On the hydra one defines a function (*amplitude*) $a(\cdot)$ as follows:

(*) for $h \in H_\gamma$ provided $\pi(h) = x \in \Omega\backslash\Gamma$ and $\rho(h) = t > 0$ we have $u^{\delta_\gamma\delta}(\cdot,t) = a\delta_x(\cdot)$ and define $a(h) = a$;

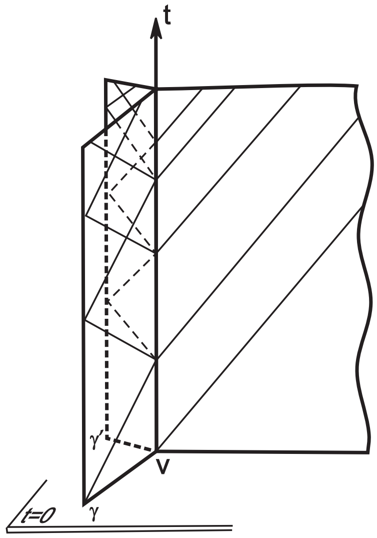

Figure 1: The hydra

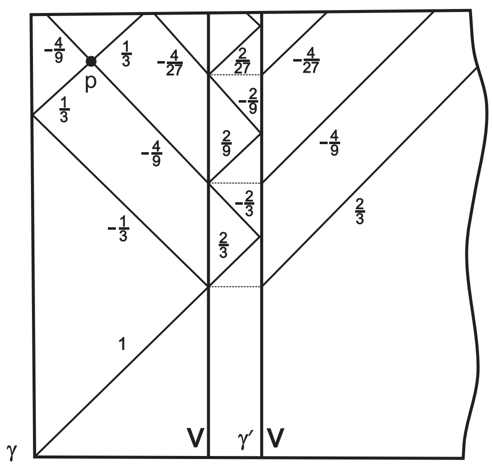

Figure 2: The amplitude on the hydra

(**) for $h \in H_\gamma$ provided $\pi(h) \in \Gamma$ and $\rho(h) > 0$, we put $a(h) = 0$;

(***) for $h \in H_\gamma$ provided $\pi(h) = \gamma$ and $\rho(h) = 0$, we put $a(h) = 1$.

So, the amplitude is a piece-wise constant function defined on the whole H_γ and determined by amplitudes of particles: see Fig. 2. As is easy to recognize, this definition is consistent with the 'dynamics of particles' $1. - 4.$ In the crossing points p, by the rule 2. one has $a(p) = -\frac{4}{9} + \frac{1}{3} = -\frac{1}{9}$. Also, for $h = (x, t) \in H_\gamma$ we write the amplitude as $a(x, t)$.

In what follows the basic object is a *truncated hydra*

$$H_\gamma^T := H_\gamma \cap \{\Omega \times [0, T]\} .$$

• Fix $\gamma \in \Gamma$ and take a control $f \in \mathscr{F}_\gamma^T$: $f(\gamma', t) = \delta_\gamma(\gamma')\varphi(t)$ with $\varphi \in L_2[0, T]$. Let u^f be the wave, i.e., the (generalized) solution to (3)–(6). As is shown in [10], the representation

$$u^f(x, T) = \sum_{t\in\rho(\pi^{-1}(x))} a(x, t)\, \varphi(T - t) , \qquad x \in \Omega \tag{14}$$

is valid. In the general case, for $f \in \mathscr{F}_\Sigma^T$: $f = \sum_{\gamma\in\Sigma} \delta_\gamma\varphi_\gamma$ with $\varphi_\gamma \in L_2[0, T]$ one has

$$u^f(x, T) = \sum_{\gamma\in\Sigma} \sum_{t\in\rho(\pi^{-1}(x))} a_\gamma(x, t)\, \varphi_\gamma(T - t) , \qquad x \in \Omega , \tag{15}$$

where a_γ are the amplitudes of the hydras H_γ^T.

Remark 1. *Representations (14) and (15) easily imply that the reachable sets $\mathscr{U}_\gamma^s$ and $\mathscr{U}_\Sigma^s$ are the (closed) subspaces in $\mathscr{H}$.*

Partition Π

Before reading this section, we'd recommend the reader to look through the paper [10], where the objects introduced here are described in more detail and provided with pictures.

Recall that $\Omega^T[\gamma]$ is the part of the graph filled at the final moment $t = T$ by waves moving from γ (see (12)). Begin with its special partition imposed by the structure of hydra H_γ^T.

• We say the points $h, h' \in H_\gamma^T$ to be the *neighbors* and write $h \sim h'$ if either or both $\pi(h) = \pi(h')$ and $\rho(h) = \rho(h')$ holds.

We put $h \overset{\gamma}{\sim} h'$ if there are $h'_1, \dots h'_k \in H_\gamma^T$ such that $h \sim h'_1 \sim \dots \sim h'_k \sim h'$, i.e. h and h' are connected via a set of neighbors. As is clear, $\overset{\gamma}{\sim}$ is an equivalence. The equivalence class

$$\mathscr{L}[h] := \{h' \in H_\gamma^T \mid h' \overset{\gamma}{\sim} h\}$$

is called a *lattice*. For a subset $B \subset H_\gamma^T$ one defines the lattice

$$\mathscr{L}[B] := \bigcup_{h\in B} \mathscr{L}[h]\,.$$

It is easy to verify that the operation $B \mapsto \mathscr{L}[B]$ possesses the following properties:

$$B \subset \mathscr{L}[B]; \qquad \mathscr{L}[\mathscr{L}[B]] = \mathscr{L}[B]; \qquad \mathscr{L}[B_1 \cup B_2] = \mathscr{L}[B_1] \cup \mathscr{L}[B_2];$$
$$\pi^{-1}(\pi(\mathscr{L}[B])) = \rho^{-1}(\rho(\mathscr{L}[B])) = \mathscr{L}[B]\,.$$

By the way, the first three properties show that this operation is a topological closure.

• With every $x \in \overline{\Omega^T[\gamma]} \setminus \Gamma$ we associate a set

$$\Lambda[x] \;:=\; \pi(\mathscr{L}[\pi^{-1}(x)]) \subset \overline{\Omega^T[\gamma]} \tag{16}$$

and name it by a *determination set* of the point x. Since $\overset{\gamma}{\sim}$ is an equivalence, the following alternative holds:

$$\text{for } x \neq x' \text{ one has either } \Lambda[x] = \Lambda[x'] \text{ or } \Lambda[x] \cap \Lambda[x'] = \emptyset\,. \tag{17}$$

• We say $h \in H_\gamma$ to be a *corner point* if either $\pi(h) \in V \cup \Gamma$ or h is a crossing point (like p on Fig.2). On the truncated hydra H_γ^T, the points of the set $\rho^{-1}(T)$ are also assigned to be corner points. By $\operatorname{Corn} H_\gamma^T$ we denote the set of all corner points of the truncated hydra.

The lattice $\mathscr{L}[\operatorname{Corn} H_\gamma^T]$ divides the hydra into a finite number of the open space-time intervals, the amplitude a taking a constant value on each interval.

The points, which constitute the set

$$\Theta \;:=\; \pi\left(\mathscr{L}[\operatorname{Corn} H_\gamma^T]\right) \subset \overline{\Omega^T[\gamma]}\,, \tag{18}$$

are called *critical*. The critical points divide $\Omega^T[\gamma]$ into parts. The set

$$\Pi \;:=\; \overline{\Omega^T[\gamma]} \setminus \Theta \tag{19}$$

is a sum of the finite number of open intervals, each interval belonging to the certain edge e. It provides the partition of $\Omega^T[\gamma]$ consistent with the structure of the hydra H_γ^T.

• Let $\omega \subset \Pi$ be a maximal interval, which does not contain critical points [3]. As is easy to see, the set

$$\Phi := \pi\left(\mathscr{L}[\pi^{-1}(\omega)]\right) \tag{20}$$

consists of the maximal intervals $\omega_1, \dots, \omega_m$ of the same length:

$$\Phi = \bigcup_{k=1}^{m} \omega_k\,, \qquad \operatorname{diam}\omega_1 = \dots = \operatorname{diam}\omega_m =: \epsilon_\Phi\,.$$

We say the intervals ω_k to be the *cells* of the *family* Φ. Comparing the definitions (16) and (20), one can easily get the representation

$$\Phi = \bigcup_{x\in\omega} \Lambda[x]\,, \tag{21}$$

where ω is any of the cells of Φ.

• Taking another maximal $\omega \subset \Pi$, which does not belong to the family Φ, one determines another family consisting of cells, and so on.

As a result, the set Π is a collection of disjoint families $\Phi^1, \dots, \Phi^J$, each family consisting of disjoint cells:

$$\Pi = \bigcup_{j=1}^{J} \Phi^j = \bigcup_{j=1}^{J}\bigcup_{k=1}^{m_j} \omega_k^j\,, \tag{22}$$

where m_j is the number of cells in Φ^j. Of course, the structure (22) changes as $T > 0$ varies.

• In parallel to the definition (16), with every $x \in \overline{\Omega^T[\gamma]} \setminus \Gamma$ one associates a set

$$\Xi[x] := \rho(\mathscr{L}[\pi^{-1}(x)]) \subset [0,T] \tag{23}$$

and verifies that for $x \neq x'$ one has either $\Xi[x] = \Xi[x']$ or $\Xi[x] \cap \Xi[x'] = \emptyset$. We put $\Xi[B] := \cup_{x\in B}\Xi[x]$.

Let $\Phi = \bigcup_{k=1}^{m_\Phi} \omega_k \subset \Pi$ be a family. As is easy to see, the set

$$\Psi := \Xi[\Phi] = \bigcup_{i=1}^{n_\Phi} \psi_i \subset [0,T]$$

[3]'maximal' means that any bigger interval $\omega' \subset \Omega^T[\gamma]\,:\,\omega' \supset \omega$ does contain critical points

consists of the time intervals $\psi_i=(t_{i-1},t_i)$ such that $0\leqslant t_1<t_2\leqslant t_3<t_4\leqslant \dots t_{n_\Phi-1}<t_{n_\Phi}\leqslant T$, the intervals being of the same length $t_i-t_{i-1}=\epsilon_\Phi$. We say Ψ also to be a family consisting of the *time cells* ψ_i.

In the sequel one makes use of the functions $\tau^i_\Phi:\Phi\to[0,T]$ defined as follows. For $x\in\Phi$ we put

$$\tau^i(x)\,:=\,\psi_i\cap\rho(\mathscr{L}[\pi^{-1}(x)])\,,\qquad i=1,\dots,n_\Phi\,. \tag{24}$$

Since $\mathscr{L}[\pi^{-1}(x)]=\mathscr{L}[\pi^{-1}(x_k)]$ for any $x_k\in\Lambda[x]$, we have $\tau^i(x)=\tau^i(x_k)$. As x varies over the cell $\omega\subset\Phi$, the value $\tau^i(x)$ sweeps the cell $\psi^i\subset\Psi$. Loosely speaking, τ^i is a piece-wise linear function on Φ. Later on this sentence will be made of rigorous meaning.

• Summarizing the above accepted definitions, one easily represents 'almost the whole' of the hydra in the form

$$H^T_\gamma\setminus\mathrm{CornH}^{\mathrm{T}}_\gamma\,=\,\bigcup_{j=1}^J\ \bigcup_{x\in\,\omega\subset\,\Phi^j}\mathscr{L}[\pi^{-1}(x)]=\bigcup_{j=1}^J\ \bigcup_{t\in\,\psi\subset\,\Psi^j}\mathscr{L}[\rho^{-1}(t)]\,,$$

which holds for any cells $\omega\subset\Phi^j$ and $\psi\subset\Psi^j$.

• The reason to introduce the partition Π and split the graph into families is that the waves u^f depend on controls f *locally* in the following sense. As one can see from (14), the values $u^f(\cdot,T)\big|_\Phi$ are determined by the values $f\big|_{\Xi[\Phi]}$. Moreover,

$$\operatorname{supp}f\subset\Xi[\Phi]\quad\text{is equivalent to}\quad\operatorname{supp}u^f(\cdot,T)\subset\Phi\,.$$

Such a locality is helpful for analysis of the reachable sets. In particular, it implies

$$\mathscr{U}^T_\gamma\,=\,\oplus\sum_{\Phi\subset\Pi}\mathscr{U}^T_\gamma\langle\Phi\rangle\,, \tag{25}$$

where $\mathscr{U}^T_\gamma\langle\Phi\rangle$ is the set of waves supported in Φ. Orthogonality of the sum is just a consequence of $\Phi^j\cap\Phi^k=\emptyset$ for $j\neq k$.

Amplitude vectors

A construction, which we introduce here, enables one to detail the representations of waves (14) and (15).

• Fix a point $x \in \Pi$; recall that such a point belongs to the metric neighborhood $\Omega^T[\gamma]$ and lies outside the set of critical points. Let $\Lambda[x] = \{x_k\}_{k=1}^{m_\Phi}$ be its determination set, $\Phi = \bigcup_{k=1}^{m_\Phi} \omega_k$ be the family which x belongs to (see (21) and (22)). So, $x_k \in \omega_k$, whereas x coincides with one of the points x_k. Let

$$\Xi[x] \;=\; \{t_i\}_{i=1}^{n_\Phi}: \;\; 0 < t_1 < \cdots < t_{n_\Phi} < T\,.$$

Note that the values t_i vary as x varies over a cell of Φ but the number n_Φ does not depend on x. Also notice the evident equality $\Xi[x] = \Xi[x_k]$ for all $x_k \in \Lambda[x]$.

• Now for an arbitrary $x \in \overline{\Omega^T[\gamma]} \setminus \Gamma$ we put

$$m[x] := \#\Lambda[x] \quad \text{and} \quad n[x] := \#\Xi[x].$$

Note that for $x \in \Phi$ on has $m[x] = m_\Phi$ and $n[x] = n_\Phi$. Then on the determination set one defines $n[x]$ functions $\alpha^i : \Lambda[x] \to \mathbb{R}$ by

$$\alpha^i(x_k) \;:=\; \begin{cases} a(x_k, t_i) & \text{if } \;(x_k, t_i) \in H^T_\gamma \\ 0 & \text{if } \;(x_k, t_i) \notin H^T_\gamma \end{cases}, \qquad k = 1, \ldots, m[x] \qquad (26)$$

and call them the *amplitude vectors.*

Return to representation (14). In terms of the amplitude vectors, (14) can be written in the form

$$u^f(x_k, T)\big|_{x_k \in \Lambda[x]} \;=\; \sum_{i=1}^{n[x]} \varphi(T - t_i)\alpha^i(x_k) \qquad\qquad (f = \delta_\gamma \varphi)\,,$$

which represents the wave not only at x but on the whole determination set $\Lambda[x]$. In particular, varying x over a cell $\omega \subset \Phi$, we represent the wave $u^f(\cdot, T)$ on the whole family Φ (see (21)).

• Let $l_2(\Lambda[x])$ be the space of functions on $\Lambda[x]$ with the inner product

$$\langle f, g\rangle \;=\; \sum_{x' \in \Lambda[x]} f(x')\, g(x') = \sum_{k=1}^{m[x]} f(x_k)\, g(x_k)\,.$$

It contains the subspace

$$\mathbb{A}[x] \;:=\; \operatorname{span}\{\alpha^1, \ldots, \alpha^{n[x]}\}\,, \qquad \dim \mathscr{A}[x] \leqslant n[x]$$

generated by the amplitude vectors.

In the sequel one makes the use of the more convenient basis in $\mathbb{A}[x]$. We redesign the system of amplitude vectors $\alpha^1, \dots, \alpha^{n[x]}$ by the Schmidt procedure:

$$\beta^i := \begin{cases} \frac{\alpha^1}{\|\alpha^1\|} & \text{if } i = 1\,, \\ \frac{\alpha^i - \sum\limits_{j=1}^{i-1} \langle \alpha^i, \beta^j \rangle\, \beta^j}{\|\alpha^i - \sum\limits_{j=1}^{i-1} \langle \alpha^i, \beta^j \rangle\, \beta^j \|} & \text{if } i \geqslant 2 \text{ and } \alpha^i \notin \operatorname{span}\{\alpha^1, \dots, \alpha^{i-1}\}\,, \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

and get a system $\beta^1, \dots, \beta^{n[x]}$. Its nonzero elements satisfy $\langle \beta^i, \beta^j \rangle = \delta_{ij}$, and $\operatorname{span}\{\beta^1, \dots, \beta^{n[x]}\} = \mathbb{A}[x]$ holds.

- For any two points $x, x' \in \omega \subset \Phi$ one has

$$\alpha^i(x) = \alpha^i(x') = \alpha^i\big|_\omega\,, \qquad \beta^i(x) = \beta^i(x') = \beta^i\big|_\omega\,.$$

Hence, $\alpha^i(\cdot)$ and $\beta^i(\cdot)$ are the piece-wise constant functions on the family Φ, these functions taking constant values on the cells.

Projection P^T_γ

Let P^T_γ be an (orthogonal) projection in $\mathscr{H} = L_2(\Omega)$ onto the reachable subspace $\mathscr{U}^T_\gamma$. Here we provide a constructive description of this projection.

- For a subset $B \subset \Omega$, by χ_B we denote its *indicator* (a characteristic function) and introduce the subspace

$$\mathscr{H}\langle B \rangle := \chi_B \mathscr{H} = \{\chi_B y \mid y \in \mathscr{H}\}$$

of functions supported on B. In accordance with (22), one has

$$\mathscr{H}\langle \Omega^T[\gamma] \rangle = \oplus \sum_{\Phi \in \Pi} \mathscr{H}\langle \Phi \rangle\,, \quad \mathscr{U}^T_\gamma \overset{(25)}{=} \oplus \sum_{\Phi \in \Pi} \mathscr{U}^T_\gamma \langle \Phi \rangle\,, \tag{28}$$

where $\mathscr{U}^T_\gamma\langle \Phi \rangle \subset \mathscr{H}\langle \Phi \rangle$ is the subspace of waves supported in Φ. Therefore,

$$P^T_\gamma = \oplus \sum_{\Phi \in \Pi} Q_\Phi\,, \tag{29}$$

where Q_Φ projects in $\mathscr{H}\langle\Phi\rangle$ onto $\mathscr{U}_\gamma^T\langle\Phi\rangle$. Hence, to characterize P_γ^T is to describe projections Q_Φ.

• As is shown in [10], the projections Q_Φ are represented via the above introduced vectors β^i as follows:

$$(Q_\Phi y)(x) \;=\; \begin{cases} \sum_{i=1}^{n_\Phi} \langle y\big|_{\Lambda[x]}, \beta^i\rangle\, \beta^i(x)\,, & x\in\Phi \\ 0\,, & x\in\Omega\backslash\Phi \end{cases}, \tag{30}$$

where $y \in \mathscr{H}$ is arbitrary. So, recalling (29), we conclude that P_γ^T is characterized.

Eikonal E_γ^T

As well as the projection P^T, the eikonal E_γ^T is reduced by the subspaces $\mathscr{H}\langle\Phi\rangle$, i.e., $E_\gamma^T\mathscr{H}\langle\Phi\rangle \subset \mathscr{H}\langle\Phi\rangle$ holds and implies

$$E_\gamma^T \;=\; \oplus \sum_{\Phi\subset\Pi} E_\gamma^T\langle\Phi\rangle \tag{31}$$

where $E_\gamma^T\langle\Phi\rangle := E_\gamma^T\big|_\Phi$ is the part of E_γ^T acting in $\mathscr{H}\langle\Phi\rangle$. As is shown in [10], the representation

$$\left(E_\gamma^T\langle\Phi\rangle y\right)(x) \;=\; \begin{cases} \sum_{i=1}^{n_\Phi} \tau^i(x)\langle y\big|_{\Lambda[x]}, \beta^i\rangle\, \beta^i(x)\,, & x\in\Phi \\ 0\,, & x\in\Omega\backslash\Phi \end{cases} \tag{32}$$

holds, where $y \in \mathscr{H}$ is arbitrary and τ^i are defined by (24).

It is seen from (32) that the eikonal is also reduced by the parts of the reachable set (the summands in (28)): one has

$$E_\gamma^T\mathscr{U}_\gamma^T\langle\Phi\rangle \;\subset\; \mathscr{U}_\gamma^T\langle\Phi\rangle\,, \qquad \Phi\subset\Pi\,.$$

More hydras for γ

For what follows it is reasonable to substitute H_γ^T for more convenient object which we call an efficient hydra and denote by $\dot H_\gamma^T$. The latter is constructed via an auxiliary extended hydra $\tilde H_\gamma^T$.

- Let $x \in \Omega^T[\gamma]$. Return to the definitions (16), (23) and introduce a *grid*

$$\mathscr{G}[x] := \Lambda[x] \times \Xi[x] \supseteq \mathscr{L}[\pi^{-1}(x)] .$$

Then define an *extended hydra*

$$\tilde{H}^T_\gamma := \overline{\bigcup_{j=1}^{J} \bigcup_{x \in \omega \subset \Phi^j} \mathscr{G}[x]} \supseteq H^T_\gamma$$

(see Fig. 3). As is easy to recognize, the neighborhood $\overset{\gamma}{\sim}$ and the lattices $\mathscr{L}[B]$ are also well defined on $\tilde{H}^T_\gamma$. Along with them, one defines the analogs of the sets (16), (23), which obviously coincide with the original $\Lambda[x]$, $\Xi[x]$ and are denoted by the same symbols.

Extend the amplitude $a(x,t)$ from H^T_γ to $\tilde{H}^T_\gamma$ by zero and denote the extension by $\tilde{a}(x,t)$. The extended amplitude is a piece-wise constant function on $\tilde{H}^T_\gamma$.

- Fix an $x \in \overline{\Omega^T[\gamma]}$ and define the new amplitude vectors $\tilde{\alpha}^i : \Lambda[x] \to \mathbb{R}$ by

$$\tilde{\alpha}^i(x_k) := \tilde{a}(x_k, t_i) \quad \text{for } (x_k, t_i) \in \tilde{H}^T_\gamma, \qquad x_k \in \Lambda[x] ,$$

where $t_i \in \Xi[x] = \Xi[x_k]$, $i = 1, \dots, n[x]$. Recall that $n[x] = n_\Phi$ for $x \in \Phi$. Comparing with (26), we see that these vectors coincide with the old ones for $x \in \Phi$.

Applying the Schmidt process (27) to the system $\{\tilde{\alpha}^i\}_{i=1}^{n[x]}$, we arrive at the vectors $\beta^i : \Lambda[x] \to \mathbb{R}$. When x varies over a cell $\omega \subset \Phi$, the points x_k also vary over the corresponding cells $\omega_k \subset \Phi$ but the values of the vector components $\beta^i(x_k)$ remain constant:

$$\beta^i(x_k) =: (\beta^i)_k = \text{const} \qquad \text{as } x_k \in \omega_k \subset \Phi . \tag{33}$$

The vectors β^i in turn determine a function b on the extended hydra by the following rule. Let $(x,t) \in \tilde{H}^T_\gamma$ be such that $x = x_k \in \Lambda[x]$ and $t = t_i \in \Xi[x]$; then we put $b(x,t) := \beta^i(x_k)$. By its definition, function b is defined on the whole $\tilde{H}^T_\gamma$. By its construction, b is a piece-wise constant function on the extended hydra.

- Now we reduce $\tilde{H}^T_\gamma$ and turn to an *efficient hydra*

$$\dot{H}^T_\gamma := \operatorname{supp} b \subset \tilde{H}^T_\gamma .$$

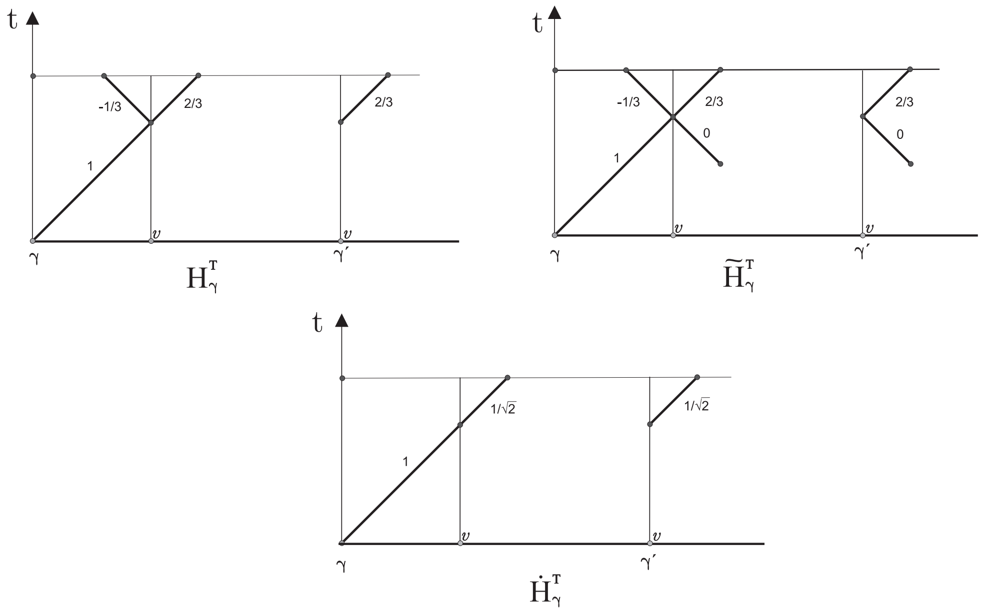

Figure 3: The original, extended, and efficient hydras

The function $b\big|_{\dot{H}^T_\gamma}$ is said to be an efficient amplitude.

For the pair $\{\dot{H}^T_\gamma, b\}$ one introduces the analogs of all the objects, which were defined for $\{H^T_\gamma, a\}$. Namely, one defines the space-time projections (13) for $\dot{H}^T_\gamma$ instead of H^T_γ, the lattices $\mathscr{L}[B] \subset \dot{H}^T_\gamma$, the neighborhood $\overset{\gamma}{\sim}$, determination sets (16) and (23), corner and critical points (18), the set (19), families (20) and their cells, and so on. In what follows we mark these analogs by dot: $\dot{\Lambda}[x]$, $\dot{\Theta}$, $\dot{\Pi}$, $\dot{\Phi}$, $\dot{\Psi}, \dots$. The analogs are of the same properties as the originals. In particular, the alternative (17) holds for $\dot{\Lambda}[x]$.

• The main reason to deal with $\dot{H}^T_\gamma$ instead of H^T_γ is that the partition $\dot{\Pi} = \cup_j \Phi^j$ (with some new families Φ^j !) provides the decompositions

$$\mathscr{H}\langle\Omega^T[\gamma]\rangle = \oplus \sum_{\Phi\in\dot{\Pi}} \mathscr{H}\langle\Phi\rangle\,, \quad \mathscr{U}^T_\gamma = \oplus \sum_{\Phi\in\dot{\Pi}} \mathscr{U}^T_\gamma\langle\Phi\rangle\,,$$

which are more natural and convenient for analysis of the eikonal algebra. One more advantage is that the amplitude vectors of $\{\dot{H}^T_\gamma, b\}$ defined by (26) (and denoted by $\dot{\beta}^i$ in the sequel), in essence, coincide with the vectors β^i (obtained by (27)) and, just by construction, constitute the orthogonal normalized bases in the spaces $\dot{\mathscr{A}}[x]$ of functions on the determination sets $\dot\Lambda[x]$.

• As one can show, in terms of the efficient hydra, the representations (29) and (30) take the form

$$P^T_\gamma \;=\; \oplus \sum_{\Phi\in\dot\Pi} Q_\Phi\,, \quad \left(Q_\Phi y\right)(x) \;=\; \begin{cases} \sum\limits_{i=1}^{n_\Phi} \langle y\big|_{\dot\Lambda[x]}, \dot\beta^i\rangle\, \dot\beta^i(x)\,, & x\in\Phi \\ 0\,, & x\in\Omega\backslash\Phi \end{cases}\,. \tag{34}$$

Respectively, for the eikonal, instead of (31) and (32) we have

$$E^T_\gamma \;=\; \oplus \sum_{\Phi\subset\dot\Pi} E^T_\gamma\langle\Phi\rangle; \qquad \left(E^T_\gamma\langle\Phi\rangle y\right)(x) \;=$$
$$=\; \begin{cases} \sum\limits_{i=1}^{n_\Phi} \dot\tau^i(x)\langle y\big|_{\dot\Lambda[x]}, \dot\beta^i\rangle\, \dot\beta^i(x)\,, & x\in\Phi \\ 0\,, & x\in\Omega\backslash\Phi \end{cases}\,, \tag{35}$$

where $\dot\tau^i$ are defined for $\dot H^T_\gamma$ in the same way as τ^i for H^T_γ: see (24).

Partition Π_Σ

Now, let $\Sigma=\{\gamma_1,\dots,\gamma_\sigma\}$ be a subset of the graph boundary Γ. Recall that $\Omega^T[\Sigma]\subset\Omega$ is a metric neighborhood of Σ of radius T. One has $\Omega^T[\Sigma]=\cup_{\gamma\in\Sigma}\Omega^T[\gamma]$, so that this neighborhood is a part of the graph filled (at the moment $t=T$) by waves, which move from all γ_k.

In what follows we deal with a collection of hydras $\dot H^T_\gamma$ for $\gamma\in\Sigma$. The objects related with single vertices are marked by the subscript: $\dot\Lambda_\gamma[x], \dot\Theta_\gamma, \dot\Pi_\gamma$ and so on.

• Recall that (20) and (22) provide the partition of $\Omega^T[\gamma]$ consistent with the structure of a single hydra. Here we describe a partition of $\Omega^T[\Sigma]$ relevant to the collection of the *efficient* hydras. Notice that it differs from the one introduced in [10] and associated with the original hydras H^T_γ.

We say the points $x,x'\in\overline{\Omega^T[\Sigma]}$ to be the space neighbors with respect to γ and write $x\overset{\gamma}{\approx}x'$ if $\dot\Lambda_\gamma[x]=\dot\Lambda_\gamma[x']$. The relation $\overset{\gamma}{\approx}$ is an equivalence.

The points x,x' are said to be the space neighbors with respect to Σ (we write $x\overset{\Sigma}{\approx}x'$) if there are the points $x_1,\dots,x_l$ and vertices $\gamma_{k_j}\in\Sigma$ such that $x\overset{\gamma_{k_1}}{\approx}x_1\overset{\gamma_{k_2}}{\approx}\dots\overset{\gamma_{k_l}}{\approx}x_l\overset{\gamma_{k_{l+1}}}{\approx}x'$. The relation $\overset{\Sigma}{\approx}$ is also an equivalence. By $\Lambda_\Sigma[x]$

we denote the equivalence class of the point x. Note the obvious relation $\dot{\Lambda}_\gamma[x] \subset \Lambda_\Sigma[x]$ following from the definitions.

• Recall that $\dot{\Theta}_\gamma \subset \overline{\Omega^T[\gamma]}$ is the set of critical points determined by the hydra $\dot{H}^T_\gamma$, and put $\Theta'_\Sigma := \cup_{\gamma\in\Sigma}\dot{\Theta}_\gamma$. Then we define

$$\Theta_\Sigma := \bigcup_{x\in\Theta'_\Sigma} \Lambda_\Sigma[x] \quad \text{and} \quad \Pi_\Sigma := \overline{\Omega^T_\Sigma} \setminus \Theta_\Sigma\,.$$

It is the set Π_Σ which provides the relevant partition of the part $\Omega^T[\Sigma]$ filled by waves from Σ. As is easy to verify, Π_Σ consists of certain (new) families Φ^j, each family consisting of the cells of equal length:

$$\Pi_\Sigma = \bigcup_{j=1}^{J_\Sigma} \Phi^j\,; \qquad \Phi^j = \bigcup_{i=1}^{m_j} \omega^j_i\,, \quad \operatorname{diam} \omega^j_i = \epsilon_j\,.$$

• Now we modify the representations (34) and (35) to make them consistent with the partition Π_Σ.

Fix a point $x \in \Omega^T[\Sigma]$ and assume that $x \in \Omega^T[\gamma]$. Let x belong to a family $\Phi \subset \dot{\Pi}_\gamma$ and let $\dot{\Lambda}_\gamma[x] \subset \Phi$ be its determination set. Let $\dot{\beta}^i_\gamma$ be the amplitude vectors on $\dot{\Lambda}_\gamma[x]$. Recall the embedding $\dot{\Lambda}_\gamma[x] \subset \Lambda_\Sigma[x]$ and extend all $\dot{\beta}^i_\gamma$ from $\dot{\Lambda}_\gamma[x]$ to $\Lambda_\Sigma[x]$ by zero. Simplifying the notation, we denote these extensions by β^i_γ. Then one can get the modified representation

$$P^T_\gamma = \sum_{\Phi\in\Pi_\Sigma} Q_\Phi\,, \quad (Q_\Phi y)(x) = \begin{cases} \sum\limits_{i=1}^{n_\Phi} \langle y\big|_{\Lambda_\Sigma[x]}, \beta^i_\gamma\rangle\, \beta^i_\gamma(x)\,, & x\in\Phi \\ 0\,, & x\in\Omega\backslash\Phi \end{cases}. \tag{36}$$

While (34) and (35) are related with $\dot{\Pi}_\gamma$, the new representation is consistent with the structure of Π_Σ that occurs for all $\gamma\in\Sigma$.

Respectively, one represents the eikonal as follows:

$$E^T_\gamma = \oplus \sum_{\Phi\subset\Pi_\Sigma} E^T_\gamma\langle\Phi\rangle; \qquad \big(E^T_\gamma\langle\Phi\rangle y\big)(x) =$$

$$= \begin{cases} \sum\limits_{i=1}^{n_\Phi} \tau^i_\gamma(x)\langle y\big|_{\Lambda_\Sigma[x]}, \beta^i_\gamma\rangle\, \beta^i_\gamma(x)\,, & x\in\Phi \\ 0\,, & x\in\Omega\backslash\Phi \end{cases}, \tag{37}$$

where $y\in\mathscr{H}$ is arbitrary and τ^i_γ is understood as $\dot{\tau}^i_\gamma$ extended from $\dot{\Lambda}_\gamma[x]$ to $\Lambda_\Sigma[x]$ by zero.

Parametrization

• Choose a family $\Phi = \cup_{k=1}^{m}\omega_k \subset \Pi_\Sigma$ [4]; let $\omega =]c, c'[\subset \Phi$ be one of the cells, which lies between critical points c and c'. Recall that all the cells are of the same length $\epsilon = \tau(c, c')$, where τ is a distance on the graph. For $x \in \omega$ we put $x = x(r)$ if $\tau(c, x) = r$.

Along with x, its determination set also turns out to be parametrized: $\Lambda_\Sigma[x(r)] = \{x_k(r)\}_{k=1}^{m}$. As r runs over $(0, \epsilon_\Phi)$, the points $x_k(r)$ vary continuously and sweep the cells ω_k. Thus, the family Φ is parametrized in whole.

By this, all elements of representations (36) and (37) are parametrized. The vectors are

$$\beta_\gamma^i = \{(\beta_\gamma^i)_k(r)\}_{k=1}^{m}, \;\; (\beta_\gamma^i)_k(r) := \beta_\gamma^i(x_k(r)) \overset{(33)}{=} (\beta_\gamma^i)_k = \text{const}, \quad 0 < r < \epsilon_\Phi \,.$$

The functions take the form $\tau_\gamma^i(r) := \tau_\gamma^i(x(r))$; as r runs over $(0, \epsilon_\Phi)$ the values $\tau_\gamma^i(r)$ sweep the proper time cell $\psi_i = (t_{i-1}, t_i)$. The definition (24) easily implies that

$$\text{either } \; \tau_\gamma^i(r) = t_{i-1} + r \quad \text{or} \quad \tau_\gamma^i(r) = t_i - r = (t_{i-1} + \epsilon) - r \tag{38}$$

holds.

In the sequel we assume that each $\Phi \subset \Pi_\Sigma$ is parametrized as described above.

• With the parametrization one associates a matrix representation of functions and operators on the graph as follows.

Let $\Phi \subset \Pi_\Sigma$ be a parametrized family, $y \in \mathscr{H}$ a function on the graph, and let $x = x(r) \in \Lambda_\Sigma[x(r)] = \{x_k(r)\}_{k=1}^{m} \subset \Phi$, $0 < r < \epsilon$. Introduce an isometry U_Φ by

$$\mathcal{H}\langle\Phi\rangle \ni y \;\; \overset{U_\Phi}{\mapsto} \;\; \begin{pmatrix} y(x_1(r)) \\ \dots \\ y(x_m(r)) \end{pmatrix} \Bigg|_{r \in (0,\epsilon)} \in L_2\left((0, \epsilon); \mathbb{R}^m\right) \,.$$

Define

$$B_\gamma := \begin{pmatrix} (\beta_\gamma^1)_1 & \dots & (\beta_\gamma^1)_m \\ (\beta_\gamma^2)_1 & \dots & (\beta_\gamma^2)_m \\ \dots & \dots & \dots \\ (\beta_\gamma^n)_1 & \dots & (\beta_\gamma^n)_m \end{pmatrix}, \;\; D_\gamma(r) := \{\tau_\gamma^i(r)\,\delta_{ij}\}_{i,j=1}^{n}\,, \quad r \in (0, \epsilon),$$

[4]For a wile, we simplify the notation: $m = m_\Phi$, $\epsilon = \epsilon_\Phi$, $n = n_\Phi$ and so on.

where $\tau^i_\gamma(r)$ are of the form (38). Note that $B^*_\gamma B_\gamma$ is the constant matrix which projects vectors in $\mathbb{R}^m$ onto the subspace $\mathbb{A}_\gamma[x(r)] = \operatorname{span}\{\beta^1_\gamma, \ldots, \beta^n_\gamma\}$. Along with its generating vectors β^i_γ, this subspace does not vary as r runs over $(0, \epsilon)$. Therefore it is reasonable to denote it by $\mathbb{A}_\gamma\langle\Phi\rangle$, what we do in the sequel. So, we have

$$\mathbb{A}_\gamma\langle\Phi\rangle \;=\; \operatorname{span}\{\beta^1_\gamma, \ldots, \beta^n_\gamma\} \;=\; \left[B^*_\gamma B_\gamma\right]\mathbb{R}^m\,.$$

• The summand in (37) corresponding to the family Φ is represented as follows:

$$\left(U_\Phi\, E^T_\gamma\langle\Phi\rangle\, y\right)(r) \;=\; \left[B^*_\gamma D_\gamma(r) B_\gamma\right](U_\Phi y)(r)\,, \qquad r \in (0, \epsilon)\,. \tag{39}$$

From here on, we assume that each family $\Phi \subset \Pi_\Sigma$ is parametrized by the proper $r \in (0, \epsilon_\Phi)$, so that the matrices entering in (39) are at our disposal. We denote them by

$$B_{\gamma\,\Phi} = \{(\beta^i_{\gamma\,\Phi})_k\}_{\substack{i=1,\ldots,n_\Phi\\ k=1,\ldots,m_\Phi}} \quad \text{and} \quad D_{\gamma\,\Phi} = \operatorname{diag}\{\tau^i_{\gamma\,\Phi}(\cdot)\}^{m_\Phi}_{i=1}\,.$$

Summary

Let us resume the previous considerations.

• Let $\Sigma \subset \Gamma$, $\#\Sigma =: \sigma$. The collection of efficient hydras $\{\dot{H}^T_\gamma \mid \gamma \in \Sigma\}$ determines the set of critical points Θ_Σ. These points divide the subdomain $\Omega^T[\Sigma]$ in parts (families):

$$\Omega^T[\Sigma] \setminus \Theta_\Sigma = \Pi_\Sigma = \bigcup_{j=1}^{J} \Phi^j\,.$$

This division determines the decomposition of subspaces

$$\mathscr{H}\langle\Omega^T[\Sigma]\rangle = \oplus\sum_{\Phi\in\Pi_\Sigma}\mathscr{H}\langle\Phi\rangle\,, \quad \mathscr{U}^T_\Sigma = \oplus\sum_{\Phi\in\Pi_\Sigma}\mathscr{U}^T_\Sigma\langle\Phi\rangle\,,$$

which reduces all the eikonals simultaneously:

$$E^T_\gamma\mathscr{H}\langle\Phi\rangle \subset \mathscr{H}\langle\Phi\rangle\,, \;\; E^T_\gamma\,\mathscr{U}^T_\Sigma\langle\Phi\rangle \subset \mathscr{U}^T_\Sigma\langle\Phi\rangle\,, \qquad \gamma \in \Sigma\,.$$

• Parametrizing, we represent the subspaces as

$$U_\Phi\mathscr{H}\langle\Phi\rangle = L_2\left([0, \epsilon_\Phi]); \mathbb{R}^{m_\Phi}\right), \quad U_\Phi\mathscr{U}^T_\Sigma\langle\Phi\rangle = L_2\left([0, \epsilon_\Phi]; \mathbb{A}_\Sigma\langle\Phi\rangle\right),$$

where $\mathbb{A}_\Sigma\langle\Phi\rangle := \operatorname{span}\{\mathbb{A}_\gamma\langle\Phi\rangle \mid \gamma\in\Sigma\}$.

In the parametric form, the parts $E^T_\gamma\langle\Phi\rangle$ of the eikonals multiply elements of $L_2((0,\epsilon_\Phi);\mathbb{R}^{m_\Phi})$ by the matrix-functions (39).

The total representation is realized by the operator $U := \oplus\sum_{\Phi\in\Pi_\Sigma} U_\Phi$ which provides

$$U\mathscr{H}\langle\Omega^T[\Sigma]\rangle = \oplus\sum_{\Phi\in\Pi_\Sigma} L_2((0,\epsilon_\Phi);\mathbb{R}^{m_\Phi})\,; \qquad UE^T_\gamma U^{-1} =$$
$$= \oplus\sum_{\Phi\in\Pi_\Sigma} U_\Phi\, E^T_\gamma\langle\Phi\rangle\, U^{-1}_\Phi = \oplus\sum_{\Phi\in\Pi_\Sigma}[B^*_{\gamma\,\Phi} D_{\,\Phi}(\cdot) B_{\gamma\,\Phi}]\,, \qquad \gamma\in\Sigma\,. \tag{40}$$

- Turning to the eikonal algebra (9), we have

$$\mathfrak{E}^T_\Sigma = \vee\{E^T_\gamma \mid \gamma\in\Sigma\} \overset{(37)}{=} \vee\left\{\oplus\sum_{\Phi\subset\Pi_\Sigma} E^T_\gamma\langle\Phi\rangle \,\middle|\, \gamma\in\Sigma\right\},$$
$$U\,\mathfrak{E}^T_\Sigma U^{-1} \overset{(40)}{=} \vee\left\{\oplus\sum_{\Phi\in\Pi_\Sigma}\left[B^*_{\gamma\,\Phi} D_{\gamma\,\Phi}(\cdot) B_{\gamma\,\Phi}\right] \,\middle|\, \gamma\in\Sigma\right\} =$$
$$\overset{(37)}{=} \vee\left\{\oplus\sum_{\Phi\in\Pi_\Sigma}\left[\sum_{i=1}^{n_\Phi}\tau^i_{\gamma\,\Phi}(\cdot)\,P^i_{\gamma\,\Phi}\right] \,\middle|\, \gamma\in\Sigma\right\}, \tag{41}$$

where $P^i_{\gamma\,\Phi} = \langle\cdot,\beta^i_{\gamma\,\Phi}\rangle\beta^i_{\gamma\,\Phi}$ are the constant (w.r.t. the parameter $r\in[0,\epsilon_\Phi]$) one-dimensional matrix projections, the projections being orthogonal by pairs: $P^i_{\gamma\,\Phi}P^{i'}_{\gamma\,\Phi} = P^{i'}_{\gamma\,\Phi}P^i_{\gamma\,\Phi} = \mathbb{O}$ for $i\neq i'$. The sum $\oplus\sum_{i=1}^{n_\Phi} P^i_{\gamma\,\Phi}$ projects in $\mathbb{R}^{m_\Phi}$ onto the subspace $\mathbb{A}_{\gamma\,\Phi} = \operatorname{span}\{\beta^i_{\gamma\,\Phi} \mid i=1,\dots,n_\Phi\}$. In more demonstrable form one has

$$U\,\mathfrak{E}^T_\Sigma U^{-1} = \vee\left\{\begin{pmatrix}\sum_{i=1}^{n_{\Phi^1}}\tau^i_{\gamma\,\Phi^1}(\cdot_1)P^i_{\gamma\,\Phi^1} & & \\ & \ddots & \\ & & \sum_{i=1}^{n_{\Phi^J}}\tau^i_{\gamma\,\Phi^J}(\cdot_J)P^i_{\gamma\,\Phi^J}\end{pmatrix} \,\middle|\, \gamma\in\Sigma\right\} \subset$$
$$\subset \begin{pmatrix} C\left([0,\epsilon_1];\mathbb{M}^{m_{\Phi^1}}\right) & & \\ & \ddots & \\ & & C\left([0,\epsilon_J];\mathbb{M}^{m_{\Phi^J}}\right)\end{pmatrix} \tag{42}$$

where the arguments $\cdot_j$ run over $[0, \epsilon_j]$. Thus, $U\, \mathfrak{E}^T_\Sigma U^{-1}$ is an operator algebra; its elements multiply the elements of the representation space

$$\mathscr{S}^T_\Sigma \,:=\, \oplus \sum_{j=1}^{J} L_2\left([0, \epsilon_j]; \mathbb{R}^{m_{\Phi^j}}\right)$$

by the continuous matrix-valued functions of the proper structure.

3 Algebra $\mathfrak{E}^T_\Sigma$

Representation (41) enables one to analyze the structure of the eikonal algebra, which is the main subject of the paper. Analysis is preceded by some general facts and results on algebras.

About C*-algebras

Recall some of the definitions. We write $\mathfrak{A} \cong \mathfrak{B}$ if the algebras $\mathfrak{A}$ and $\mathfrak{B}$ are isometrically isomorphic.

• A C*-algebra is a Banach algebra with an involution $x \mapsto x^*$ obeying $(x^*)^* = x$, $(x+y)^* = x^* + y^*$, $(\lambda x)^* = \lambda x^*$, $(xy)^* = y^* x^*$, and $\|x^*\| = \|x\|$, $\|x^* x\| = \|x\|^2$ holds [11, 12].

A C*-algebra $\mathfrak{M}$ is *elementary* if there is a Hilbert space $\mathscr{R}$ such that $\mathfrak{M} \cong \mathfrak{S}_\infty(\mathscr{R})$ holds, where $\mathfrak{S}_\infty(\mathscr{R})$ is the compact operator algebra in $\mathscr{R}$ [11]. We'll deal with $\mathscr{R} = \mathbb{R}^m$ and the matrix algebras $\mathfrak{M}$.

Let $\mathscr{T}$ be a topological space, $\{\mathfrak{A}(t)\}_{t\in\mathscr{T}}$ a family of C*-algebras. The elements $x \in \prod_{t\in\mathscr{T}} \mathfrak{A}(t)$, i.e., the functions on $\mathscr{T}$ provided $x(t) \in \mathscr{T}$, are called the *vector fields*.

A *continuous field of algebras* is a family $\{\mathfrak{A}(t)\}_{t\in\mathscr{T}}$ endowed with a set of vector fields $F \subset \{\mathfrak{A}(t)\}_{t\in\mathscr{T}}$ such that
1. F is a linear space in $\{\mathfrak{A}(t)\}_{t\in\mathscr{T}}$
2. for any $t \in \mathscr{T}$, the set $\{x(t)\}_{x\in F}$ is dense in $\mathfrak{A}(t)$
3. for any $x \in F$, a function $t \mapsto \|x(t)\|$ is continuous in $\mathscr{T}$
4. if an element $x \in \prod_{t\in\mathscr{T}} \mathfrak{A}(t)$ is such that for all $\varepsilon > 0$, $t \in \mathscr{T}$ there is a $\phi \in F$ providing $\|x(t) - \phi(t)\| < \varepsilon$, then $x \in F$.

• The following fact plays the key role (see [11], sec 10.5.3).

Theorem 1. *Let $\mathscr{T}$ be a locally compact space, $(\mathfrak{A}(t), F)$ a continuous field of elementary C^*-algebras on $\mathscr{T}$, $\mathfrak{A}$ the C^*-algebra determined by this field. Let $\mathfrak{B} \subset \mathfrak{A}$ be a C^*-algebra such that for any $t, t' \in \mathscr{T}$ and arbitrary $\alpha \in \mathfrak{A}(t)$, $\alpha' \in \mathfrak{A}(t')$ there is an element $f \in \mathfrak{B}$ such that $f(t) = \alpha$, $f(t') = \alpha'$ holds. Then $\mathfrak{B} = \mathfrak{A}$.*

We say that algebra $\mathfrak{B}$, possessing such a property, *strongly separates* the points of $\mathscr{T}$.

In our case the fields f will be the matrix-valued functions given on a finite segment $\mathscr{T} = [0, \epsilon]$.

Standard algebras

By $\mathfrak{B}(\mathscr{G})$ we denote an algebra of bounded operators acting in a Hilbert space $\mathscr{G}$. Let $\mathbb{M}^n = \mathfrak{B}(\mathbb{R}^n)$ be the algebra of $n \times n$ - matrices with the norm $\|M\| = \sup\{\|M\xi\|_{\mathbb{R}^n} \mid \|\xi\|_{\mathbb{R}^n} = 1\}$ and involution (conjugation) $M \mapsto M^*$;

• In the sequel we make use of the following concrete algebras, which we call *standard*:

algebra $C[0, \epsilon] \equiv C([0, \epsilon]; \mathbb{M}^1)$ of continuous functions with the norm $\|f\| = \sup\limits_{0 \leqslant t \leqslant \epsilon} |f(t)|$ and its subalgebra $C_0[0, \epsilon] := \{f \in C[0, \epsilon] \mid f(0) = 0\}$;

algebra $C([0, \epsilon]; \mathbb{M}^n)$ of continuous matrix-valued functions with the norm $\|f\| = \sup\limits_{0 \leqslant t \leqslant \epsilon} \|f(t)\|$, the point-wise (matrix) multiplication and involution $f(t) \mapsto [f(t)]^*$;

(sub)algebra $\dot{C}([0, \epsilon]; \mathbb{M}^3) := \left\{f \in C([0, \epsilon]; \mathbb{M}^3) \;\middle|\; f(0) \in \mathbb{M}^1 \oplus \mathbb{M}^2\right\}$;

operator algebra $C^{\rm op}([0, \epsilon]; \mathbb{M}^n) \subset \mathfrak{B}(L_2([0, \epsilon]; \mathbb{R}^n))$ $(n \geqslant 1)$: its elements $f^{\rm op}$ multiply vector-functions $u \in L_2([0, \epsilon]; \mathbb{R}^n)$ by $f \in C([0, \epsilon]; \mathbb{M}^n)$;

operator (sub)algebra $\dot{C}^{\rm op}([0, \epsilon]; \mathbb{M}^n)$; its elements $f^{\rm op}$ multiply vector-functions $u \in L_2([0, \epsilon]; \mathbb{R}^n)$ by $f \in \dot{C}([0, \epsilon]; \mathbb{M}^n)$.

As is well known, $f \mapsto f^{\rm op}$ is an isometric isomorphism of C*-algebras: $C([0, \epsilon]; \mathbb{M}^n) \cong C^{\rm op}([0, \epsilon]; \mathbb{M}^n)$ holds. By isometry, in what follows we identify $C(\dots) \equiv C^{\rm op}(\dots)$.

• More generally, we say a C*-algebra $\mathfrak{A}$ to be standard if

$$\mathfrak{A} \cong \left\{f \in C([\alpha, \beta]; \mathbb{P}) \;\middle|\; f(\alpha) \in \mathbb{P}_\alpha,\ f(\beta) \in \mathbb{P}_\beta\right\},$$

where $\mathbb{P}_\alpha$ and $\mathbb{P}_\beta$ are the C*-subalgebras of a matrix C*-algebra $\mathbb{P} \subseteq \mathbb{M}^n$. General properties of such algebras are well known: see [1, 11, 12]. In what follows they play the role of the 'building blocks' which the eikonal algebra consists of.

Block structure

- Return to (41), (42) and define

$$E_{\gamma\,\Phi}(r) := \sum_{i=1}^{n_\Phi} \tau^i_{\gamma\,\Phi}(r) P^i_{\gamma\,\Phi}; \;\; \mathfrak{b}_\Phi(r) := \vee\{E_{\gamma\,\Phi}(r) \mid \gamma \in \Sigma\}, \;\; r \in [0, \epsilon_\Phi]\,. \quad (43)$$

Looking at the form of (41) and (42), it is reasonable to say the algebra $\mathfrak{b}_\Phi(\cdot)$ to be a *block* of the algebra $U\,\mathfrak{E}^T_\Sigma U^{-1}$ corresponding to the family Φ. Denoting $\mathbb{P}_\Phi := \vee\{P^i_{\gamma\,\Phi} \mid i = 1, \dots, n_\Phi;\; \gamma \in \Sigma\} \subset \mathbb{M}^{m_\Phi}$, we have the evident relations

$$\mathfrak{b}_\Phi(\cdot) \subset C\left([0, \epsilon_\Phi]; \mathbb{P}_\Phi\right) \subset C\left([0, \epsilon_\Phi]; \mathbb{M}^{m_\Phi}\right)\,.$$

By $\mathfrak{b}_\Phi\big|_K$ we denote the set of restrictions of matrix-functions $b \in \mathfrak{b}_\Phi$ onto the subset $K \subset [0, \epsilon_\Phi]$.

Lemma 1. *For any* $[a, b] \subset (0, \epsilon_\Phi)$ *the relation* $\mathfrak{b}_\Phi\big|_{[a,b]} = C\left([a, b]; \mathbb{P}_\Phi\right)$ *is valid.*

□ For a function $\tau = \tau(s)$, by $\tau(K) \subset \mathbb{R}$ we denote the range of its values as s varies over K.

$*$ Fix Φ, γ and i for a while. Take the values r', r'' of the parameter r provided $0 < a \leqslant r' < r'' \leqslant b < \epsilon_\Phi$. The property (38) implies

$$\tau^i_{\gamma\,\Phi}([a, b]) \cap \tau^l_{\gamma\,\Phi}([a, b]) = \emptyset \qquad \text{for } \; l \neq i\,.$$

Hence, there is a polynomial $q = q(\lambda)$ $(\lambda \in \mathbb{R})$ such that

$$q(0) = 0; \quad q\big|_{\lambda = \tau^l_{\gamma\,\Phi}(r')} = 0 \;\text{ for all }\; l = 1, \dots, n_\Phi; \quad q\big|_{\lambda = \tau^l_{\gamma\,\Phi}(r'')} = 0 \;\text{ for } l \neq i;$$
$$q\big|_{\lambda = \tau^i_{\gamma\,\Phi}(r'')} = 1\,.$$

$*$ By orthogonality of the projections $P^i_{\gamma\,\Phi}$ in (43) we have

$$\left(q\left(E_{\gamma\,\Phi}\right)\right)(r) := \sum_{l=1}^{n_\Phi} q\left(\tau^l_{\gamma\,\Phi}(r)\right) P^l_{\gamma\,\Phi}$$

that leads to

$$(q\,(E_{\gamma\,\Phi}))\,(r') = \mathbb{O} \text{ and } (q\,(E_{\gamma\,\Phi}))\,(r'') = P^i_{\gamma\,\Phi} \tag{44}$$

owing to the choice of the polynomial.

$*$ By arbitrariness of a, b, r'' the second relation in (44) means that each single projection $P^i_{\gamma\,\Phi}$ belongs to the algebra $\mathfrak{b}_\Phi(r)$ generated by 'sums of such projections'. Therefore we get

$$\mathfrak{b}_\Phi(r) = \vee\{P^i_{\gamma\,\Phi} \mid i = 1, \dots, n_\Phi;\ \gamma \in \Sigma\} = \mathbb{P}_\Phi, \quad r \in (0, \epsilon_\Phi)\,. \tag{45}$$

The second consequence of (44) is the following. It is easy to see that the proper choice of the polynomial q enables one to change the roles of r', r'' and get the relation $(q\,(E_{\gamma\,\Phi}))\,(r'') = \mathbb{O}$ and $(q\,(E_{\gamma\,\Phi}))\,(r') = P^i_{\gamma\,\Phi}$. Then, combining the relations, one can find a polynomial $\tilde{q}$ provided $\tilde{q}(0) = 0$, which obeys $\tilde{q}(r) = p$ and $\tilde{q}(r') = p'$ for *any* $r, r' \in (0, r_\Phi)$ and $p, p' \in \mathbb{P}_\Phi$. The latter means that the (sub)algebra $\mathfrak{b}_\Phi\big|_{[a,b]} \subset C\,([a,b]; \mathbb{P}_\Phi)$ strongly separates points of the compact $\mathscr{T} = [a, b]$. Applying Theorem 1, we conclude that the subalgebra exhausts the algebra, what the Lemma claims. ■

Notice in addition that (45) may be invalid at the endpoints $r = 0, \epsilon_\Phi$. For instance, if $\tau^i_{\gamma\,\Phi}(0) = 0$ then the projection $P^i_{\gamma\,\Phi}$ drops out of the generators of $\mathfrak{b}_\Phi(0)$ that may lead to $\mathfrak{b}_\Phi(0) \neq \mathbb{P}_\Phi$. Similarly, if $\tau^i_{\gamma\,\Phi}(\epsilon_\Phi) = \tau^{i+1}_{\gamma\,\Phi}(\epsilon_\Phi)$ then in (43) one gets

$$E_{\gamma\,\Phi}(\epsilon_\Phi) = \dots + \tau^i_{\gamma\,\Phi}(\epsilon_\Phi)[P^i_{\gamma\,\Phi} + P^{i+1}_{\gamma\,\Phi}] + \dots$$

that also reduces the list of generators and may lead to $\mathfrak{b}_\Phi(\epsilon_\Phi) \neq \mathbb{P}_\Phi$. One more occasion is the equalities like $\tau^i_{\gamma\,\Phi'}(\epsilon_{\Phi'}) = \tau^i_{\gamma\,\Phi''}(0)$ which provide certain connections between the different families. These effects do occur in the known examples and will be demonstrated in section 4.

• So, each block-algebra $\mathfrak{b}_\Phi$ consists of the continuous $\mathbb{P}_\Phi$-valued functions satisfying certain conditions at the endpoints of $[0, \epsilon_\Phi]$. Respectively, the bulk of the algebra $U\mathfrak{E}^T_\Sigma U^{-1} \cong \mathfrak{E}^T_\Sigma$ is exhausted by the sum $\oplus \sum\limits_{\Phi\subset\Pi_\Sigma} C\,((0, \epsilon_\Phi); \mathbb{P}_\Phi)$. More subtle considerations are required to clarify possible connections between the summands at the endpoints $r = 0, \epsilon_\Phi$. As a result, with some abuse of terms, we can claim that the eikonal algebra consists of the *standard algebras.*

On matrix algebras

Dealing with concrete examples, we have to reveal the structure of the algebras $\mathbb{P}_\Phi \subset \mathbb{M}^{m_\Phi}$. By doing so, we regard $\mathbb{M}^n$ as an algebra of operators acting in $\mathbb{R}^n$ and make use of the following simple facts.

1. Any C*-subalgebra of the algebra $\mathbb{M}^n$ is isometrically isomorphic to a sum $\oplus \sum_k \mathbb{M}^{n_k}$, where $\sum n_k \leqslant n$.

2. The only irreducible C*-subalgebra of the algebra $\mathbb{M}^n$ is $\mathbb{M}^n$ itself.

3. Let $P_1, \dots, P_\sigma \subset \mathbb{M}^n$ be a set of projections: $P_l^* = P_l = P_l^2$. Then the equality $\vee\{P_1, \dots, P_\sigma\} = \mathbb{M}^n$ is valid if and only if no one nonzero vector in $\mathbb{R}^n$ is an eigenvector for all P_j simultaneously.

4. If $E_l = \sum_{i=1}^{n_l} \lambda_l^i P_l^i$ $(l = 1, \dots, \sigma;\ \lambda_l^i \neq 0)$ is the spectral decomposition, then the equality $\vee\{E_1, \dots, E_\sigma\} = \vee\{P_l^i \mid i = 1, \dots, n_l;\ \ l = 1, \dots, \sigma\}$ holds.

4 Simple graph

As a simple (but not trivial!) example, we deal with a graph Ω, which consists of three edges e_1, e_2, e_3, three boundary vertices $\gamma_1, \gamma_2, \gamma_3$, and a single interior vertex v. The lengths of edges satisfy a generic condition

$$l_1 < l_2 < l_3\,.$$

We take $\Sigma = \{\gamma_1, \gamma_2\}$ and study the structure of $\mathfrak{E}_\Sigma^T$ and evolution of this structure w.r.t. T. Namely, we search the cases $T = T_{1,2,3,4}$ such that

- $T_1 < l_1$
- $l_1 < T_2 < \frac{l_1+l_2}{2}$
- $\frac{l_1+l_2}{2} < T_3 < l_2$
- $l_2 < T_4 < l_1 + l_2\,.$

In the rest of the paper we omit technical details and just present the results. All of them are simply verifiable.

On default, the absent matrix entries are assumed equal to zero.

The moment T_1

At this moment the domains on the graph filled by waves are disposed so that $\Omega^{T_1}[\gamma_1] \cap \Omega^{T_1}[\gamma_2] = \emptyset$ and v is not captured by waves: $v \notin \Omega^{T_1}[\gamma_{1,2}]$. The partition of the filled domain is $\Pi_\Sigma = \Phi^1 \cup \Phi^2$: see Fig. 4, 5

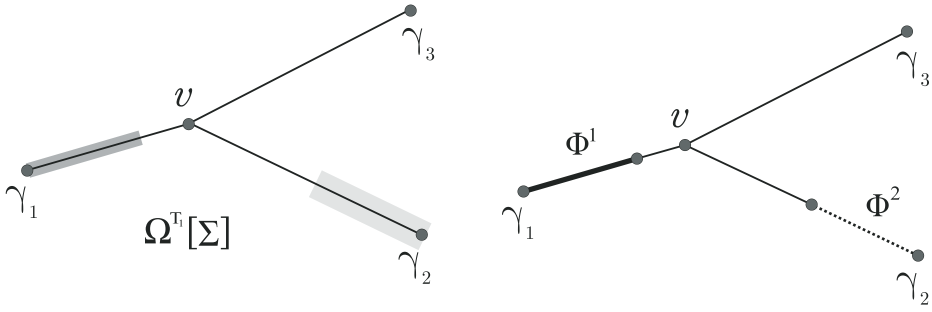

Figure 4: $T = T_1$; the graph

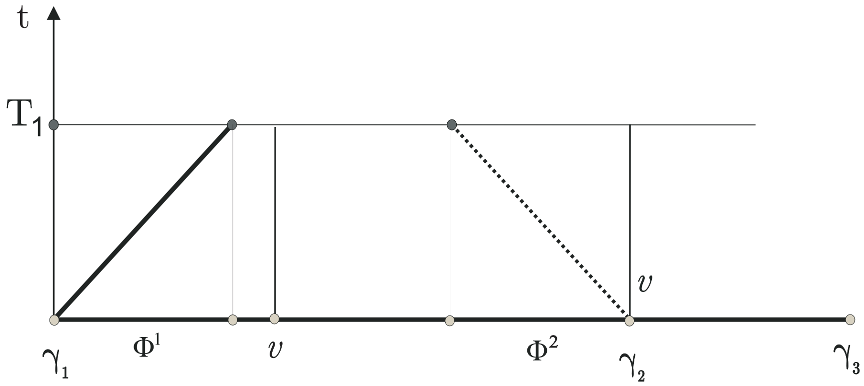

Figure 5: $T = T_1$; the hydras

The families are parametrized so that
$\tau^1_{\gamma_1\,\Phi^1}(r_1) = r_1 : \quad r_1 \in [0, T_1]$
$\tau^1_{\gamma_2\,\Phi^2}(r_2) = r_2 : \quad r_2 \in [0, T_1]$
holds. Note that here we have $\tau^i_{\gamma_l\,\Phi^j}(0) = 0$.

Respectively, the matrices (43) which represent the eikonals, take the form

$$E_{\gamma_1\,\Phi^1} = \begin{pmatrix} r_1 & \\ & 0 \end{pmatrix}, \quad E_{\gamma_2\,\Phi^2} = \begin{pmatrix} 0 & \\ & r_2 \end{pmatrix}.$$

As a result, for the eikonal algebra we easily get

$$\mathfrak{E}^{T_1}_{\{\gamma_1,\,\gamma_2\}} = \vee\{E^{T_1}_{\gamma_1},\, E^{T_1}_{\gamma_2}\} \cong C_0[0, T_1] \oplus C_0[0, T_1].$$

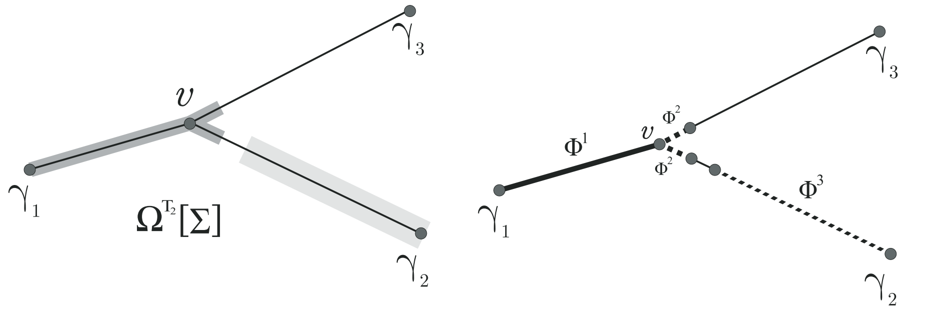

Figure 6: $T = T_2$; the graph

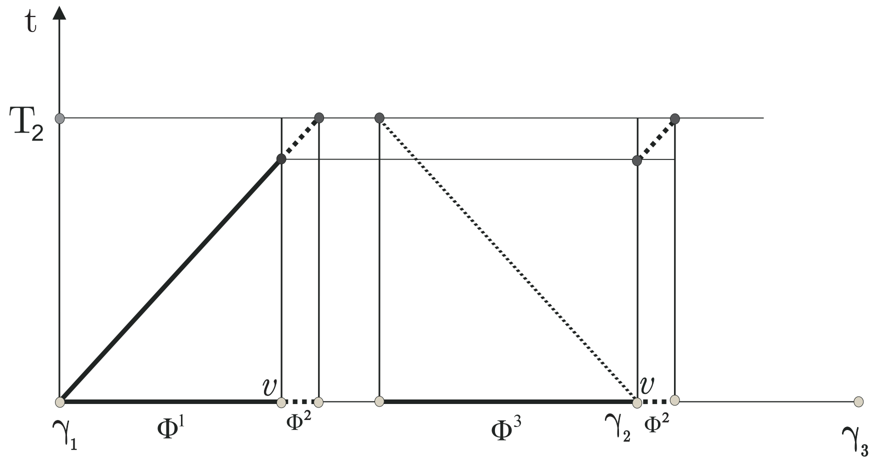

Figure 7: $T = T_2$; the hydras

The moment T_2

At this moment one has $\Omega^{T_2}[\gamma_1] \cap \Omega^{T_2}[\gamma_2] = \emptyset$, $v \in \Omega^{T_2}[\gamma_1]$ and $v \notin \Omega^{T_2}[\gamma_2]$, so that the waves from γ_2 do not reach the interior vertex yet. In this case one has $\Pi_\Sigma = \Phi^1 \cup \Phi^2 \cup \Phi^3$: see Fig. 6, 7.

- The proper parametrization provides

$\tau^1_{\gamma_1\,\Phi^1}(r_1) = r_1: \quad r_1 \in [0, l_1]$;
$\tau^1_{\gamma_1\,\Phi^2}(r_2) = l_1 + r_2: \quad r_2 \in [0, T_2 - l_1]$;
$\tau^1_{\gamma_2\,\Phi^3}(r_3) = r_3: \quad r_3 \in [0, T_2]$.

Notice the equalities

$$\tau^1_{\gamma_1\,\Phi^1}(0) = \tau^1_{\gamma_2\,\Phi^3}(0) = 0; \qquad \tau^1_{\gamma_1\,\Phi^1}(l_1) = \tau^1_{\gamma_1\,\Phi^2}(0)\,, \tag{46}$$

which will impose certain matching conditions on the matrix-functions.

• The matrices (43) are of the form

$$E_{\gamma_1\,\Phi^1} = r_1 P^1_{\gamma_1\,\Phi^1}, \quad E_{\gamma_1\,\Phi^2} = (l_1 + r_2) P^1_{\gamma_1\,\Phi^2}, \quad E_{\gamma_2\,\Phi^3} = r_3 P^1_{\gamma_2\,\Phi^3}\,,$$

where

$$P^1_{\gamma_1\,\Phi^1} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \; P^1_{\gamma_1\,\Phi^2} = \begin{pmatrix} 0 & & & \\ & \frac{1}{2} & \frac{1}{2} & \\ & \frac{1}{2} & \frac{1}{2} & \\ & & & 0 \end{pmatrix}, \; P^1_{\gamma_2\,\Phi^3} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}.$$

Then one gets the representatives of the eikonals:

$$\begin{aligned} U E^{T_2}_{\gamma_1} U^{-1} &= E_{\gamma_1\,\Phi^1} + E_{\gamma_1\,\Phi^2} = \\ &= r_1 P^1_{\gamma_1\,\Phi^1} + (l_1 + r_2) P^1_{\gamma_1\,\Phi^2} = \begin{pmatrix} r_1 & & & \\ & \frac{l_1+r_2}{2} & \frac{l_1+r_2}{2} & \\ & \frac{l_1+r_2}{2} & \frac{l_1+r_2}{2} & \\ & & & 0 \end{pmatrix}, \end{aligned} \tag{47}$$

$$U E^{T_2}_{\gamma_2} U^{-1} = E_{\gamma_2\,\Phi^3} = r_3 P^1_{\gamma_2\,\Phi^3} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & r_3 \end{pmatrix}. \tag{48}$$

• By orthogonality of the projections $P^1_{\gamma\,\Phi}$, for any polynomial $q = q(\lambda)$ provided $q(0) = 0$ we have

$$U q(E^{T_2}_{\gamma_1}) U^{-1} = q(r_1) P^1_{\gamma_1\,\Phi^1} + q(l_1 + r_2) P^1_{\gamma_1\,\Phi^2}$$

that easily leads to

$$\begin{aligned} & U \left[\vee\{E^{T_2}_{\gamma_1}\}\right] U^{-1} \cong \\ & \cong \left\{ \begin{pmatrix} \phi(r_1) & \\ & \psi(r_2) \end{pmatrix} \middle|\ \phi \in C_0[0, l_1],\, \psi \in C[0, T_2 - l_1],\ \phi(l_1) \overset{(46)}{=} \psi(0) \right\} \cong \\ & \cong C_0[0, T_2]\,. \end{aligned}$$

In the mean time one has

$$U \left[\vee\{E^{T_2}_{\gamma_2}\}\right] U^{-1} \cong C_0[0, T_2]\,.$$

Summarizing, we represent the eikonal algebra in the form

$$\mathfrak{E}^{T_2}_{\{\gamma_1,\,\gamma_2\}} \cong C_0[0, T_2] \oplus C_0[0, T_2]\,.$$

The moment T_3

At this moment one has $\Omega^{T_3}[\gamma_1] \cap \Omega^{T_3}[\gamma_2] \neq \emptyset$, $v \in \Omega^{T_3}[\gamma_1]$, $v \notin \Omega^{T_3}[\gamma_2]$. The waves from γ_2 do not reach the interior vertex yet but overlap with the waves going from γ_1. So, at $t = T_3$ the waves from different boundary vertices begin to interact. As will be seen, interaction leads to the curious effect: while $\mathfrak{E}_\Sigma^{T_1}$ and $\mathfrak{E}_\Sigma^{T_2}$ are commutative, for $T \geqslant T_3$ the algebra $\mathfrak{E}_\Sigma^T$ becomes *noncommutative*.

• Now the partition of the domain filled by waves is $\Pi_\Sigma = \Phi^1 \cup \Phi^2 \cup \Phi^3 \cup \Phi^4$: see Fig. 8, 9.

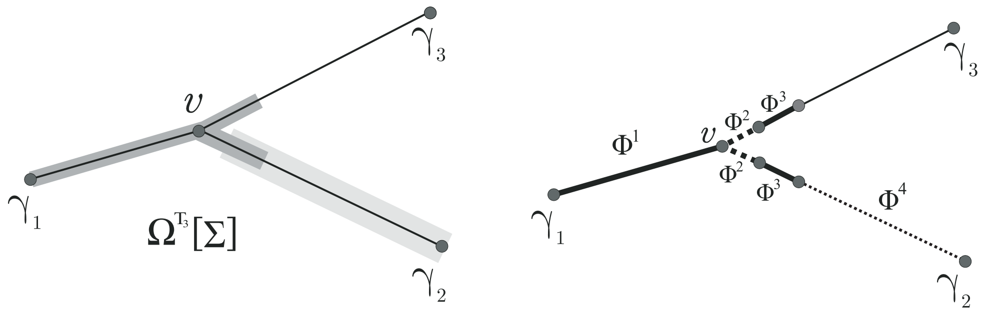

Figure 8: $T = T_3$; the graph

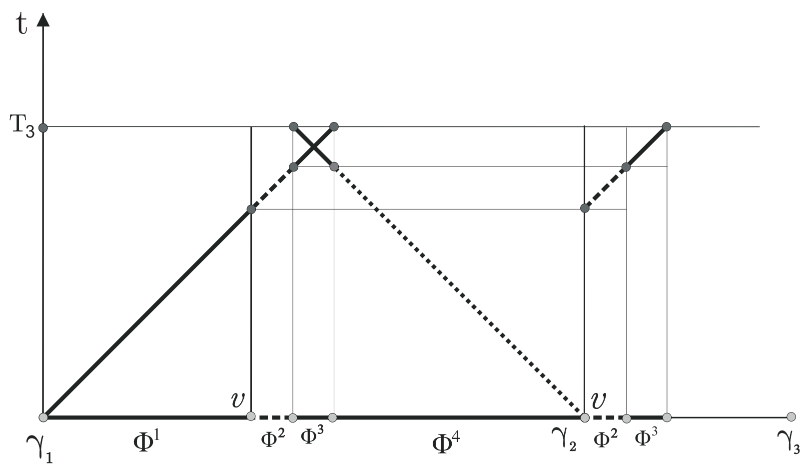

Figure 9: $T = T_3$; the hydras

The proper parametrization provides

$\tau^1_{\gamma_1\,\Phi^1}(r_1) = r_1\colon\quad r_1 \in [0, l_1];$
$\tau^1_{\gamma_1\,\Phi^2}(r_2) = l_1 + r_2\colon\quad r_2 \in [0, l_2 - T_3];$
$\tau^1_{\gamma_1\,\Phi^3}(r_3) = (l_1{+}l_2{-}T_3) + r_3,\ \ \tau^1_{\gamma_2\,\Phi^3}(r_3) = T_3 - r_3\colon\quad r_3 \in [0, 2T_3{-}l_1{-}l_2];$
$\tau^1_{\gamma_2\,\Phi^4}(r_4) = r_4\colon\quad r_4 \in [0, l_1{+}l_2{-}T_3]$

- As one can derive, the generators of the algebra $UE^{T_3}_{\Sigma}U^{-1}$ take the form

$$UE^{T_3}_{\gamma_1}U^{-1} = \begin{pmatrix} r_1 & & & & & \\ & \frac{l_1+r_2}{2} & \frac{l_1+r_2}{2} & & & \\ & \frac{l_1+r_2}{2} & \frac{l_1+r_2}{2} & & & \\ & & & \frac{(l_1+l_2-T_3)+r_3}{2} & \frac{(l_1+l_2-T_3)+r_3}{2} & \\ & & & \frac{(l_1+l_2-T_3)+r_3}{2} & \frac{(l_1+l_2-T_3)+r_3}{2} & \\ & & & & & 0 \end{pmatrix}, \tag{49}$$

$$UE^{T_3}_{\gamma_2}U^{-1} = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & 0 & \\ & & & 0 & T_3{-}r_3 & \\ & & & & & r_4 \end{pmatrix}. \tag{50}$$

Looking at their structure, one can select the blocks $\{(\cdot)_{i\,j}\}^3_{i,j=1}$ and $\{(\cdot)_{6\,6}\}$, which act in perfect analogy to (47) and (48). These blocks generate the 'commutative part' of $UE^{T_3}_{\Sigma}U^{-1}$, the part being isometrically isomorphic to $C_0[0, l_1{+}l_2{-}T_3] \oplus C_0[0, l_1{+}l_2{-}T_3]$.

The origin of noncommutativity is the presence of the bloks $\{(\cdot)_{i\,j}\}^5_{i,j=4}$ proportional to the projections

$$p = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad p' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

which do not commute. These projections nave no mutual eigenvectors. Therefore one has $\vee\{p,\, p'\} = \mathbb{M}^2$ by the property 3. of the matrix algebras (see the last item of sec. 3).

- As a result, one can arrive at the final representation

$$\mathfrak{E}^{T_3}_{\{\gamma_1,\,\gamma_2\}} \cong C_0[0, l_1{+}l_2{-}T_3] \oplus C_0[0, l_1{+}l_2{-}T_3] \oplus ([0, 2T_3{-}l_1{-}l_2], \mathbb{M}^2).$$

Notice that some work has to be done to check that the summands are independent, i.e., there are no matching conditions at the endpoints of the segments $[0, \epsilon_j]$ which might connect the values of matrix-functions belonging to different summands.

The moment T_4

• Here the partition of the domain $\Omega^{T_4}[\Sigma]$ is $\Pi_\Sigma = \Phi^1 \cup \Phi^2 \cup \Phi^3 \cup \Phi^4$: see Fig. 10, 11.

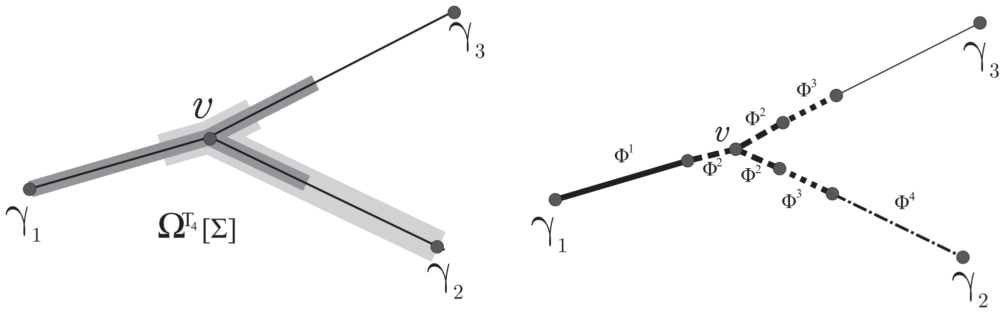

Figure 10: $T = T_4$; the graph

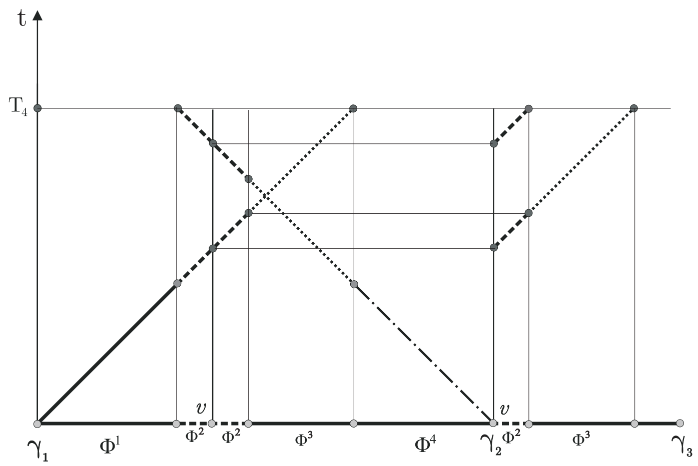

Figure 11: $T = T_4$; the hydras

The proper parametrization provides

$\tau^1_{\gamma_1\,\Phi^1}(r_1) = r_1: \quad r_1 \in [0, l_1 + l_2 - T_4]$;

$\tau^1_{\gamma_1\,\Phi^2}(r_2) = l_1 - r_2, \ \tau^2_{\gamma_1\,\Phi^2}(r_2) = l_1 + r_2: \ \ r_2 \in [0, T_4 - l_2]$;

$\tau^1_{\gamma_2\,\Phi^2}(r_2) = l_2 - r_2, \ \tau^2_{\gamma_2\,\Phi^2}(r_2) = l_2 + r_2: \quad r_2 \in [0, T_4 - l_2]$;

$\tau^1_{\gamma_1\,\Phi^3}(r_3) = (l_1 - l_2 + T_4) + r_3, \ \ \tau^1_{\gamma_2\,\Phi^3}(r_3) = (2l_2 - T_4) - r_3: \ \ r_3 \in [0, l_2 - l_1]$;

$\tau^1_{\gamma_2\,\Phi^4}(r_4) = r_4: \ \ r_4 \in [0, l_1 + l_2 - T_4]$.

• As one can show, the generators of the eikonal algebra are of the form

$$UE_{\gamma_1}^{T_4}U^{-1} = \begin{pmatrix} r_1 & & & & & & \\ & l_1-r_2 & 0 & 0 & & & \\ & 0 & \frac{l_1+r_2}{2} & \frac{l_1+r_2}{2} & & & \\ & 0 & \frac{l_1+r_2}{2} & \frac{l_1+r_2}{2} & & & \\ & & & & \frac{(l_1-l_2+T_4)+r_3}{2} & \frac{(l_1-l_2+T_4)+r_3}{2} & \\ & & & & \frac{(l_1-l_2+T_4)+r_3}{2} & \frac{(l_1-l_2+T_4)+r_3}{2} & \\ & & & & & & 0 \end{pmatrix},$$

$$UE_{\gamma_2}^{T_4}U^{-1} = \begin{pmatrix} 0 & & & & & & \\ & \frac{l_2+r_2}{2} & \frac{l_2+r_2}{2} & 0 & & & \\ & \frac{l_2+r_2}{2} & \frac{l_2+r_2}{2} & 0 & & & \\ & 0 & 0 & l_2-r & & & \\ & & & & 0 & 0 & \\ & & & & 0 & (2l_2-T_4)-r_3 & \\ & & & & & & r_4 \end{pmatrix}.$$

The blocks $\{(\cdot)_{11}\}$ and $\{(\cdot)_{77}\}$, which correspond to the families Φ^1 and Φ^4, generate the commutative part of the algebra $UE_\Sigma^{T_4}U^{-1}$ of the form $C_0[0,l_1+l_2-T_4]\oplus C_0[0,l_1+l_2-T_4]$.

The blocks $\{(\cdot)_{ij}\}_{i,j=5}^6$ of the generators, which correspond to the family Φ^3, produce the algebra $([0,l_2-l_1];\mathbb{M}^2)$ in perfect analogy to the moment T_3.

A new type algebra appears owing to the blocks $\{(\cdot)_{ij}\}_{i,j=2}^4$, which correspond to the family Φ^2 and are of the form

$$\begin{pmatrix} l_1-r_2 & 0 & 0 \\ 0 & \frac{l_1+r_2}{2} & \frac{l_1+r_2}{2} \\ 0 & \frac{l_1+r_2}{2} & \frac{l_1+r_2}{2} \end{pmatrix} = (l_1-r_2)P^1_{\gamma_1\,\Phi^2} + (l_1+r_2)P^2_{\gamma_1\,\Phi^2} \tag{51}$$

and

$$\begin{pmatrix} \frac{l_2+r_2}{2} & \frac{l_2+r_2}{2} & 0 \\ \frac{l_2+r_2}{2} & \frac{l_2+r_2}{2} & 0 \\ 0 & 0 & l_2-r_2 \end{pmatrix} = (l_2+r_2)P^2_{\gamma_2\,\Phi^2} + (l_2-r_2)P^1_{\gamma_2\,\Phi^2}\,. \tag{52}$$

Further analysis makes use of the properties 1.−4. of the matrix algebras. First, one can check that $\vee\{P^i_{\gamma_l\,\Phi^2}\mid i,l=1,2\}=\mathbb{M}^3$. Then, for $r_2=0$ the generators (51) and (52) turn out to be proportional to the projections

$$P_1 := P^1_{\gamma_1\,\Phi^2} + P^2_{\gamma_1\,\Phi^2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \ P_2 := P^2_{\gamma_2\,\Phi^2} + P^1_{\gamma_2\,\Phi^2} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which do have the mutual eigenvector $\{1,1,1\}^t \in \mathbb{R}^3$. Therefore, one easily gets $\vee\{P_1,\, P_2\} \cong \mathbb{M}^1 \oplus \mathbb{M}^2$ that is a proper subalgebra in $\mathbb{M}^3$. As a result, the contribution of the blocks (51) and (52) to the eikonal algebra turns out to be the standard algebra $\dot{C}([0, T_4 - l_2]; \mathbb{M}^3)$.

Also one can verify that there are no more connections at the endpoints $r = 0, \epsilon_\Phi$ between the blocks, which enter in the generators $UE^{T_4}_{\gamma_1}U^{-1}$ and $UE^{T_4}_{\gamma_2}U^{-1}$.

• Summarizing, we arrive at the final representation

$$\mathfrak{E}^{T_4}_{\Sigma} \cong C_0[0, l_1{+}l_2{-}T_4] \oplus C_0[0, l_1{+}l_2{-}T_4] \oplus C([0, l_2{-}l_1]; \mathbb{M}^2) \oplus \dot{C}([0, T_4{-}l_2]; \mathbb{M}^3).$$

References

[1] W.Arveson. An Invitation to C^*-Algebras. *Springer-Verlag Inc.*, 1976.

[2] M.I.Belishev. Boundary spectral inverse problem on a class of graphs (trees) by the BC-method. *Inverse Problems*, 20 (2004), No 3: 647–672.

[3] M.I.Belishev. On the boundary controllability of dynamical system governed by the wave equation on a class of graphs (trees). *Zapiski Nauch. Semin. POMI*, 308: 23–47, 2004 (in Russian); English translation: *J. Math. Sciences*, v. 132, no 1, 2006.

[4] M.I.Belishev. Recent progress in the boundary control method. *Inverse Problems*, 23 (2007), No 5, R1–R67.

[5] M.I.Belishev. Algebras in reconstruction of manifolds. Spectral Theory and Partial Differential Equations, G.Eskin, L.Friedlander, J.Garnett Eds. *Contemporary Mathematics, AMS*, 640 (2015), 1–12. DOI: http://dx.doi.org/10.1090/conm/640 . ISSN: 0271-4132.

[6] M.I.Belishev and M.N.Demchenko. Elements of noncommutative geometry in inverse problems on manifolds. *Journal of Geometry and Physics*, Volume 78, April 2014, 29–47.

[7] M.I.Belishev, M.N.Demchenko, and A.N.Popov. Noncommutative geometry and tomography of manifolds. *Proceedings of the Moscow Math. Society*, vol 75, no 2, 2014, 159–180. (in Russian). English translation: *Transactions of the Moscow Mathematical Society* 11/2014; 75:133-149. DOI:10.1090/S0077-1554-2014-00239-9.

[8] M.I.Belishev, A.F.Vakulenko. Inverse problems on graphs: recovering the tree of strings by the BC-method. *Journal of Inverse and Ill-Posed Problems*, 14 , no 1: 29–46, 2006.

[9] M.I.Belishev, N.Wada. On revealing graph cycles via boundary measurements. *Inverse Problems.*, 25 (2009), no 10, 105011, 1–25.

[10] M.I.Belishev and N.Wada. A C*-algebra associated with dynamics on a graph of strings. *J. Math. Soc. Japan*, Vol. 67, No. 3 (2015) pp. 1239-1274. doi: 10.2969/jmsj/06731239.

[11] J.Dixmier. Les C^*-algebrae et leurs represebtations. *Gauthier-Villars Ed, Paris*, 1969.

[12] G.J.Murphy. C^*-Algebras and Operator Theory. *Academic Press, San Diego*, 1990.