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Some approaches to studying the stability of solutions of stochastic multi-objective optimization

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Abstract. This article focuses on the issue of another way to study of the stability of solutions of stochastic multi–objective optimization problem. It presents some approaches to the study of the stability of solutions of stochastic multi–objective optimization with linear and nonlineral convolution.

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1. Introduction

The problem of stohastic optimization stability is quite important and has been studied by many authors.

In [5], the authors present stability and sensitivity analysis of a stochastic optimization problem with stochastic second order dominance constraints. Authors of [1] consider convex optimization problems with k order stochastic dominance constraints for $k \geq 2$, discuss distances of random variables that are relevant for the dominance relation and establish quantitative stability results for optimal values and solution sets of the optimization problems in terms of a suitably selected probability metric.

In [3], the authors present different types of stability of stochastic optimization problem such as ϵ- stability of solution in the mean, stability with respect to the i^{th} constraint, absolute solution stability and other. In [7], the authors consider a parametric stochastic quasi-variational inequality problem.

The research work presented in [6] considers distributionally robust formulations of a two stage stochastic programming problem with the objective of minimizing a distortion risk of the minimal cost incurred at the second stage. In [2], the authors consider distributionally robust formulations of a two stage stochastic programming problem with the objective of minimizing a distortion risk of the minimal cost incurred at the second stage.

This paper is a generalization of the results obtained in [3] for the scalar stochastic optimization problems to the multi-objective case and represents a continuation of the study initiated by the authors in [4], where they considered the ϵ – stability in the mean multiobjective optimization problem and the concepts of the region of admissibility and scope of optimality are also taken into consideration.

2. Brief information about stochastic multi-objective optimization

Let us introduce the concept of multi–objective optimization in the following form:

Definition 1.

$$\min_{x \in X} f(x) \tag{1}$$

where $X \subset R^n$ is some given set of alternatives, $f(x)$ – vector-valued objective function,

$$f : X \to R^m, f(x) = (f_1(x), f_2(x), \ldots, f_m(x)).$$

If a multi-objective optimization problem involve random parameter, it makes sense to talk about stochastic multi-objective optimization. A conceptual generalization of the problem of stochastic multi-objective optimization:

Definition 2.

$$\begin{aligned} &\min_{x \in X(\omega)} f(x, \omega), \\ &X(\omega) \subset S(\omega) \subset R^n, \omega \in \Omega. \end{aligned} \tag{2}$$

where $X(\omega)$ is some given set of alternatives, which depends on the random parameter ω, Ω – set of random parameters ω, $f(x, \omega)$ – vector-valued objective function,

$$\begin{aligned} &f : X, \Omega \to R^m, \\ &f(x, \omega) = (f_1(x, \omega), f_2(x, \omega), \ldots, f_m(x, \omega)). \end{aligned}$$

In practical problems, usually each component of vector-valued objective function has its own dimension. To reduce the components to a dimensionless form, the following methods of normalization used:

1. Change of direction of the goal $f_n(x, \omega) = -f(x, \omega)$
2. Natural normalization $f_n(x, \omega) = \frac{f(x,\omega)}{\max_{x \in X} f(x,\omega) - min_{x \in X} f(x,\omega)}$
3. Complete normalization $f_n(x, \omega) = \frac{f(x,\omega) - \min_{x \in X} f(x,\omega)}{\max_{x \in X} f(x,\omega) - min_{x \in X} f(x,\omega)}$
4. Change of ingredient $f_n = \frac{1}{f(x,\omega)}$
5. Normalization of comparison $f_n(x, \omega) = \frac{f(x,\omega)}{\max_{x \in X} f(x,\omega)}$
6. Normalization of Savage $f_n(x, \omega) = \max_{x \in X} f(x, \omega) - f(x, \omega)$

7. Normalization of averaging $f_n(x,\omega) = \frac{f(x,\omega)}{\sum_i^m f_i(x,\omega)}$

We call the $F : f(x,\omega) \to R^1$ component of the stochastic multi–objective exponent $f(x,\omega)$ a *"convolution"* that converts the set of components of the stochastic multi–objective indicator f corresponding to the target terms i to the scalar target the exponent. When solving practical problems of stochastic multi–objective optimization the following convolutions of the criteria are usually used:

1. Linear(additive) convolution - $F(f(x,\omega)) = \sum_{i=1}^{m} p_i f_i(x,\omega)$, where p_i are real numbers.
2. Multiplicative convolution $F(f(x,\omega)) = \prod_{i=1}^{m} p_i f_i(x,\omega)$, where p_i are real numbers.
3. The ideal point method $F(f(x,\omega)) = \rho(Z, f(x,\omega))$ where Z — ideal point, $\rho(x,y)$ — distance between points x, y
4. Convolution of Cobb — Douglas $F(f(x,\omega)) = \prod_{i=1}^{m} [p_i f(x,\omega)]^{q_i}$ where p_i and q_i are real numbers.

Let us consider some approaches to the study of stability of solutions of stochastic multi–objective optimization problems with linear and nonlinear convolutions in the next sections.

3. Some approaches to the study of the stability of solutions of stochastic multi-objective optimization with linear convolution

Let us introduce the following concepts:

1. **Feasibility region.** Consider a fixed realization of random event $\omega_0 \in \Omega$. Suppose we have a deterministic multi-objective optimization

with linear convolution problem of the form:

$$\min_{X \in S(\omega_0)} F(f_n(x, \omega_0)), \tag{3}$$

where the criteria $f_{n_i}(x, \omega_0)$ of the multi-objective function $f_n(x, \omega_0)$ are linear.

Let $l_k (k = 1, 2, \ldots, K_{\omega_0})$ denote the extreme points of the convex set $S(\omega_0)$, each of which is produced by intersection of n cutting planes, where K_{ω_0} is the total number of extreme points when $\omega_0 \in \Omega$ random parameters are realized.

The region $V^k_{\omega_0} \in \Omega$ is called the *"feasibility"* region of the point l_k, if for any $\omega \in V^k_{\omega_0}$ the intersection of the cutting planes producing this point determines an extreme point of the corresponding set $S(\omega)$.

2. **Optimality region.** Let l_{k_0} be an optimal extreme point of problem (3), i.e., for any $k \neq k_0, Z^k_{\omega_0} \geq Z^{k_0}_{\omega_0}$, where $Z^k_{\omega_0}$ is the linear convolution value of problem (3) at the extreme point l_k.

   The region $W^{k_0}_{\omega_0} \in V^{k^0}_{\omega_0}$ is the optimality region of the point l_{k_0} if for any $\omega \in W^{k_o}_{\omega_0}, Z^k_\omega \geq Z^{k_0}_\omega$.

We will prove that for any extreme point l_k the function Z^k_ω is the continuous function of ω.

Lemma 1. *The function Z^k_ω is the continuous function of ω for any extreme point l_k.*

Proof. The functional value at the extreme point l_k of the convex set $S(\omega_0)$ for the fixed realization $\omega_0 \in \Omega$ in view of the linearity $F(\cdot)$ and $f_{n_i}(x, \omega_0)$ is

$$Z^k_{\omega_0} = F(f_n(l_k)) = c^0_1 x^{0k}_1 + c^0_2 x^{0k}_2 + \ldots + c^0_2 x^{0k}_2,$$

where x_j^{0k} are coordinates of extrime point.

Denote by $\hat{x}_j^{0k}$ nonzero values of x_j^{0k}. These values are determined from the relation $D^{0k}\hat{X}^{0k} = b^0$, where D^{0k} is the relevant basic matrix.

By the nonsingularity of this matrix, the values $\hat{x}_j^{0k}$ are continuos functions of a_{ij}^0 and b_i^0, respectively, Z_ω^k is a continuous function of $\omega \in \Omega$.□

We will prove that for any $\omega_0 \in \Omega$ there exist a stability region of W_{ω_o} such that for all $\omega \in W_{\omega_o}$ the corresponding problem (3) has same optimal basis.

Theorem 1. *Let* $Z_{\omega_0}^k > Z_{\omega_0}^{k_0}(k = \overline{1, k_{\omega_0}}; k \neq k_0)$ *at the point* $\omega_0 \in \Omega$. *Then there exists a neighborhood* $O(\omega_0)$ *of the point* ω_0 *such that for all* $\omega \in O(\omega_0) \subset \Omega$

$$Z_\omega^k > Z_\omega^{k_0}(k = \overline{1, k_{\omega_0}}; k \neq k_0)$$

Proof. At the point $\omega_0 \in \Omega$

$$Z_{\omega_0}^k > Z_{\omega_0}^{k_0} \tag{4}$$

By Lemma 1, the functions Z_k^ω and $Z_{k_0}^\omega$ are continuous in ω, hence their difference also is a continuous function. From the properties of continuous functions it follows that there exists a neighborhood $O(\omega_0)$ such that for $\omega \in O(\omega_0)$ the inequality (4) is preserved, i.e., $Z_\omega^k > Z_\omega^{k_0}$, $k = \overline{1, k_{\omega_0}}$. □

4. Some approaches to the study of the stability of solutions of stochastic multi-objective optimization with nonlinear convolution

In applied problems of stochastic programming, random parameters have optimistic and pessimistic boundaries for their variations, i.e., random variables are distributed in a finite (continuous or discrete) way.

In such cases it is possible under certain conditions to determine the range of the objective function and localize the set of optimal solutions for all realizations $\omega \in \Omega$.

Suppose we have a stochastic programmipg problem, in which the objective function $F(f_n(x,\omega)) = F(f_n(x))$ is a continuous deterministic function, and the functions defining the problem conditions $g(x,\omega), \forall i = \overline{l,m}$ are quadratic or linear in X with nonpositive definite matrices for any realization $\omega \in \Omega$, i.e., we have the problem

$$\begin{cases} \min_x F(f_n(x)), \\ x^T H_i(\omega)x + p_i(\omega)x - b_i(\omega) \geq 0, i = \overline{1,m}, \\ x \geq 0, \end{cases} \tag{5}$$

where all the elements of matrices $H_i(\omega)$, vectors $p_i(\omega)$ and components $b_i(\omega)$, $i = \overline{1,m}$ can be random.

Let us introduce the following sets:

$$\begin{aligned} S(\omega) &= \left\{x : x^T H_i(\omega)x + p_i(\omega)x \geq b_i, x \geq 0, i = \overline{1,n}\right\}, \\ S^-(\omega) &= \left\{x : x^T H_i^-(\omega)x + p_i^- x \geq b_i^-, x \geq 0, i = \overline{1,n}\right\}, \\ S^+(\omega) &= \left\{x : x^T H_i^+(\omega)x + p_i^+ x \geq b_i^+, x \geq 0, i = \overline{1,n}\right\}. \end{aligned} \tag{6}$$

where in $H_i^+, p_i^+, b_i^+, H_i^-, p_i^-, b_i^-$, $i = \overline{l,m}$ random variables representing realizations of vector ω are replaced by optimistic and pessimistic boundaries, respectively.

We have the following statement.

Theorem 2. *If the set of permanent solutions S^- is nonempty, then the following relation holds for the objective function of problem (5):*

$$\min_{x \in S^+} F(f_n(x)) \leq \min_{x \in S(\omega)} F(f_n(x)) \leq \min_{x \in S^-} F(f_n(x)) \tag{7}$$

Proof. If is sufficient to show that $S^+ \supset S(\omega) \supset S^-$, whence it follows that the minimum of continous function in some set is less than or equal to the minimum of this same function in any part of this set.

If $x \in S^-$, it follows that for all random realizations ω and with $\forall i$

$$\begin{aligned} x^T H_i^+ x + p_i^+ x &\overset{(***)}{\geq} x^T H_i(\omega) x + p_i(\omega) x \overset{(**)}{\geq} x^T H_i^- x + p_i^- x \overset{(*)}{\geq} \\ \overset{(*)}{\geq} b_i^+ &\overset{(**)}{\geq} b_i(\omega) \overset{(***)}{\geq} b_i^-, x \geq 0 \end{aligned}$$

From this it follows that any feasible solution from S^- satisfying inequality $(*)$ also satisfies inequality $(**)$, i.e., $S(\omega) \supset S^-$. Additionaly, any feasible solution from the set $S(\omega)$ satisfying $(**)$ also satisfies $(***)$, i.e., $S^+ \supset S(\omega)$. Finally,

$$S^+ \supset S(\omega) \supset S^-$$

and if $S^- \neq \emptyset$, then the expression (5) from conditions of the theorem is meaningful. □

Corollary. *Suppose we have a quadratic convolution in problem (5), i.e.* $F(f_n(x)) = F(f_n(x), \omega)$ *is a random x-quadratic function with a nonnegative definite matrix* $H_0(\omega)$ *for* $\forall\ \omega$

$$F(f_n(x), \omega) = f_n(x)^T H_0(\omega) f_n(x) + p_0(\omega) f_n(x) + b_0(\omega)$$

where among the elements of H_0, p_0, b_0 *are random components with finite distribution. Then expression (7) in Theorem 2 becomes*

$$\begin{aligned} F^-(f_n(x^-)) &= \min_{x \in S^+} F^-(f_n(x)) \\ &\leq \min_{x \in S(\omega)} F(f_n(x), \omega) \\ &\leq \min_{x \in S^-} F^+(f_n(x)) \\ &= F^+(f_n(x^+)) \end{aligned}$$

where x^- *and* x^+ *are optimal solutions for respective set* S^+ *and* S^-*, while* $F^-(f_n(x))$ *and* $F^+(f_n(x))$ *are objective functions, in which random data are replaced by their pessimistic and optimistic boundaries, respectively.*

Proof. In accordance with the statement of Theorem 2 the system of manual inclusions $S^+ \supset S(\omega) \supset S^-$ is valid. If $x^0(\omega)$ is the optimal solution of problem (5) for realization of ω, then for any ω

$$\begin{aligned} F^-(f_n(x^-)) &= f_n(x^-)^T H_0^- f_n(x^-) + p_0^- f_n(x^-) + b_0^-(\omega) \leq \\ &\leq F(f_n(x),\omega) \\ &= f_n(x)^T H_0(\omega) f_n(x) + p_0(\omega) f_n(x) \\ &+ b_0(\omega) \leq f_n(x^+)^T H_0^+ f_n(x^+) \\ &+ p_0^+ f_n(x^+) + b_0^+(\omega) \\ &= F^+(f_n(x^+)) \end{aligned}$$

whence comes the required result. □

We introduce the following set

$$S^-_> = \left\{x : x^T H_i^-(\omega)x + p_i^- x > b_i^-, x > 0, i = \overline{1,n}\right\}.$$

Theorem 3. *Let the functions $g_i(x,\omega), i = \overline{l,m}$ in problem (5) be concave or linear, the set of feasible S^- nonempty, and the set S^+ bounded. If the objective function of problem (5) is concave and deterministic, then for any ω there is an optimal solution $x^0(\omega)$, which does not belong to $S^-_>$, and the set of all such optimal solutions satisfies the conditions*

$$\{x^0(\omega)\}_{\forall\omega} \subset \Delta S = S^+ \backslash S^-_>.$$

Proof. Since the minimum of the concave function $F(f_n(x))$ on the boundary is not less than the minimum of $F(f_n(x))$ inside the convex region of feasible solutions, by the continuity of $F(f_n(x))$ and by the closedness and boundedness of the region of feasible solutions for each realization of ω there exist a boundary point $X^0(\omega)$ such that

$$F(f_n(x^0(\omega))) \leq F(f_n(x(\omega))), \forall x(\omega) \in S(\omega).$$

This means that for each realization of ω at least one of the inequalities

$$g_i(x,\omega) \geq 0$$

or

$$x_j \geq 0$$

becomes an equality, i.e. either

$$x^{0T}(\omega)H_i(\omega)x^0(\omega) + p_i(\omega)x^0(\omega) = b_i(\omega)$$

for some i_ω or $x_j^0(\omega) = 0$ for some j_ω. Such $x^0(\omega)$ does not belong to $S_>^-$, because the set $S_<^-$ includes none of the x which could transform some constraint of problem (5) into equality.

But $x^0(\omega) \in S^+$, because S^+ includes all feasibility region for any realization of ω, and hence

$$\{x^0(\omega)\}_{\forall\omega} \subset S^+ \backslash S_>^-$$

$\square$

Suppose the functions of original problem are all quadratic in x with nonpositive definite matrices $H_i(\omega), i = \overline{1,m}$ and nonnegative definite matrix $H_0(\omega)$ for $\forall\omega$, i.e., we have the quadratic programming problem for each realization:

$$\begin{cases} \min_x F(f_n(x),\omega) = \min_x \Big(f_n^T(x)H_0(\omega)f_n(x) + p_0(\omega)f_n(x) + b_0(\omega)\Big), \\ x^T H_i(\omega)x + p_i(\omega)x - b_i(\omega) \geq 0, i = \overline{1,m}, \\ x \geq 0, \end{cases} \tag{8}$$

where the elements $H_i(\omega), p_i(\omega)$ and $b_i(\omega), i = 0,1,\ldots,m$ can be random variables.

Let $\overline{h_l^i}$, $\overline{p_j^i}$ and $\overline{b_i}$ be the mathematical expectations, and let σ_l^i, σ_j^i and σ_i be variances of the elements $H_i(\omega)$, $p_i(\omega)$ and $b_i(\omega)$, $i = 0,1,\ldots,m$, respectively.

Introduce the following set and notations:

$$\begin{aligned}
S(\omega) &= \{x : x^T H_i(\omega)x + p_i(\omega)x \geq b_i(\omega), i = \overline{1,m}, x \geq 0\},\\
S_\lambda^+ &= \{x : x^T H_i^+ x + p_i^+ x \geq b_i^-, i = \overline{1,m}, x \geq 0\},\\
S_\nu^- &= \{x : x^T H_i^- x + p_i^- x \geq b_i^+, i = \overline{1,m}, x \geq 0\},\\
\Delta_{\nu,\lambda} &= \{x : x \in \{S_\lambda^+ \backslash \{x : x^T H_i^- x + p_i^- x > b_i^+, i = \overline{1,m}, x \geq 0\}\}\},\\
F_{\lambda,\nu}^+ &= \{\min_{x \in S_\nu^-} F_\lambda(f_n(x))\} = \min_{x \in S_\nu^-} (f_n^T(x) H_0^+ f_n(x) + p_0^+ f_n(x) + b_0^+),\\
F_{\nu,\lambda}^- &= \{\min_{x \in S_\lambda^+} F_\nu(f_n(x))\} = \min_{x \in S_\lambda^+} (f_n^T(x) H_0^- f_n(x) + p_0^- f_n(x) + b_0^-),
\end{aligned}$$

where $H_i^+, H_i^-, p_i^+, p_i^-$, and b_i^+, b_i^- are determined form the following relations for any $i = \overline{1,m}$ and $l, j = \overline{1,n}$:

$$\begin{aligned}
H_i^- &= (\overline{h_{lj}^i} - \nu\sigma_{lj}^i), H_i^+ = (\overline{h_{lj}^i} + \nu\sigma_{lj}^i),\\
p_i^- &= (\overline{p_j^i} - \nu\sigma_j^i), p_i^+ = (\overline{p_j^i} + \nu\sigma_j^i),\\
b_i^- &= (\overline{b^i} - \nu\sigma^i), b_i^+ = (\overline{b^i} + \nu\sigma^i).
\end{aligned}$$

Suppose ν and λ are strictly greater than zero, $S_\nu^- \neq \emptyset$, while the absolute minimum in problem (8) does not belong to S_ν^-, and considered for $S(\omega)$ are all ω for which there exists at least one feasible solution.

Theorem 4. *The probability that the optimal value of the objective function* $F(f_n(x^0), \omega) \in [F_{\nu,\lambda}^-, F_{\nu,\lambda}^+]$ *and the solution of problem (8),* x^0*, belongs to the region* $\Delta_{\nu,\lambda}$ *is equal at least to*

$$\begin{aligned}
P_{\nu,\lambda} &= P\{\omega : H_i^- \leq H_i(\omega) \leq H_i^+; p_i^- \leq p_i(\omega) \leq p_i^+;\\
&\quad b_i^- \leq b_i(\omega) \leq b_i^+, \forall i = 0, 1, \ldots, m\}
\end{aligned}$$

Proof. It is sufficient to show that the requirement of the theorem holds for the event

$$\{\omega : H_i^- \leq H_i(\omega) \leq H_i^+; p_i^- \leq p_i(\omega) \leq p_i^+; b_i^- \leq b_i(\omega) \leq b_i^+\}(*).$$

Proceeding in the same manner as in Theorem 3, we have

$$S_\nu^- \subset S(\omega) \subset S_\lambda^+ \text{ for } \text{ all } \omega$$

from the event $(*)$, while the set of optimal solutions for these ω is contained $\Delta_{\lambda,\nu}$.

According to Corollary of Theorem 2 we have then that for all ω from the event $(*)$ the required relations are satisfied for the objective functions:

$$F_{\nu,\lambda}^- = \min_{x \in S_\lambda^+} F_\nu(f_n(x)) \leq \min_{x \in S(\omega), \omega \in (*)} F(f_n(x), \omega) \leq \min_{x \in S_\nu^-} F_\lambda(f_n(x)) = F_{\lambda,\nu}^+$$

Now it is easy to show, e.g., by shifting a nonbasic constraint, that depending on the form of objective function and constraint functions there can be realizations such that the optimal solution $x^0 \in \Delta_{\nu\lambda}$ and, respectively, $F(f_n(x^0), \omega) \in [F_{\nu,\lambda}^-, F_{\nu,\lambda}^+]$; but the strict requirement of event $(*)$ is not satisfied for such realizations of ω.

From this it follows that $p_{\nu,\lambda}$ is the lower boundary for the probability that the optimal solution $x^0 \in \Delta_{\nu,\lambda}$ while $F(f(x^0), \omega) \in \left[F_{\nu,\lambda}^-, F_{\lambda,\nu}^+\right]$. This is what we set out to prove. □

5. Conclusion

In the present article we consider problems of stability of solutions of stochastic multi-optimization with normalizations introduced for them and linear and nonlinear convolutions.

In Section 3, we present a study of stability of solutions of stochastic multi–objective optimization problems with linear components and linear convolutions, introduce feasibility and optimality regions, show that any optimal solution of this problem has optimality region.

Problems of stability of solutions of stochastic multi-optimization with normalizations with nonlinear convolution are also considered in Section 4;

for them it is shown that the solution of such problems is bounded above and below by a solution with a pessimistic and optimistic value of the random parameter.

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