Воронежский государственный университет
Московский государственный университет
имени М. В. Ломоносова
Математический институт имени В. А. Стеклова
Российской академии наук

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В юбилейном сборнике, посвященном 100-летию ВГУ, представлены статьи участников Международной молодежной конференции «Воронежская зимняя математическая школа С. Г. Крейна – 2018», содержащие новые результаты по функциональному анализу, дифференциальным уравнениям, краевым задачам математической физики, истории математики, а также другим фундаментальным разделам математики.

Предназначен для научных работников, аспирантов и студентов.

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question was posed: does A^{-1} generate a bounded C_0-semigroup or at least generate a C_0-semigroup?

In [1,2] it was shown that the answer is positive for generators of bounded analytic C_0 semigroup. For other classes of semigroups, as multiplication semigroups or contraction semigroups in a Hilbert space, the answer is positive again. When the semigroup generated by A is exponentially stable, then A^{-1} is a bounded operator and accordingly it generates a C_0 semigroup, but in general it is not uniformly bounded. This situation is analysed by an explicit representation of the semigroup generated A^{-1} in [7,8], including growth estimates. In [4] a sufficient condition on the resolvent map of A under which A^{-1} is the generator of a bounded C_0-semigroup is provided. Several equivalences for A^{-1} generating a C_0-semigroup are given in [3]. There is a way to show that even if A^{-1} does not generate bounded C_0-semigroup it can generate integrated semigroup [5,6]. Such fact can be used to solve ill-posed problems.

The main result of this note is the following statement: if A is the generator of a tempered β-times integrated α-resolvent operator function and is injective, then the inverse operator A^{-1} is the generator of a tempered γ-times integrated α-resolvent operator function for $\gamma > \beta + 1/2$, and it is also the generator of a tempered δ-times resolvent operator function for $\delta < \alpha$.

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APPROXIMATION PROPERTIES ASSOCIATED WITH QUASI-NORMED OPERATOR IDEALS OF (r,p,q)-NUCLEAR OPERATORS

© 2018 *O. I. Reinov*
(Saint Petersburg; *orein51@mail.ru*)

We consider quasi-normed tensor products lying between Lapresté tensor products and the spaces of (r,p,q)-nuclear operators. We define and investigate the corresponding approximation properties for Banach spaces. An intermediate aim is to answer a question of Sten Kaijser. In the end we present two results in connection with a question posed by Hinrichs A. and Pietsch A. in [2].

Throughout, we denote by $X, Y, \dots$ Banach spaces over a field $\mathbb{K}$ (which is either $\mathbb{R}$ or $\mathbb{C}$); $X^*, Y^*, \dots$ are Banach dual to $X, Y, \dots$. By $x, y, x', \dots$ (maybe with indices) we denote elements of $X, Y, Y^* \dots$ respectively. $\pi_Y : Y \to Y^{**}$ is a natural isometric imbedding. It is denoted by $F(X,Y)$ a vector space of all finite rank operators from X to Y. By $X \otimes Y$ we denote the algebraic tensor product of the spaces X and Y. $X \otimes Y$ can be considered as a subspace of the vector space $F(X^*, Y)$ (namely, as a vector space of all linear weak*-to-weak continuous finite rank operators). We can identify also the tensor product (in a natural way) with a corresponding subspace of $F(Y^*, X)$. If $X = W^*$, then $W^* \otimes Y$ is identified with $F(X, Y^{**})$ (or with $F(Y^*, X^*)$. If $z \in X \otimes Y$, then $\widetilde{z}$ is the corresponding finite rank operator. If $z \in X^* \otimes X$ and e.g. $z = \sum_{k=1}^n x'_k \otimes x_k$, then trace $z := \sum_{k=1}^n \langle x'_k, x_k \rangle$ does not depend on representation of z in

$X^*\otimes X$. $L(X,Y)$ is a Banach space of all linear continuous mappings («operators») from X to Y equipped with the usual operator norm.

If $A\in L(X,W)$, $B\in L(Y,G)$ and $z\in X\otimes Y$, then a linear map $A\otimes B: X\otimes Y\to W\otimes G$ is defined by $A\otimes B((x\otimes y) := Ax\otimes By$ (and then extended by linearity). Since $\widetilde{A\otimes B(z)} = B\widetilde{z}A^*$ for $z\in X\otimes Y$, we will use notation $B\circ z\circ A^*\in W\otimes G$ for $A\otimes B(z)$. In the case where X is a dual space, say F^*, and $T\in L(W,F)$ (so, $A=T^*: F^*\to W^*$), one considers a composition $B\widetilde{z}T$; in this case $T^*\otimes B$ maps $F^*\otimes Y$ into $W^*\otimes Y$ and we use notation $B\circ z\circ T$ for $T^*\otimes B(z)$.

If ν is a tensor quasi-norm (see [3, 0.5]), then $\nu(A\otimes B(z))\leqslant ||A||\,||B||\,\nu(z)$ and we can extend the map $A\otimes B$ to the completions of the tensor products with respect to the quasi-norm ν, having the same inequality. The natural map $(X\otimes Y,\nu)\to L(X^*,Y)$ is continuous and can be extended to the completion $X\widehat{\otimes_\nu}Y$; for a tensor element $z\in X\widehat{\otimes_\nu}Y$, we still denote by $\widetilde{z}$ the corresponding operator. The natural mapping $X\widehat{\otimes_\nu}Y\to L(X^*,Y)$ need not to be injective; *if it is injective for a fixed Y and for all X, then we say that Y has the ν-approximation property.*

A projective tensor product $X\widehat{\otimes}Y$ of Banach spaces X and Y is defined as a completion of $X\otimes Y$ with respect to the norm $||\cdot||_\wedge$: if $z\in X\otimes Y$, then $||z||_\wedge := \inf\sum_{k=1}^n ||x_k||\,||y_k||$, where infimum is taken over all representation of z as $\sum_{k=1}^n x_k\otimes y_k$. We can try to consider $X\widehat{\otimes}Y$ also as operators $X^*\to Y$ or $Y^*\to X$, but this correspondence is, in general, not one-to-one. Note that $X\widehat{\otimes}Y = Y\widehat{\otimes}X$ in a sense. If $z\in X\widehat{\otimes}Y, \varepsilon>0$, then one can represent z as $z=\sum_{k=1}^\infty x_k\otimes y_k$ with $\sum_{k=1}^\infty ||x_k||\,||y_k|| < ||z||_\wedge+\varepsilon$. For $z\in X^*\widehat{\otimes}X$ with a «projective representation» $z=\sum_{k=1}^\infty x_k'\otimes x_k$, trace of z, trace $z := z = \sum_{k=1}^\infty\langle x_k, y_k\rangle$, does not depend of representation of z. The Banach dual $(X\widehat{\otimes}Y)^* = L(Y,X^*)$ by $\langle T,z\rangle = \text{trace } T\circ z$.

Finally, $l_p(X)$ (resp. $l_p^w(X)$) are the Banach spaces of all sequences $(x_i)\subset X$ so that the norm $||(x_i)||_p := \big(\sum ||x_i||^p\big)^{1/p}$ (resp. $||(x_i)||_{w,p} := \sup_{||x'||\leqslant 1}\big(\sum |\langle x', x_i\rangle|^p\big)^{1/p}$) is finite.

Below $0<r,s\leqslant 1$, $0<p,q\leqslant\infty$ **and** $1/r+1/p+1/q = 1/\beta\geqslant 1$.

1. The tensor products $X\widehat{\otimes}_{r,p,q}Y$. We use partially notations from [3]. For $z\in X\otimes Y$ we put

$$\mu_{r,p,q}(z) := \inf\{||(\alpha_k)||_r||(x_k)||_{w,p}||(y_k)||_{w,q}:\ z=\sum_{k=1}^n \alpha_k x_k\otimes y_k\};$$

$X\otimes_{r,p,q}Y$ is the tensor product, equipped with this quasi-norm $\mu_{r,p,q}$. Note that $\mu_{1,\infty,\infty}$ is the projective tensor norm of A. Grothendieck [1].

Let us denote by $X\widehat{\otimes_{r,p,q}}Y$ the completion of $X\otimes Y$ with respect to this quasi-norm $\mu_{r,p,q}$ (in [3] — $X\underset{r,p,q}{\widehat{\otimes}}Y$). Every tensor element $z\in X\widehat{\otimes_{r,p,q}}Y$ admits a representation of type $z=\sum_{k=1}^\infty\alpha_k x_k\otimes y_k$, where $||(\alpha_k)||_r||(x_k)||_{w,p}||(y_k)||_{w,q}<\infty$, and $\mu_{r,p,q}(z) := \inf||(\alpha_k)||_r||(x_k)||_{w,p}||(y_k)||_{w,q}$ (inf. is taken over all such finite or infinite representations) [3, Proposition 1.3, p. 52]. Note that $X\widehat{\otimes_{1,\infty,\infty}}Y = X\widehat{\otimes}Y$.

The topological dual to $(X\widehat{\otimes_{r,p,q}}Y, \mu_{r,p,q})$ is the Banach space $\Pi_{\infty,p,q}(X,Y^*)$ of absolutely (∞,p,q)-summing operators from X to Y^* [3, Theorem 1.3, p. 57] (recall that $0<r\leqslant 1$) : If $\tau\in(X\widehat{\otimes_{r,p,q}}Y)^*$ and $x\otimes y\in X\otimes Y$, then the corresponding operator T is defined by $\langle\tau, x\otimes y\rangle = \langle Tx, y\rangle$ [3, pp. 56-57]. Recall that, by definition, an operator $T: X\to F$ is absolutely (∞,p,q)-summing if for any finite sequences (x_k) and (f_k') (from X and F^* respectively) one has

$$\sup_k|\langle Tx_k, f_k'\rangle|\leqslant C\,||(x_k)||_{w,p}||(f_k')||_{w,q}.$$

With a norm $\pi_{\infty,p,q}(T) := \inf C$, the space $\Pi_{\infty,p,q}(X,F)$ is a Banach space and in duality above (for $F=Y^*$) $\pi_{\infty,p,q}(T) = ||\tau||$ (on the right, the norm of the functional $\tau\in(X\widehat{\otimes_{r,p,q}}Y)^*$). Futhermore, taking a sequence in $X\times F^*$, consisting of one nonzero element (x,f'), we obtain: If $T\in\Pi_{\infty,p,q}(X,F)$, then $|\langle Tx,f'\rangle|\leqslant\pi_{\infty,p,q}(T)\,||x||\,||f'||$; thus, $||T||\leqslant\pi_{\infty,p,q}(T)$. On the other hand, if $T\in L(X,F)$, then for any finite sequences (x_k) and (f_k'), $\sup_k|\langle Tx_k, f_k'\rangle|\leqslant||T||\,||(x_k)||_{w,p}\,||(f_k')||_{w,q}$. Therefore, $\Pi_{\infty,p,q}(X,F) = L(X,F)$.

I do not know whether the dual space $\Pi_{\infty,p,q}(X,Y^*)$ separates points of $\widehat{X \otimes_{r,p,q} Y}$. If so, then a natural map $\widehat{X \otimes_{r,p,q} Y} \to X\widehat{\otimes}Y$ is one-to-one. As a matter of fact, it follows from the above considerations, that *the space* $\Pi_{\infty,p,q}(X,Y^*)$ *separates points of* $\widehat{X \otimes_{r,p,q} Y}$ *iff the natural map* $j_{r,p,q} : \widehat{X \otimes_{r,p,q} Y} \to X\widehat{\otimes}Y$ *is one-to-one.*

Definition 1.1.1. *We define a tensor product* $X\widehat{\otimes}_{r,p,q}Y$ *as a linear subspace of the projective tensor product* $X\widehat{\otimes}Y$, *consisting of all tensor elements* z, *which admit representations of type* $z = \sum_{k=1}^{\infty} \alpha_k x_k \otimes y_k$, $(\alpha_k) \in l_r$, $(x_k) \in l_{w,p}$, $(y_k) \in l_{w,q}$ *and equipped with the quasi-norm* $||z||_{\wedge;r,p,q} = \inf ||(\alpha_k)||_r \, ||(x_k)||_{w,p} \, ||(y_k)||_{w,q}$, *where the infimum is taken over all representations of* z *in the above form.*

Remark 1.1. We can define $X\widehat{\otimes}_{r,p,q}Y$ also as a quotient of the space $\widehat{X \otimes_{r,p,q} Y}$ by the kernel of the map $j_{r,p,q}$ (i.e. by the annihilator $L(X,Y^*)_\perp$ of $L(X,Y^*)$ in the space $\widehat{X \otimes_{r,p,q} Y}$). Therefore:

(i) The tensor product $X\widehat{\otimes}_{r,p,q}Y$ is complete, i.e. a quasi-Banach space. This, with the injectivity of the natural map $X\widehat{\otimes}_{r,p,q}Y \to X\widehat{\otimes}Y$ answers a question of Sten Kaijser («Why the last map is one-to-one for the «completion» $X\widehat{\otimes}_{r,p,q}Y$?»).

(ii) If the dual of $\widehat{X \otimes_{r,p,q} Y}$ separates points of this space, then we can write $\widehat{X \otimes_{r,p,q} Y} = X\widehat{\otimes}_{r,p,q}Y$. In this case «finite nuclear» quasi-norm $\mu_{r,p,q}$ coincides with the tensor quasi-norm $||z||_{\wedge;r,p,q}$ (compare with [4, 18.1.10.]).

(iii) The dual space to $X\widehat{\otimes}_{r,p,q}Y$ is still $\Pi_{\infty,p,q}(X,Y^*)$ of absolutely (∞,p,q)-summing operators from X to Y^* with its natural quasi-norm.

Proposition 1.1 *Let 1)* $0 < r_1 \leqslant r_2 \leqslant 1$, $p_1 \leqslant p_2$ *and* $q_1 \leqslant q_2$ *or 2)* $0 < r_1 < r_2 \leqslant 1$, $p_1 \geqslant p_2$, $q_1 \geqslant q_2$ *and* $1/r_2 + 1/p_2 + 1/q_2 \leqslant 1/r_1 + 1/p_1 + 1/q_1$. *If* $z \in X \otimes Y$, *then* $||z||_{\wedge;r_2,p_2,q_2} \leqslant ||z||_{\wedge;r_1,p_1,q_1}$. *In particular,* $||z||_{\wedge;1,\infty,\infty} \leqslant ||z||_{\wedge;r_1,p_1,q_1}$. *Consequently, a natural mappings* $X\widehat{\otimes}_{r_1,p_1,q_1}Y \to X\widehat{\otimes}_{r_2,p_2,q_2}Y \to X\widehat{\otimes}Y$ *are continuos injections of quasi-norms 1.*

Proposition 1.2.2. *If* X *or* Y *has the bounded approximation property, then* $\mu_{r,p,q} = ||\cdot||_{\wedge;r,p,q}$ *on* $X \otimes Y$. *Hence, in this case the dual of* $\widehat{X \otimes_{r,p,q} Y}$ *separates points,* $j_{r,p,q}$ *is injective and* $\widehat{X \otimes_{r,p,q} Y} = X\widehat{\otimes}_{r,p,q}Y$ *(and equals to the corresponding space of* (r,p,q)*-nuclear operators; see below Corollary 2.1).*

Remark 1.2.2. For an «operator» situation, see Corollary 2.1 below and (for $1 \leqslant p, q, \leqslant \infty$) [4, pp. 249-251].

2. Approximation properties. We begin with the main definition.

Definition 2.1.1. *A Banach space* X *has the approximation property* $AP_{r,p,q}$ *if for every Banach space* Y *the canonical mapping* $Y\widehat{\otimes}_{r,p,q}X \to L(Y^*,X)$ *is one to one.*

Proposition 2.1.1. *The following conditions are equivalent:*

1) X *has the* $AP_{r,p,q}$.

2) For every space W *the natural map from* $W^*\widehat{\otimes}_{r,p,q}X$ *to* $L(W,X)$ *is one-to-one.*

3) The natural map $X^*\widehat{\otimes}_{r,p,q}X \to L(X) := L(X,X)$ *is one-to-one.*

Proposition 2.2.2. *If* X^* *has the* $AP_{r,p,q}$, *then* X *has the* $AP_{r,q,p}$.

Remark 2.1.1. The inverse statement is not true. Examples are given in [7, Remark 6.1].

Recall that a linear map $T : X \to Y$ is called (r,p,q)-nuclear if it has a representation $T = \sum_{k=1}^{\infty} \alpha_k \langle x'_k, \cdot \rangle y_k$, where $(\alpha_k) \in l_r$, $(x'_k) \in l_{w,p}(X^*)$ and $(y_k) \in l_{w,q}(Y)$. Every such a map is continuous. The space $N_{r,p,q}(X,Y)$ of all (r,p,q)-nuclear operators from X to Y can be considered as a quotient of the tensor product $X^*\widehat{\otimes}_{r,p,q}Y$ (as well as a quotient of $\widehat{X^* \otimes_{r,p,q} Y}$) by the kernel of the natural map $X^*\widehat{\otimes}_{r,p,q}Y \to L(X,Y)$. We equip this space with the induced quasi-norm (β-norm) denoted by $\nu_{r,p,q}$. If the corresponding quotient map has a trivial kernel, then we write $N_{r,p,q}(X,Y) = X^*\widehat{\otimes}_{r,p,q}Y$ Thus, X has the $AP_{r,p,q}$ iff for every space Y the equality $N_{r,p,q}(Y,X) = Y^*\widehat{\otimes}_{r,p,q}X$ holds.

It follows from Proposition 1.1:

Proposition 2.3.3. *Let 1)* $0 < r_1 \leqslant r_2 \leqslant 1$, $p_1 \leqslant p_2$ *and* $q_1 \leqslant q_2$ *or 2)* $0 < r_1 < r_2 \leqslant 1$, $p_1 \geqslant p_2$, $q_1 \geqslant q_2$ *and* $1/r_2 + 1/p_2 + 1/q_2 \leqslant 1/r_1 + 1/p_1 + 1/q_1$. *If* X *has the* AP_{r_2,p_2,q_2}, *then* X *has the* AP_{r_1,p_2,q_3}. *In particular, the AP of A. Grothendieck implies any* $AP_{r,p,q}$.

Corollary 2.1.1. *(i) If* X *has the bounded approximation property, then for all* r, p, q *and* Y *the equalities* $N_{r,p,q}(Y, X) = Y^*\widehat{\otimes}_{r,p,q}X = Y^*\, \widehat{\otimes_{r,p,q}}\, X$ *hold (with the same quasi-norms). (ii) If* Y^* *has the bounded approximation property, then for all* r, p, q *and* X *the equalities* $N_{r,p,q}(Y, X) = Y^*\widehat{\otimes}_{r,p,q}X = Y^*\, \widehat{\otimes_{r,p,q}}\, X$ *hold (with the same quasi-norms).*

The first part of the following fact is partially known (cf. [4, 18.11.15-18.1.16] for $1 \leqslant p, q \leqslant \infty$).

Proposition 2.4.4. *For any spaces* X, Y *the equalities*

$$N_{r,p,2}(Y, X) = Y^*\widehat{\otimes}_{r,p,2}X = Y^*\, \widehat{\otimes_{r,p,2}}\, X,$$

$$N_{r,2,q}(Y, X) = Y^*\widehat{\otimes}_{r,2,q}X = Y^*\, \widehat{\otimes_{r,2,q}}\, X$$

hold (with the same quasi-norms). In particular, every Banach space has the $AP_{r,p,2}$ *and the* $AP_{r,2,p}$.

Remark 2.2.2. The fact that every X has the $AP_{1,2,\infty}$ is essentially contained in [4, 27.44.10, Proposition]. It is strange, but it seems that a corresponding fact for $AP_{1,\infty,2}$ appears here for the first time. Note that this fact follows also from the preceding by virtue of Proposition 2.2: if every X has the $AP_{1,2,\infty}$, then X^* possesses this property, and by Proposition 2.2 X has the $AP_{1,\infty,2}$.

Many of the above approximation properties were considered earlier, e.g. in the papers [5, 6, 7]: (i) For $p = q = \infty$, we get the AP_r from [6, 7]. (ii) For $p = \infty$, we get the $AP_{[r,q]}$ from [5, 7]. (iii) For $q = \infty$, we get the $AP^{[r,p]}$ from [5, 7].

Following notations from [7] (see also [5]), we denote $N_{r,\infty,\infty}$ by N_r, $N_{r,\infty,q}$ by $N_{[r,q]}$, $N_{r,p,\infty}$ by $N^{[r,p]}$. The corresponding notations are used also for $AP_{r,p,q}$ (cf. (i)–(iii) above). Almost all the information about Banach spaces without (or with) the properties AP_r, $AP_{[r,q]}$ and $AP^{[r,p]}$ which is known to us by now, can be found in [5, 6, 7].

3. On regularity of $N_{r,p,q}$. The following question was posed by A. Hinrichs and A. Pietsch in [2]: suppose T is a (bounded linear) operator acting between Banach spaces X and Y, and let $s \in (0, 1)$. Is it true that if T^* is s-nuclear then T is s-nuclear too? We present here two results (answering the question in negative):

Theorem 3.1.1. *Let* $T \in L(X, Y)$ *and assume that either* $X^* \in AP_{r,q,p}$ *or* $Y^{***} \in AP_{r,q,p}$. *If* $\pi_Y T \in N_{r,p,q}(X, Y^{**})$, $\nu_{r,p,q}(T) < 1$, *then* $T \in N_1(X, Y)$ *and* $\nu_1(T) \leqslant 1$.

This theorem is sharp. For example:

Theorem 3.2.2. *Let* $r \in (2/3, 1]$, $p \in (1, 2]$, $1/r - 1/p = 1/2$. *There exists a separable Banach space* Z *so that* Z^{**} *has a basis and there is an operator* $U : Z^{**} \to Z$ *such that*

(α) $\pi_Z U \in N^{[r,p_0]}(Z^{**}, Z^{**})$, $\forall\, p_0 \in [1, p)$;

(β) *U is not nuclear as a map from Z^{**} into Z*

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