ISSN 1063-4541, Vestnik St. Petersburg University, Mathematics, 2018, Vol. 51, No. 2, pp. 144–163. © Pleiades Publishing, Ltd., 2018.
Original Russian Text © M.A. Lifshits, Ya.Yu. Nikitin, V.V. Petrov, A.Yu. Zaitsev, A.A. Zinger, 2018, published in Vestnik Sankt-Peterburgskogo Universiteta: Matematika, Mekhanika, Astronomiya, 2018, Vol. 63, No. 2, pp. 231–264.

MATHEMATICS

Toward the History of the Saint Petersburg School of Probability and Statistics. I. Limit Theorems for Sums of Independent Random Variables

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Received July 15, 2017; in final form, September 21, 2017

Abstract—This is the first in a series of reviews devoted to the scientific achievements of the Leningrad—St. Petersburg school of probability and statistics in the period from 1947 to 2017. It is devoted to limit theorems for sums of independent random variables—a traditional subject for St. Petersburg. It refers to the classical limit theorems: the law of large numbers, the central limit theorem, and the law of the iterated logarithm, as well as important relevant problems formulated in the second half of the twentieth century. The latter include the approximation of the distributions of sums of independent variables by infinitely divisible distributions, estimation of the accuracy of strong Gaussian approximation of such sums, and the limit theorems on the weak almost sure convergence of empirical measures generated by sequences of sums of independent random variables and vectors.

Keywords: sums of independent random variables, central limit theorem, law of large numbers, law of the iterated logarithm, infinitely divisible distributions, concentration functions, Littlewood—Offord problem, empirical measure, almost sure limit theorem.

DOI: 10.3103/S1063454118020115

1. INTRODUCTION

This article, written at the kind invitation of the editorial board of the journal "Vestnik of St. Petersburg University: Mathematics. Mechanics. Astronomy", opens a series of reviews of the achievements of the Leningrad—St. Petersburg School of Probability and Statistics in the period from 1947 to the present day. Main attention will be paid to the achievements of the staff of the Department of Probability Theory and Mathematical Statistics of the Faculty of Mathematics and Mechanics of the Leningrad State University (LSU, SPbSU), founded by Yu. V. Linnik in 1948, as well as its alumni who continued their work in Leningrad—St. Petersburg. This kind of extensive description of the activities of this mathematical school is being done for the first time. The well-known review by Yu. V. Linnik [1] was brief and covered only the period until 1969.

Studies on probability theory in St. Petersburg were started by V. Ya. Bunyakovsky (1804—1889), a disciple of Cauchy, who received an excellent education in Lausanne and Paris and defended a doctoral thesis at the University of Paris. He was the first who began to teach a course of probability theory at St. Petersburg University; he also wrote the first Russian textbook on this subject [2]. This textbook—"excellent for that time," according to B. V. Gnedenko [3], p. 28—became very popular; there is an evidence that Gauss and Bienaymé studied Russian to read it [4]. It gave an original presentation of both the probability theory and its application to insurance, demography, etc.

In review [1], Yu. V. Linnik wrote: “Bunyakovsky gave an important formulation of the issues concerning the acceptance inspection, e.g., the reception of a lot of sacks of sugar on the basis of sampling. In the history of mathematics, this is the first publication on the statistical control of the quality of industrial products. A number of later articles by Bunyakovsky are applications to various practical problems in probability and statistics, e.g., calculating the probable strength of the Russian army after the draft and determining observation errors. Beginning in 1858, Bunyakovsky was the chief expert of the government on statistics and insurance. From 1864 to 1889, he was vice-president of the Academy of Sciences.

A new phase of research on probability began in St. Petersburg after the relocation there of P. L. Chebyshev (1821–1894), who, in 1860, replaced V. Ya. Bunyakovsky as a lecturer in the course of probability at the university. In the works of Chebyshev and his disciples—beginning with A. A. Markov (1856–1922) and A. M. Lyapunov (1857–1918)—numerous brilliant results were obtained; they are well-known and described in detail in textbooks on probability and statistics, as well as in works on the history of mathematics. An idea of the level of research and teaching is also given by Chebyshev’s lectures, recorded by A. M. Lyapunov [5], and by Markov’s textbook [6].

The prewar period in research on probability is associated with the name of the outstanding mathematician, academician S. N. Bernstein (1880–1968), who, in 1933–1941, lectured at the university as a professor at the Faculty of Mathematics and Mechanics. In this period, in particular, he laid the foundations of stochastic differential equations (in the discrete version), proved the central limit theorem for new classes of dependent random variables, and found one of the first variants of characterizing the normal law by the independence of the sums and differences of random variables. In 1937, Bernstein gave a number of far-reaching generalizations of Chebyshev’s inequality, now called Bernstein inequalities and widely used in the theory of large deviations and other issues of limit theorems. In the period from 1927 to 1946, four editions of the remarkable textbook on probability by Bernstein were published [7]. (Concerning Bernstein’s works in the field of probability, see also [8, 9].)

Bernstein then moved to Moscow, and the revival of the research on probability and statistics at the Leningrad University in the post-war period is associated with the name of Academician Yu. V. Linnik (1915–1972), by that time a renowned author of exceptionally strong results in analytic number theory. According to Linnik himself, he was recommended to conduct research in probability by the outstanding Moscow mathematician A. Ya. Khinchin, who believed that mathematicians should work in at least two different fields of their discipline. This prompted Linnik to begin intensive research in the field that was new for him. On his initiative, in 1948, the Department of Probability Theory and Mathematical Statistics at the Faculty of Mathematics and Mechanics was inaugurated.

Linnik’s interests gradually extended to statistics and led to a number of remarkable books and articles, which can be found in his collected works [10, 11] (see also the jubilee article [12]).

Linnik’s bright personality attracted many talented disciples, some of whom (V. V. Petrov, V. P. Skitovich, and I. A. Ibragimov) began working at the Department of Probability Theory and Mathematical Statistics, while others worked at the Leningrad Division of the Mathematical Institute of the Academy of Sciences (LDMI, now PDMI) or at universities and research institutions in the city. They also acquired disciples, mainly students and postgraduate students of the Department. Eventually, a considerable school formed, which, in the 1990s, under the leadership of Academician I. A. Ibragimov, was officially recognized by the Ministry of Education and Science of the Russian Federation and the Russian Federation Presidential Council for Grants.

The aforementioned series of articles is intended to give a brief account of the main achievements of this scientific school. In this issue, we focus mainly on the limit theorems for sums of independent random variables. Section 2–4 were written by V. V. Petrov (SPbSU), Section 5, by A. Yu. Zaitsev (PDMI RAS and SPbSU), Section 6, by A. A. Zinger (SUAI), and Section 7, by M. A. Lifshits (SPbSU). The introduction was compiled by Ya. Yu. Nikitin (SPbSU).

2. CENTRAL LIMIT THEOREM FOR SUMS OF INDEPENDENT RANDOM VARIABLES

Among the works of the founders of the St. Petersburg School of Probability, P. L. Chebyshev, A. A. Markov, and A. M. Lyapunov, whose works became classics of our discipline, an important place belongs to research on limit theorems for sums of independent random variables. In the 1930s, their works were continued at Leningrad University by S. N. Bernstein and, from the end of the 1940s, the founder of the Department of Probability Theory and Mathematical Statistics, Yu. V. Linnik, his numerous disciples, and the disciples of their disciples.

In his first publication on probability theory [13], Linnik obtained nonuniform estimates of the deviation of the distribution function of the sum of independent non-identically distributed random variables from the normal distribution on an arbitrary finite interval. Along with the traditional technique of characteristic functions, he used new methods [13], previously used in the analytic numbers theory, a field of mathematics in which Linnik had made outstanding discoveries.

An extensive literature is devoted to estimating the convergence rate of the distributions of sums of independent random variables to the normal distribution. Classical results in this field are the Lyapunov, Berry–Esseen, and Esseen inequalities. Generalizations and strengthening of these results, containing both uniform and more accurate nonuniform estimates, were obtained by V. V. Petrov [14], I. A. Ibragimov [15], L. V. Osipov [16], L. V. Osipov and V. V. Petrov [17], and B. A. Lifshits [18]. Let us formulate one result of I. A. Ibragimov.

Let $\{X_n\}$ be a sequence of independent random variables with the common distribution function $V(x)$, $S_n = X_1 + \ldots + X_n$, and suppose that there exist sequences of constants $\{a_n\}$ and $\{b_n\}$ such that $F_n(x) \to \Phi(x)$ as $n \to \infty$ for any x, where $F_n(x) = \mathbf{P}(S_n/a_n - b_n < x)$ and $\Phi(x)$ is the standard normal distribution function. Set $r_n = \inf_{a_n, b_n} \sup_x |F_n(x) - \Phi(x)|$. For $r_n = O(n^{-\delta/2})$, where $0 < \delta < 1$, it is necessary and sufficient that

$$\int_{-\infty}^{\infty} x^2 dV(x) < \infty, \qquad \int_{|x| \geq z} x^2 dV(x) = O(z^{-\delta}) \quad \text{at} \quad z \to \infty. \tag{1}$$

For $r_n = O(n^{-1/2})$, it is necessary and sufficient to satisfy conditions (1) for $\delta = 1$ and the condition

$$\int_{-z}^{z} x^3 dV(x) = O(1) \quad \text{at} \quad z \to \infty. \tag{2}$$

L. V. Osipov and V. V. Petrov [17] obtained an estimate for the deviation of the distribution function of an arbitrarily normalized sum of n independent non-identically distributed random variables from a normal distribution function without assuming the existence of any moments. Heyde [19] found the optimal asymptotic behavior of this estimate. Despite the relative simplicity of the formulations of the estimates obtained, their implications are the Lyapunov and Esseen inequalities.

For a sequence of independent equally distributed random variables with finite variance, L. V. Osipov [20] joined the upper and lower bounds of the remainder term in the central limit theorem.

Considering a sequence of independent random variables with a common distribution function, V. A. Egorov [21] and L. V. Rozovsky [22] studied the relationship between the moment properties of this function and various forms of the asymptotic normality of the distribution function of an arbitrarily normalized sum of n random variables from an original sequence.

An important place in the literature on limit theorems of probability theory belongs to the theorems on the asymptotic expansions for distributions of sums of independent random variables. For a sequence of independent identically distributed random variables, I. A. Ibragimov [23] obtained the necessary and sufficient conditions for the classical form of the asymptotic expansion in the central limit theorem. L. V. Osipov [24] obtained nonuniform estimates of the remainder term in the asymptotic expansion of the distribution function $F_n(x)$ of the normalized sum of n independent identically distributed random variables with a finite absolute moment of order $k \geq 3$; no other additional assumptions were made. These estimates are valid for all x and n. The simple implications of the results obtained are the known nonuniform estimates.

V. V. Petrov [25] found explicit formulas for any terms of the classical asymptotic expansion in the central limit theorem. In [26], theorems on the asymptotic expansions of the distribution function of the normalized sum of independent non-identically distributed random variables, as well as the derivatives of any order of this distribution function, were obtained. Some of these results refer to local theorems for the distribution densities of normalized sums of such random variables.

V. V. Petrov [27, 28] obtained local limit theorems for the distribution densities of the sums of independent non-identically distributed random variables with estimates of the remainder term and with asymptotic expansions.

A prominent place in the literature belongs to the limit theorems for probabilities of large deviations of sums of independent random variables. In many applications, an important role is played by information about the probabilities of the form $\mathbf{P}(Z_n \geq x)$, where Z_n is the normalized sum of n independent random variables as $x = x_n \to \infty$ $(n \to \infty)$. The fundamental result in the field of the limit theorems for probabilities of large deviations is Cramér's theorem [29] for a sequence of independent identically distributed random variables $\{X_n\}$, under the condition that the generating function of the moments $\mathbf{E}e^{tX_1}$ is finite in the domain $|t| < H$ for some $H > 0$ (Cramér's condition). This theorem contains asymptotic equalities for the ratios $(1 - F_n(x))/(1 - \Phi(x))$ and $F_n(-x)/\Phi(-x)$ for $x = x_n = o(\sqrt{n}/\log n)$ $(n \to \infty)$, where $F_n(x) = \mathbf{P}(S_n < x\sigma\sqrt{n})$ under the conditions $\mathbf{E}X_1 = 0$ and $\mathbf{D}X_1 = \sigma^2$. The simplest implication of Cramér's theorem is the assertion that, under Cramér's condition, each of these relations tends to 1 for $x = o(n^{1/6})$.

V. V. Petrov [30] obtained a generalization of Cramér's limit theorem for sequences of independent non-identically distributed random variables with the condition $x = o(\sqrt{n}/\log n)$ replaced by the condition $x = o(\sqrt{n})$, which is optimal in this situation, and with the corresponding improvement of the order of the remainder term in the asymptotic expansions for the ratios of tail probabilities. The limit theorems for large deviations of sums of independent non-identically distributed random variables had previously been studied by Feller [31], but Feller's results, who considered only sequences of bounded non-identically distributed random variables, do not imply Cramér's theorem.

W. Richter [32] obtained local limit theorems for large deviations of sums of independent random variables under Cramér's condition.

Cramér's condition is rather restrictive; it implies the analyticity of the characteristic function of random variables in some neighborhood of zero and, therefore, in some strip containing the real axis. Yu. V. Linnik [33, 34] developed new methods that made it possible to study the probabilities of large deviations of sums of independent random variables when Cramér's condition is violated.

Let us present one of Yu. V. Linnik's results. Let $\{X_n\}$ be a sequence of independent identically distributed random variables with zero mean and a variance $\sigma^2 > 0$. Set $S_n = X_1 + \ldots + X_n$ and $F_n(x) = \mathbf{P}(S_n < x\sigma\sqrt{n})$. Let $\rho(n)$ be a function satisfying the condition $\rho(n) \to \infty$ as $n \to \infty$. If $0 < \alpha \le 1/6$, then the condition

$$\mathbf{E}\exp\left\{|X_1|^{\frac{4\alpha}{2\alpha+1}}\right\} < \infty \tag{3}$$

is sufficient for

$$\frac{1 - F_n(x)}{1 - \Phi(x)} \to 1, \quad \frac{F_n(-x)}{\Phi(-x)} \to 1 \tag{4}$$

uniformly in x in the domain $0 \le x \le n^{\alpha}/\rho(n)$, and it is necessary that relations (4) be uniform in x in the domain $0 \le x \le n^{\alpha}\rho(n)$. If, on the other hand, $1/6 < \alpha < 1/2$, then conditions (3) and

$$\gamma_m = 0 \quad (m = 3, \ldots, s + 2) \tag{5}$$

are sufficient for relations (4) to take place uniformly in x in the domain $0 \le x \le n^{\alpha}/\delta(n)$ and are necessary for these relations to take place uniformly in x in the domain $0 \le x \le n^{\alpha}\rho(n)$. Here, γ_m is the cumulant of order m of the random variable X_1, and s is the nonnegative integer number defined by the inequalities $s/(2(s + 2)) < \alpha < (s + 1)/(2(s + 3))$. It should be noted that, for $\alpha = 1/2$, condition (3) coincides with Cramér's condition and, for $\alpha < 1/2$, it represents a weakened Cramér's condition.

Yu. V. Linnik also obtained the corresponding local limit theorems for large deviations when Cramér's condition is weakened and the limit theorems for large deviations on the whole axis, i.e., theorems on the asymptotic behavior of probabilities of the form $\mathbf{P}(S_n \ge x)$ for sums of independent random variables, without any restrictions on the order of growth of x.

Yu. V. Linnik's works on the limit theorems for large deviations of sums of independent random variables have found a broad response in the domestic and foreign literature. V. V. Petrov [35], under conditions (3) and $\alpha < 1/2$, studied zones in which normal convergence (4) was replaced by more general asymptotic relations connected with segments of Cramér's series. L. V. Osipov [36] found the necessary and sufficient conditions for relations (4) (or the asymptotic relations just mentioned) to be satisfied in the domain $0 \le x \le bn^{\alpha}$ uniformly in x, where $\alpha < 1/2$ and b are positive constants. The limit theorems for large deviations of sums of independent non-identically distributed random variables, generalizing Linnik's theorems, were obtained by V. V. Petrov [35]. In [37], asymptotic representations were found for the probabilities $\mathbf{P}(S_n \ge nx)$ and $\mathbf{P}(S_n = nx)$, uniform in x in the domain $\mathbf{E}X_1 + \varepsilon \le x \le A - \varepsilon$, where S_n is the sum of n independent random variables with the same non-lattice or lattice distribution, respectively, satisfying the one-sided analogue of Cramér's condition, ε is an arbitrary positive constant, and the constant A is given in explicit form. Interesting supplements to these results were obtained by L. V. Rozovsky (see, e.g., [38, 39]).

Along with the limit theorems, inequalities for the probabilities of large deviations of the sums S_n of independent random variables that are valid for any number of summands can be useful. V. V. Petrov obtained the following result for the distributions of sums of n independent random variables. Suppose that there exist positive constants $g_1, \ldots, g_n$, and T such that $\mathbf{E}e^{tX_k} \le e^{g_k t^2/2}$ $(k = 1, \ldots, n)$ for $0 \le t \le T$. Set $G_n = \sum_{k=1}^n g_k$. Then, we have the following inequalities:

$$\mathbf{P}(S_n \ge x) \le \exp\{-x^2/(2G_n)\}, \quad \text{if} \quad 0 \le x \le G_n T,$$

$$\mathbf{P}(S_n \ge x) \le \exp\{-Tx/2\}, \quad \text{if} \quad x \ge G_n T.$$

There is a left-sided analogue of this assertion and a corollary of these assertions, in which the condition on the moment generating function is assumed to be satisfied in the domain $|t| \le T$. The set of formulated conditions implies the fulfillment of Cramér's condition. The result obtained is stronger than the more complex formulation of S. N. Bernstein's inequalities and is much easier to prove.

Books by I. A. Ibragimov and Yu. V. Linnik [40] and V. V. Petrov [41, 42] contain the results obtained by the authors and other materials concerning the limit theorems for sums of independent random variables.

3. THE LAW OF LARGE NUMBERS

A prominent place in modern studies belongs to works on strong limit theorems of probability theory, including various forms of the strong law of large numbers and the law of the iterated logarithm for sequences of independent random variables and the case where the condition of independence is replaced by some condition of dependence or where the condition of independence is removed.

For a sequence of independent non-identically distributed random variables, A. I. Martikainen [43] found the necessary and sufficient conditions for the strong law of large numbers with an arbitrary (not necessarily monotonic) sequence of normalizing constants. If $\{X_n\}$ is a sequence of independent random variables and $\{a_n\}$ is a sequence of positive numbers such that $\liminf a_{n+1}/a_n > 1$, then $S_n/a_n \to 0$ a.s. if and only if $\sum \mathbf{P}(|X_n| \ge \varepsilon a_n) < \infty$ for any $\varepsilon > 0$, as shown by L. V. Rozovsky [44].

It is of interest to estimate the growth of sums of random variables almost surely in terms of the sum of the moments of these variables. To formulate results of this type obtained by V. V. Petrov [45], some additional notation is required. The set of functions $\psi(x)$ such that each $\psi(x)$ is positive and non-decreasing in the domain $x > x_0$ for some x_0 (not necessarily the same for different functions ψ) and the series $\sum \frac{1}{n\psi(n)}$ converges (diverges) will be denoted by Ψ_c (respectively, Ψ_d). For example, $\psi(x) = x^\alpha \in \Psi_c$ for any $\alpha > 0$, $\psi(x) = (\log x)^{1+\delta} \in \Psi_c$ for every $\delta > 0$, and $\psi(x) = \log x \in \Psi_d$.

Let $g(x)$ be an even continuous function that is positive and strictly increasing in the domain $x > 0$, where $g(x) \to \infty$ as $x \to \infty$, and $\{X_n\}$ be a sequence of independent random variables such that $\mathbf{E}g(X_n) < \infty$ for all n. Suppose that one of the following two conditions are satisfied:

(A) the function $x/g(x)$ does not decrease in the domain $x > 0$;

(B) $x/g(x)$ and $g(x)/x^2$ do not increase in the domain $x > 0$.

In case (B), it is additionally assumed that $\mathbf{E}X_n = 0$ for all n.

Next, suppose that $M_n = \sum_{k=1}^n \mathbf{E}g(X_k) \to \infty$ as $n \to \infty$. Then,

$$S_n = o(g^{-1}(M_n \psi(M_n))) \text{ a.s.,} \tag{6}$$

for any function $\psi \in \Psi_c$, where g^{-1} is the inverse function of g.

If, instead of $\psi \in \Psi_c$, we take a more slowly increasing function $\psi \in \Psi_d$, then, as shown in [45], the assertion may be incorrect.

The formulations are significantly simplified if we set $g(x) = |x|^p$, where $0 < p \le 2$. In particular, for $p = 2$, we obtain the following assertion.

Let X_n be a sequence of independent random variables with finite variances. Set $B_n = \mathbf{D}S_n$. If $B_n \to \infty$ as $n \to \infty$, then $S_n - \mathbf{E}S_n = o(\sqrt{B_n \psi(B_n)})$ a.s. for any $\psi \in \Psi_c$. This assertion may be wrong for a slower increasing function $\psi \in \Psi_d$.

Hence, for the sum S_n of independent random variables with finite variances and an unlimited increase of the variance of the sum, $B_n = \mathbf{D}S_n$, we have the following estimates of the growth order, each of which is stronger than the previous one: for any $\varepsilon > 0$, we have

$$S_n - \mathbf{E}S_n = o(B_n^{1/2+\varepsilon}) \text{ a.s.},$$

$$S_n - \mathbf{E}S_n = o(B_n^{1/2}(\log B_n)^{1/2+\varepsilon}) \text{ a.s.},$$

$$S_n - \mathbf{E}S_n = o(B_n^{1/2}(\log B_n)^{1/2}(\log\log B_n)^{1/2+\varepsilon}) \text{ a.s.}$$

and so on. In these estimates, ε cannot be replaced by zero without introducing additional conditions.

V. A. Egorov in [46] generalized these results.

As was shown in [47], with complete abandoning the independence assumption and under the condition that the absolute moments of order $p \le 1$ of these quantities are finite, the estimates are valid if $S_n - \mathbf{E}S_n$ is replaced by S_n and B_n is replaced by $M_n = \sum_{k=1}^{n} \mathbf{E}|X_k|^p$. It was shown in [48] that, under this condition, we can additionally replace the sum S_n by the sum $T_n = \sum_{k=1}^{n} |X_k|$.

In terms of the classes Ψ_c and Ψ_d, we can describe the behavior of $\liminf b(n)S_n$, where $b(n)$ is a given function and S_n is the sum of n independent random variables. In [42, Section 6.6], one can find the following results, which generalize and strengthen some results of Chung and Erdös. Let $\{X_n\}$ be a sequence of independent identically distributed random variables with the characteristic function $f(t)$ satisfying the condition $\limsup|f(t)| < 1$ as $|t| \to \infty$. Then, $\lim\sqrt{n}\psi(n)|S_n| = \infty$ a.s. for any function $\psi \in \Psi_c$ and, under the additional conditions $\mathbf{E}X_1 = 0$ and $\mathbf{D}X_1 < \infty$, we have $\liminf\sqrt{n}\psi(n)|S_n| = 0$ a.s. for any function $\psi \in \Psi_d$.

As A. A. Markov remarked, the Chebyshev inequality immediately implies the following proposition: if $\{X_n\}$ is an arbitrary sequence of random variables with finite variances and the condition $\mathbf{D}S_n/n^2 \to 0$ as $n \to \infty$ is satisfied, then $(S_n - \mathbf{E}S_n)/n \to 0$ in probability. It was shown in [49] that some strengthening of the Markov condition leads to the strong law of large numbers. Namely, if $\{X_n\}$ is a sequence of nonnegative random variables with finite variances satisfying conditions

$$\mathbf{D}S_n = O(n^2/\psi(n)) \quad \text{for some function} \quad \psi \in \Psi_c, \tag{7}$$

and the condition $\mathbf{E}(S_n - S_m) \le C(n - m)$ for all sufficiently large $n - m$, where C is a constant, then $(S_n - \mathbf{E}S_n)/n \to 0$ a.s. In this proposition, one cannot replace condition (7) by the weaker condition $\mathbf{D}S_n = O(n^2/\psi(n))$ for some function $\psi \in \Psi_d$.

V. V. Petrov [50] proved the following theorem, in which the conditions of independence and nonnegativity of the initial random variables are absent. If $\mathbf{E}X_n = 0$, $\mathbf{E}|X_n|^p < \infty$ for all n and some $p > 1$, and the condition $\mathbf{E}|S_n - S_m|^p \le C(n - m)^{pr-1}$ holds for all n and m such that $n > m \ge 0$, where $r \ge 1$ and C is a constant, then $S_n/n^r \to 0$ a.s. Hence, for a sequence of random variables with finite variances, the condition $\mathbf{D}(S_n - S_m) \le C(n - m)^{2r-1}$ for all n and m such that $n > m$, where $r \ge 1$, implies the relation $(S_n - \mathbf{E}S_n)/n^r \to 0$ a.s.

4. THE LAW OF THE ITERATED LOGARITHM

The best-known results among the theorems on the law of the iterated logarithm are Kolmogorov's and Hartman–Wintner's theorems concerning the sequences of independent non-identically distributed and identically distributed random variables, respectively. The formulations of these theorems and of a number of their generalizations can be found, e.g., in [41].

The applicability of the central limit theorem to a sequence of independent random variables with finite variances does not imply the applicability of the law of the iterated logarithm but a rather weak estimate of the convergence rate in the central limit theorem already ensures this applicability, as shown by the following V. V. Petrov's theorem [51].

Let $\{X_n\}$ be a sequence of independent random variables such that $\mathbf{E}X_n = 0$ and $\mathbf{D}X_n < \infty$ for all n. Set

$$S_n = \sum_{k=1}^{n} X_k, \quad B_n = \sum_{k=1}^{n} \mathbf{D}X_k, \quad R_n = \sup_x \left|\mathbf{P}(S_n < x\sqrt{B_n}) - \Phi(x)\right|,$$

where $\Phi(x)$ is the standard normal distribution function. Under the conditions $B_n \to \infty$ $(n \to \infty)$,

$$B_{n+1}/B_n \to 1, \tag{8}$$

$$R_n = O((\log B_n)^{-1-\delta}) \quad \text{for some} \quad \delta > 0, \tag{9}$$

we have the relation

$$\limsup S_n/(2B_n \log\log B_n)^{1/2} = 1 \text{ a.s.} \tag{10}$$

V. A. Egorov [52, 53] showed that, under the hypotheses of this theorem, one cannot replace the positive number δ by zero.

As shown in [54], if condition (8) is removed from the hypotheses of the above theorem, then we obtain relation (10) with the sign of equality replaced by the sign $\le$. There are publications in which relations of type (10) are obtained for an arbitrary nondecreasing numerical sequence $\{B_n\}$ and a sequence $\{X_n\}$ of independent random variables without assumptions on the existence of any moments of these quantities.

A. Martikainen, A. Rosalsky, and W.E. Pruitt independently and almost simultaneously published the following result: if a sequence of independent identically distributed random variables $\{X_n\}$ satisfies the equality $\limsup S_n/(2n\log\log n)^{1/2} = 1$ a.s., then $\mathbf{E}X_1 = 0$ and $\mathbf{D}X_1 = 1$ (the references can be found in [42]).

Many works have been devoted to the generalized law of the iterated logarithm for sequences of random variables without assuming the independence and existence of any moments of the random variables. In these works, the conditions under which the relations $\limsup S_n/a_n \le 1$ a.s. or $\limsup S_n/a_n = 1$ a.s. takes place were studied, where $\{a_n\}$ is a sequence of positive numbers such that $a_n \to \infty$ $(n \to \infty)$; it is not always assumed that the normalizing numerical sequence is nondecreasing. This kind of normalizing sequence occurs in the study of the law of the iterated logarithm for sequences of m-dependent or m-orthogonal random variables. There is an extensive literature on the limit theorems for sequences of m-dependent random variables and for sequences of orthogonal random variables. The concept of a sequence of m-orthogonal random variables was introduced in [55]; in [55, 56], theorems on the iterated logarithm law for these sequences were obtained.

For a sequence of independent random variables, A. I. Martikainen and V. V. Petrov found the conditions necessary and sufficient for the applicability of the generalized law of the iterated logarithm with a nondecreasing normalizing numerical sequence (see, e.g., [42, Section 7.3]). More simply formulated sufficient conditions without the assumption of independence can be found in [56].

Much attention in the literature on limit theorems of probability theory has been given to strong limit theorems for increments of sums of independent random variables. One of the objects of study in this field is the conditions under which equalities of the type $\limsup U_n/b_n = \limsup W_n/b_n = 1$ a.s. take place, where

$$U_n = \max_{0\le k\le n-a_n} (S_{k+a_n} - S_k),$$

$$W_n = \max_{0\le k\le n-a_n} \max_{1\le j\le a_n} (S_{k+j} - S_k), \qquad S_n = \sum_{k=1}^{n} X_k,$$

$\{X_n\}$ is a sequence of independent identically distributed random variables, and $\{a_n\}$ is a sequence of positive integers, $a_n \le n$. For $a_n = n$, we have $U_n = S_n$ and $W_n = \max_{1\le k\le n} S_k$. In a series of works by A. N. Frolov (see [57, 58] and references therein), the dependence of the asymptotic behavior of U_n and W_n on the growth rate of the sequence $\{a_n\}$ was studied. In particular, generalizations of the Erdös–Rényi and Csörgő–Revesz theorems were obtained.

5. APPROXIMATION OF DISTRIBUTIONS OF SUMS OF INDEPENDENT VARIABLES

In the early 1960s, I. A. Ibragimov was interested in two problems on the accuracy of the infinitely divisible approximation of distributions of sums of independent random variables, formulated in the mid-1950s by A. N. Kolmogorov [59]. In the book by I. A. Ibragimov and Yu. V. Linnik [40], a special chapter is devoted to this subject. A number of results were obtained in the joint article by I. A. Ibragimov and E. L. Presman [60]. In particular, the optimal (up to a logarithm) estimate was proven for the proximity of n-fold convolutions F^n of symmetric one-dimensional probability distributions F with accompanying infinitely divisible laws $e(nF)$ of the form

$$\rho(F^n, e(nF)) \le cn^{-1/2}(\log n + 1). \tag{11}$$

Here, $\rho(\cdot, \cdot)$ is the classical uniform Kolmogorov distance between the corresponding distribution functions, and $e(nF)$ is the infinitely divisible distribution with the characteristic function $\exp\{n(\hat{F}(t) - 1)\}$, $t \in \mathbf{R}$, where $\hat{F}(t)$ is the characteristic function of the probability distribution F. Hereinafter, the symbols c and $c(\cdot)$ (sometimes with indices) denote, in general, various positive absolute constants and variables that depend only on the argument in parentheses. The concentration function of a random variable Y with the distribution $F = \mathcal{L}(Y)$ is defined by the equality

$$Q(F, \tau) = \sup_{x \in \mathbf{R}} \mathbf{P}(Y \in [x, x + \tau]), \qquad \tau \geq 0.$$

This subject was also interesting and important because it concerned the case of arbitrary distributions of variables without standard assumptions on the moments. In light of this, I. A. Ibragimov began to offer Kolmogorov's problems to his disciples. As a result, both problems were solved by his disciples, graduates of the Leningrad University T. Arak and A. Yu. Zaitsev. In 1986, a joint monograph by Arak and Zaitsev [61], containing an account of these results, was published in the Proceedings of the Steklov Mathematical Institute.

In the early 1980s, T. Arak obtained the complete solution of the first Kolmogorov's problem, having proven in [62] the following remarkable result: *There exists an absolute constant c such that, for any one-dimensional probability distribution F and any natural number n, there exists an infinitely divisible distribution* D_n *such that*

$$\rho(F^n, D_n) \leq cn^{-2/3}. \tag{12}$$

Therefore,

$$\varphi(n) = \sup_F \rho(F^n, \mathfrak{D}) \leq cn^{-2/3}, \tag{13}$$

where $\mathfrak{D}$ is the set of all one-dimensional infinitely divisible distributions. Arak [63] also established that there is a similar lower bound:

$$\varphi(n) \geq cn^{-2/3}. \tag{14}$$

In 1986, these results were reported by T. Arak in an invited talk at the International Mathematical Congress in Berkeley.

A multidimensional analogue of (13) has not yet been obtained. E. L. Presman [64] obtained in a d-dimensional situation an estimate of the form

$$\varphi_d(n) = \sup_F \rho_d(F^n, \mathfrak{D}_d) \leq c(d)n^{-1/3}. \tag{15}$$

Here, $\rho_d(\cdot, \cdot)$ is the uniform distance between the corresponding d-dimensional distribution functions and $\mathfrak{D}_d$ is the set of all d-dimensional infinitely divisible distributions.

Somewhat earlier, Arak [65] showed that, if F is a symmetric one-dimensional distribution with a non-negative characteristic function for all $t \in \mathbf{R}$, then

$$\rho(F^n, e(nF)) \leq cn^{-1}. \tag{16}$$

Therefore, for particular distributions F, the decay of $\rho(F^n, \mathfrak{D})$ can be much faster than $O(n^{-2/3})$. In the mid-1990s, A. Yu. Zaitsev [66] formulated a conjecture that, for any one-dimensional distribution F, there exists a variable $c(F)$ depending on F and such that $\rho(F^n, \mathfrak{D}) \leq c(F)n^{-1}$ for any positive integer n. Earlier, E. L. Presman [67] showed that this is true for the binomial distribution if the distribution F is concentrated at two points. For some distributions, this conjecture was confirmed in the works of Čekanavičius [68, 69] and Čekanavičius and Wang [70]. In particular, it was shown in [69] that the conjecture is valid for discrete distributions concentrated at a finite number of points.

To solve the aforementioned problems, Arak used new estimates for the concentration functions of the sums of independent random variables. These estimates were formulated in terms of the arithmetic structure of the supports of the distributions of summands. It was shown that, if the concentration function of the sum is large, then the supports of the distributions of the summands are concentrated near some set with a nontrivial arithmetic structure.

In the recently published work by F. Götze, Yu. S. Eliseeva, and A. Yu. Zaitsev [71], it was shown that Arak's results make it possible to obtain estimates of the concentration functions of the weighted sums of independent identically distributed random variables $S_a = \sum_{k=1}^{n} a_k X_k$ in the Littlewood–Offord problem, first considered in [72, 73]. In this case, we are dealing with sums of non-identically distributed random variables with distributions of a special kind. Estimates of a non-asymptotic nature were obtained that are valid without additional assumptions, expressed in terms of the number of terms n, of the type of the condition $Q(\mathcal{L}(S_a), \tau) \geq n^{-A}$ assumed in the formulation of the so-called inverse principle in the Littlewood–Offord problem, introduced in works by Nguyen, Tao, and Vu [74–78]. The interrelation of these estimates was studied. It was shown in [71] that Arak's results have implications that can be interpreted as manifestations of the inverse principle for the Littlewood–Offord problem. Some of them have a nonempty intersection with the results of Nguyen, Tao, and Vu, in which the arithmetic structure of the coefficients $a_1, \ldots, a_n$ under the condition $Q(F_a, \tau) \geq n^{-A}$, where A is a positive constant, is discussed.

In addition, monograph [61] contains some structural results that imply assertions apparently new in the Littlewood–Offord problem and have no analogues in the literature.

Another disciple of I. A. Ibragimov, A. Yu. Zaitsev, at the beginning of his research activity, was engaged in solving the second problem formulated by Kolmogorov in [59]. He succeeded in obtaining an estimate of the order of accuracy of infinitely divisible approximation of the distributions of sums of independent random variables whose distributions are concentrated on an interval of small length τ up to a small probability p. It turned out that the accuracy of the approximation in the Lévy metric is of the order $p + \tau\log(1/\tau)$, which is much more accurate than Kolmogorov's original result $p^{1/5} + \tau^{1/2}\log(1/\tau)$ (see [59]) and the later results of other authors (see, e.g., [60]). For the approximation, the so-called accompanying infinitely divisible distributions were used. Moreover, as shown by T. Arak, the estimate proved to be order correct. These results can be found in [79] and in monograph [61]. Later, in [80], it was shown that a similar estimate holds also in the multidimensional case and, instead of an absolute constant, the estimate acquires a factor $c(d)$ depending only on the dimension d. In the course of the proof, it was established that, for $p = 0$ (i.e., if the norms of the terms are bounded by a constant τ with probability one), for any fixed $\lambda > 0$, the random vector X, which has the same distribution as the sum under consideration, can be constructed on the same probabilistic space with the corresponding Gaussian vector Y such that

$$\mathbf{P}(\|X - Y\| > \lambda) \leq c_1 d^2 \exp\{-\lambda/c_2 d^2 \tau\} \tag{17}$$

(see [81, 82]). Moreover, A. Yu. Zaitsev [82] proved that the same result holds for vectors with the distributions from the class $\mathcal{A}_d(\tau)$—which he introduced—of distributions with sufficiently slowly increasing semi-invariants.

The class $\mathcal{A}_d(\tau)$ (with a fixed $\tau \geq 0$) consists of d-dimensional distributions F for which function

$$g(z) = g(F, z) = \log \int_{\mathbf{R}^d} e^{\langle z, x\rangle} F\{dx\} \quad (g(0) = 0) \tag{18}$$

is defined and analytic for $\|z\|\tau < 1$, $z \in \mathbf{C}^d$, and

$$\left| d_u d_v^2 g(z) \right| \leq \|u\| \tau \langle \mathbb{D} v, v \rangle \tag{19}$$

for all $u, v \in \mathbf{R}^d$ and $\|z\|\tau < 1$, where $\mathbb{D} = \operatorname{cov} F$ and $d_u g$ is the derivative of the function g in the direction u.

The class $\mathcal{A}_d(\tau)$ is closed with respect to convolution and contains, in particular, all possible convolutions of distributions concentrated on a ball of radius $c\tau$ centered at zero. It also contains arbitrary infinitely divisible distributions with spectral measures concentrated on the same ball. Applying a linear operator $\mathbb{A} : \mathbf{R}^d \to \mathbf{R}^m$ to a random vector with distribution from the class $\mathcal{A}_d(\tau)$, we obtain a vector with distribution from the class $\mathcal{A}_m(\|\mathbb{A}\|\tau)$. If some d-dimensional random vector ξ has finite exponential moments $\mathbf{E}e^{\langle h, \xi\rangle} < \infty$ for all $h \in V$, where $V \subset \mathbf{R}^d$ is some neighborhood of zero, then $F = \mathcal{L}(\xi) \in \mathcal{A}_d(c(F))$.

Inequality (17) implies known results for the probabilities of large deviations. Assuming that the independent identically distributed random variables $X_1, X_2, \ldots$ with finite exponential moments have zero means and unit variances, it is easy to deduce from (17) that

$$\frac{\mathbf{P}(n^{-1/2}(X_1 + \ldots + X_n) > x)}{\mathbf{P}(\eta > x)} \to 1 \quad \text{as} \quad n \to \infty \tag{20}$$

if $0 < x = x_n = o(n^{1/6})$ (see (4)). Here, η is a standard normal random variable. Thus, inequality (17) can be interpreted as a simple formulation of the multidimensional analogue of relation (9).

Another important special case for estimating the accuracy of infinitely divisible approximation is obtained for $\tau = 0$, when the right-hand side of the estimate of the uniform distance between the distribution function, $\rho_d(\cdot, \cdot)$, has the form $c(d)p$ (see [80, 83]). In [84], this result was interpreted as a general estimate of the accuracy of the approximation of a sample composed of non-identically distributed rare events of the general form by a Poisson point process. For any measurable function $f: \mathscr{X} \to \mathbf{R}^d$, we have the following inequality:

$$\rho_d\left(\mathscr{L}\left(\sum_j f(X_j)\right), \mathscr{L}\left(\sum_k f(Y_k)\right)\right) \le c(d)p.$$

Here, $\mathscr{X}$ is the space of rare events, X_j are independent rare events occurring with probabilities not exceeding p, and Y_k are points of the corresponding Poisson point process.

Some optimal estimates were obtained in other papers for the uniform distance $\rho(\cdot, \cdot)$ in the general case. In particular, in [85, 86], A. Yu. Zaitsev succeeded in strengthening the results of [60, 87] and obtaining simple one-dimensional formulation of the results, which imply order correct estimates of the infinitely divisible approximation of convolutions by the accompanying laws and very general estimates in the central limit theorem. Since the "tails" of the distributions of terms are arbitrary, the results also cover the case—actively studied in recent years—of "heavy tails" of distributions of summands. In [88], the result of Le Cam [87] on the approximation of convolutions of one-dimensional probability distributions F_j by convolutions of infinitely divisible distributions $e(F_j)$ was refined.

In proving these results, A. Yu. Zaitsev used the methods used by Arak in proving inequality (2). He managed to modify these methods, adapting them to the multidimensional case (see [89–91]). In particular, in [90], a multidimensional analogue of inequality (2) was obtained. By similar methods, the following paradoxical result was obtained (see [92–94]). There exists a variable $c(d)$ depending only on the dimension d and such that, for any symmetric distribution F and any natural number n, the uniform distance between powers in the sense of convolution F^n admits the estimates $\rho_d(F^n, F^{n+1}) \le c(d)n^{-1/2}$ and $\rho_d(F^n, F^{n+2}) \le c(d)n^{-1}$, and both estimates are unimprovable with respect to order. In [93], A. Yu. Zaitsev also succeeded in removing the logarithmic factor in inequality (1).

Arak's methods were used by V. Čekanavičius in his studies on infinitely divisible approximation of convolutions of one-dimensional probability distributions. Recently, he published a monograph [95] presenting the results and the methods.

In [96], A. Yu. Zaitsev also succeeded in giving a negative answer to Kolmogorov and Yu. V. Prokhorov's question on the possibility of infinitely divisible approximation of the distributions of sums of independent identically distributed random variables in terms of the variation distance. A one-dimensional probability distribution was constructed whose all n-fold convolutions are uniformly separated from the set of infinitely divisible laws in terms of the distance in variation by at least 1/14.

The accuracy of strong Gaussian approximation for sums of independent random vectors is usually estimated in two different but closely related situations. The estimation of the accuracy of strong approximation in the invariance principle can be reduced to these problems.

One of these can be formulated as follows. It is required to construct—on the same probability space—independent random vectors $X_1, \ldots, X_n$ (with given, generally non-identical distributions, $\mathbf{E}X_j = 0$ and $\mathbf{E}\|X_j\|^2 < \infty$) and independent Gaussian random vectors $Y_1, \ldots, Y_n$ in such a way that $\mathbf{E}Y_j = 0$, $\operatorname{cov}Y_j = \operatorname{cov}X_j$, $j = 1, \ldots, n$, and such that the quantity

$$\Delta_n(X,Y) = \max_{1\le s\le n}\left\|\sum_{j=1}^{s} X_j - \sum_{j=1}^{s} Y_j\right\| \tag{21}$$

be small with a sufficiently high probability.

In the second problem, it is required to construct—on the same probability space—a sequence of independent identically distributed random vectors $X_1, X_2, \ldots$ (with a given distribution $\mathscr{L}(X)$ with zero mean and $\mathbf{E}\|X\|^2 < \infty$) and a sequence of independent Gaussian random vectors $Y_1, Y_2, \ldots$ such that

$$\mathscr{L}(X_j) = \mathscr{L}(X), \quad \mathbf{E}Y_j = 0, \quad \operatorname{cov}Y_j = \operatorname{cov}X, \quad j = 1,2,\ldots, \tag{22}$$

and

$$\left\|\sum_{j=1}^{n} X_j - \sum_{j=1}^{n} Y_j\right\| = O(f(n)) \quad \text{or} \quad o(f(n)) \text{ a.s.}$$

as $n \to \infty$ with a sequence $f(n)$ tending to infinity as slowly as possible.

The most significant result obtained by A. Yu. Zaitsev in the 1990s is the multidimensional version of the classical one-dimensional result of Komlós, Major, and Tusnády [97] on estimating the accuracy of strong Gaussian approximation of sums of independent identically distributed random variables under the existence of exponential moments of the summands. By analogy with the one-dimensional result of A. I. Sakhanenko [98], the result of [99] is formulated as an estimate of the exponential moment of the quantity $\Delta_n(X, Y)$. In this case, the dependence of the constants on the dimension and the distributions of the summands is given explicitly (see [98–100]). This dependence is formulated in terms of the membership of the distributions of the summands in the class $\mathcal{A}_d(\tau)$ defined above. In [99], A. Yu. Zaitsev succeeded in getting rid of the unnecessary logarithmic factor in the result of Einmahl [100]. In the second problem, this corresponds to the estimate of the order $O(\log n)$ (instead of $O(\log^2 n)$) for vectors with finite exponential moments, obtained in the one-dimensional case in [97]. Somewhat later, A. Yu. Zaitsev [101] transferred the result to the case of non-identically distributed terms and obtained a complete multidimensional analogue of the one-dimensional result of A. I. Sakhanenko [98], who generalized and refined the results of [97]. In 2002, these results were reported by A. Yu. Zaitsev in an invited talk at the International Mathematical Congress in Beijing [102].

At the end of the last decade, in [103–105], estimates were studied of the accuracy of the strong Gaussian approximation of the sums of independent d-dimensional random vectors X_j with finite moments of the form $\mathbf{E}H(\|X_j\|)$, where H is a monotonic function increasing not slower than x^2 and not faster than $\exp(cx)$. Multidimensional generalizations of the results of Komlós, Major, and Tusnády [97] and Sakhanenko [106] were obtained. In particular, for the second problem in [105], an estimate of the form $O(H^{-1}(n))$ was obtained, where H^{-1} is the inverse of the function H. This refines the multidimensional result of Einmahl [100], who proved the same assertion for a narrower class of functions H.

When considering the first problem in a special case of $H(x) = c^{\gamma}$, $\gamma > 2$, in the joint papers [107] and [108], F. Götze and A. Yu. Zaitsev obtained estimates that proved to be order optimal for identically distributed summands. In the case where $X_1, \ldots, X_n$ are d-dimensional independent random vectors identically distributed with a random vector X with the standard unit covariance operator $\operatorname{cov} X = \mathbb{I}_d$, it was shown in [108] that there exists a construction ensuring the inequality

$$\mathbf{E}(\Delta_n(X,Y))^{\gamma} \leq c(\gamma) A n \mathbf{E}\|X\|^{\gamma} \quad \text{for all} \quad n = 1, 2, \ldots, \tag{23}$$

where

$$A = A(\gamma, d) = \max\{d^{11\gamma}, d^{\gamma(\gamma+2)/4}(\log d + 1)^{\gamma(\gamma+1)/2}\}. \tag{24}$$

This assertion is a multidimensional version of the result of A. I. Sakhanenko [106] for a special case of identically distributed summands. In the general case, A. I. Sakhanenko [106] proved that, for $d = 1$, there is a construction for the first problem such that

$$\mathbf{E}(\Delta_n(X,Y))^{\gamma} \leq c\gamma^{2\gamma} \sum_{j=1}^{n} \mathbf{E}\|X_j\|^{\gamma}. \tag{25}$$

In [107], it was established that, in the multidimensional case, the analogous assertion holds for non-identically distributed summands but under the additional conditions that the covariance operators of partial sums are not degenerate and the partial sums of order γ moments of the norms of the summands are regular. With the help of the results of [107], the infinite-dimensional case was considered in [109] and [110].

On the basis of the above results on the strong Gaussian approximation, review [111] was published in the journal "Russian Mathematical Surveys".

In [112], for any $\varepsilon > 0$, A. Yu. Zaitsev constructed two-dimensional distributions such that the distance in variation between their projections on an arbitrary one-dimensional direction does not exceed ε, although the uniform distance between the corresponding two-dimensional distribution functions is 1/2. This shows the instability of the inversion of the Radon transform of multidimensional probability distributions. There are distributions that are almost indistinguishable by tomography methods and, at the same time, very distant from one another.

Under the assumption that the independent identically distributed d-dimensional random summands $X, X_1, X_2, \ldots$ have zero mathematical expectations and finite fourth-order moments, in joint works [113, 114], it was shown that, for sets bounded by second-order surfaces, the accuracy of approximation by short asymptotic expansions in the central limit theorem is of the order $O(1/n)$, where n is the number of summands, provided that the there are no less than five dimensions of the space. Earlier, similar statements were obtained in the joint work by F. Götze and V. Bentkus [115] under the assumption that the space dimension is at least nine. In [114], nine was replaced by five and no further reduction in the dimensions is possible. Estimates are uniform with respect to isometric operators participating in the definition of the surfaces. In [114], explicit simple expressions were obtained for the power-law dependence of the corresponding constants on the fourth-order moments and on the eigenvalues of the covariance operator of finite-dimensional summands. In particular, it was proven that, for $5 \le d < \infty$, we have the inequality

$$\sup_{x\in\mathbf{R}}\left|\mathbf{P}(n^{-1/2}\|X_1+\ldots+X_n\|<x)-\mathbf{P}(\|\eta\|<x)\right| \le c(d)\sigma^d(\det\mathbb{C})^{-1/2}\mathbf{E}\left\|\mathbb{C}^{-1/2}X\right\|^4/n.$$

Here, $\mathbb{C}$ is the covariance operator of the random vector X, $\sigma^2 = \mathbf{E}\|X\|^2$, and η is a centered Gaussian vector with the covariance operator $\mathbb{C}$. Note that the results of [113] are not overlapped by the results of [114]. The quantity $\sigma^d(\det\mathbb{C})^{-1/2}$ for $\sigma^2 = 1$ is replaced in [113] by a quantity that depends only on the five maximal eigenvalues of the operator $\mathbb{C}$.

In recent years, A. Yu. Zaitsev has published several joint works on estimating the concentration functions of the distributions of sums of independent random variables. In addition to the aforementioned work [71], in recent papers [116–118], the inequalities for estimating the concentration functions of the weighted sums of independent identically distributed random variables $S_a = \sum_{k=1}^n a_k X_k$ in the Littlewood–Offord problem from [119–121] and [122] were refined. These results reflect the dependence of the estimates on the arithmetic structure of the weight coefficients a_k and on the common distribution of the random variables X_k.

Esseen [123] showed that $Q(F^n, \lambda) = o(n^{-1/2})$ for fixed $\lambda > 0$ if and only if $\mathbf{E}Y^2 = \infty$ and $F = \mathscr{L}(Y)$. In [124, 125], quantitative refinements of this result were obtained.

In [126], the connection between the rate of decay of $Q(F^n, \lambda)$ and the assumptions on the existence of finite moments $\mathbf{E}\psi(Y)$ of the functions $\psi(Y)$ was studied. It was shown that no condition of the infiniteness of the moments can provide a decay of the concentration functions $Q(F^n, \lambda)$ which is much faster than $o(n^{-1/2})$.

6. ON A CLASS OF LIMIT DISTRIBUTIONS FOR NORMALIZED SUMS OF INDEPENDENT RANDOM VARIABLES

Let $\xi_1, \xi_2, \ldots$ be a sequence of mutually independent random variables, and

$$S_n = \frac{1}{B_n}\sum_{i=1}^n \xi_i - A_n, \quad n = 1, 2, \ldots,$$

be a sequence of normalized sums having—with an appropriate choice of the normalizing constants B_n ($B_n \to \infty$)—its own limit distribution. In the mid-1950s, Gnedenko [127] formulated the problem of characterizing the class of limit distributions of such sums when, among the laws of distribution of random variables ξ_n, there are no more than r different. Denote this class by P_r.

This problem aroused a keen interest from the very beginning, and a hypothesis was suggested that class P_r must coincide with the composition of stable distributions, which was facilitated by certain facts such as, e.g., the descriptions of P_1 and P_2 obtained by V. M. Zolotarev and V. S. Korolyuk [128]. However, further studies have shown that the hypothesis about the nature of P_r should be substantially refined.

The corresponding results are contained in the papers of A. A. Zinger [129, 130] in three theorems formulated there that allow one to describe the laws belonging to P_r with the help of the character of the spectral measures in their Lévy–Khinchin representation. Theorem 3 gives a condition under which the Gnedenko hypothesis is true. As follows from the formulation of Theorem 1, the laws of the class P_r are a special case of the laws of a more general nature, first studied by Yu. V. Linnik [131] in connection with the laws admitting identically distributed linear statistics in repeated samples.

7. ALMOST SURE LIMIT THEOREMS

An almost sure limit theorem (ASLT) is an assertion about the weak convergence with probability one (almost surely) of empirical measures generated by a sequence of random variables. This type of convergence was independently discovered by Schatte and Brosamler in 1988 and aroused great interest (see, e.g., review [132]). Here, we describe only the results obtained in St. Petersburg, where I. A. Ibragimov, M. A. Lifshits, and E. S. Stankevich [133–138] worked on this subject.

We begin with a sufficient condition for the ASLTs for sums of independent non-identically distributed random vectors [135].

Let $\{\xi_j\}$ be a sequence of independent random vectors taking values in a separable normed space $(\mathcal{X}, \|\cdot\|)$. Consider the normalized sums

$$\zeta_k = \frac{1}{B_k}\sum_{j=1}^{k}\xi_j - A_k, \quad B_k > 0, \quad A_k \in \mathcal{X}, \quad k \geq 1,$$

and assume that they satisfy the limit theorem

$$\zeta_k \Rightarrow G, \quad k \to \infty, \tag{26}$$

with some limit distribution law G in $\mathcal{X}$.

Define empirical measures

$$Q_n = \frac{1}{\gamma_n}\sum_{k=1}^{n} b_n \delta_{\zeta_k}, \tag{27}$$

where $\{b_k\}$ is a positive bounded sequence satisfying the condition

$$b_k \leq \log(B_k/B_{k-1}), \quad k \geq 2,$$

and $\gamma_n = \sum_{k=1}^{n} b_k$. For $u \geq 1$ and a function $H: [1, \infty) \to (0, \infty]$, set

$$M_H(u) = \sup_{n:\, u<\gamma_n<H(u)} \mathbf{P}\{\log\|\zeta_n\| > u\}.$$

Then we have the following theorem on the almost sure convergence.

Theorem 7.1. *Suppose that* $B_n \nearrow \infty$ *and condition* (26) *is satisfied. If there exists a function* H *such that*

$$\int_1^\infty \frac{du}{H(u)} < \infty \quad \textit{and} \quad \int_1^\infty \frac{M_H(u)du}{u} < \infty,$$

then $\mathbf{P}\{Q_n \Rightarrow G\} = 1$.

This result holds for any sequence of random vectors ζ_k admitting the representation

$$\zeta_k = \frac{B_\ell}{B_k}\pi_{k,\ell}(\zeta_\ell) + \eta_{k,\ell}$$

for all $\ell \leq k$, where ζ_ℓ and $\eta_{k,\ell}$ are independent and the family of linear operators $\{\pi_{k,\ell} : \mathcal{X} \mapsto \mathcal{X}\}$ is uniformly bounded. In this form, it can be applied, e.g., to sample trajectories of the partial sum processes arising in the classical invariance principle.

It should be noted that, later, this type of results was developed in the work by Berkes and Csáki [139], who considered the nonlinear functionals of sequences of independent variables.

Further on, in [135], the delicate difference between the central limit theorem (CLT) and its almost sure analogue (ASCLT) was studied. Berkes and Dehling [140] established the equivalence of the CLT and the ASCLT under the assumption

$$\sup_n \mathbf{E}(\log_+ \log_+ |\zeta_n|)^{1+h} < \infty$$

with an arbitrarily small $h > 0$. The following result shows that the parameter h in this statement cannot be excluded.

Theorem 7.2. *Let* $a > 1$, $\alpha \in \left(\frac{1}{2}, \frac{a}{2}\right)$, *and* $\xi_j = Y_j + X_j$, *where* $\{Y_j, X_j\}$ *are independent collectionwise*, Y_j *are standard normal variables*, *and* $X_j = 0$ *for all* x_j *except for the sequence* $n_m = [\exp\{a^m\}]$. *Let* $X_{n_m} = \pm U_m$ *with equal probabilities* $\frac{1}{2m}$ *and an amplitude* $U_m = n_m^{\alpha}$. Set $X_{n_m} = 0$ *with the residual probability* $1 - \frac{1}{m}$.

Then, *the normalized sums*

$$\zeta_n = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \xi_j$$

satisfy the condition

$$\sup_n \mathbf{E} \log_+ \log_+ |\zeta_n| < \infty$$

and the central limit theorem $\zeta_n \Rightarrow \mathcal{N}(0, 1)$ *but the almost sure limit theorem for them is not satisfied*.

In [138], the applicability of ASCLT to the normalized values of martingales with discrete time was studied. It turns out that, unlike the sums of independent variables, even under relatively strong moment conditions, the ASCLT does not follow from the classical limit theorem. Nevertheless, the ASCLT can be proven under conditions fairly close to those under which the classical limit theorems are proven. It is only necessary to make sure that the increments of the martingale be asymptotically smaller than the normalizing sequence. It is interesting that, in the limit, random limiting distributions G can arise.

Let $\{X_j\}$ be a square-integrable martingale difference with zero expectation and $B_k \nearrow \infty$ be a sequence of positive numbers. Set $\zeta_k = \frac{1}{B_k} \sum_{j \le k} X_j$. Define the empirical measures Q_n by relation (27) and assume that the corresponding weights $\{b_k\}$ satisfy the condition

$$b_k \le \frac{B_k - B_{k-1}}{B_k}.$$

Let $\tilde{Q}_n$ be analogous empirical measures obtained by replacing ζ_n by the self-normalized sums $\frac{1}{U_k} \sum_{j \le k} X_j$, where $U_k^2 = \sum_{j \le k} X_j^2$.

Finally, if η is a nonnegative random variable, then denote by $\mathcal{N}_\eta$ the η-mixture of normal laws, i.e., a distribution with the characteristic function $\phi(t) = \mathbf{E}\exp\{-\eta t^2/2\}$.

Theorem 7.3. *Let* X_j, B_k, ζ_k, U_k, *and* b_k *be defined as above and be supposed to satisfy the condition*

$$\sup_k \mathbf{E} \max_{j \le k} |X_j|^2 / B_k^2 < \infty.$$

Suppose that the limit relations $X_j/B_j \to 0$ *and*

$$\frac{1}{\gamma_n} \sum_{k=1}^{n} b_k \delta_{U_k^2/B_k^2} \Rightarrow \delta_\eta$$

are satisfied a.s.

Then, *almost surely*, $Q_n \Rightarrow \mathcal{N}(0, \eta)$ *and* $\tilde{Q}_n \Rightarrow \mathcal{N}(0, 1)$.

Some results were obtained for the classical statement of the problem that appears as follows. Let $\{S_k\}$ be partial sums of a sequence of independent identically distributed random variables with zero mean and a unit variance: $X_1, X_2, \ldots$ Denote $\zeta_k = \frac{1}{\sqrt{k}} S_k$. Define the corresponding empirical measures as

$$Q_n = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{\zeta_k}. \tag{28}$$

The ASCLT states that $\mathbf{P}\{Q_n \Rightarrow \mathcal{N}\} = 1$, where $\mathcal{N}$ is the standard normal distribution. Therefore, for any continuous bounded function h, we almost surely have

$$\int h dQ_n = \frac{1}{\log n}\sum_{k=1}^{n}\frac{1}{k}h(\zeta_k) \to \int h d\mathcal{N}.$$

The following theorem from [136] shows that this assertion holds for an unbounded function, provided that its increase is subject to a minimal condition.

Theorem 7.4. *Let* $\{X_j\}$ *be a sequence of independent identically distributed variables with zero mean and a unit variance. Let* Q_n *be empirical measures from* (28) *and* $\mathcal{N}$ *be the standard normal distribution. Let* A, $H_0 > 0$ *and the function* $f: [A, \infty) \mapsto \mathbf{R}_+$ *be nondecreasing, the function* $x \mapsto f(x)\exp\{-H_0x^2\}$ *be nonincreasing, and*

$$\int_A^{\infty} f d\mathcal{N} < \infty. \tag{29}$$

Then, for any continuous functions h satisfying the estimate

$$|h(x)| \le f(|x|), \quad |x| \ge A,$$

we have

$$\mathbf{P}\left\{\lim_{n\to\infty}\int h dQ_n = \int h d\mathcal{N}\right\} = 1.$$

The assertion of the theorem, which is naturally interpreted as the convergence of generalized moments, will become false if we remove the assumptions on the regularity and leave only basic condition (29).

For the same classical ASCLT scheme, the necessary conditions for the principle of large deviations were found in [137]. As shown by Heck [141] and P. March and T. Seppäläinen [142], if $\mathbf{E}|X_1|^m < \infty$ for all $m > 0$, then the measures Q_n satisfy the strong principle of large deviations in the space $\mathcal{M}(\mathbf{R})$ of finite non-negative measures equipped with the topology of weak convergence. In other words, for all closed sets $F \subset \mathcal{M}(\mathbf{R})$ and all open sets $G \subset \mathcal{M}(\mathbf{R})$, we have the relations

$$\limsup_{n\to\infty}\frac{1}{\log n}\log \mathbf{P}\{Q_n \in F\} \le -\inf_{\mu\in F} I(\mu),$$

$$\liminf_{n\to\infty}\frac{1}{\log n}\log \mathbf{P}\{Q_n \in G\} \ge -\inf_{\mu\in G} I(\mu).$$

Here, the deviation function I for probability distributions is equal to the entropy of the Ornstein–Uhlenbeck process I_{OU} and is equal to infinity on the remaining measures. It should be noted that $I(\mu) = 0$ is equivalent to $\mu = \mathcal{N}(0, 1)$.

It appears that, in this statement, the moment conditions are optimal (although, in the classical principle of large deviations for empirical measures, moment restrictions are not required at all).

Theorem 7.5. *Let a sequence of empirical measures* Q_n *corresponding to independent identically distributed variables* $\{X_j\}$ *satisfy the strong principle of large deviations with a deviation function I such that the level set* $\{\mu : I(\mu) \le r\}$ *is compact for each* $r \ge 0$. *Then,* $\mathbf{E}|X_j|^m < \infty$ *at all* $m > 0$.

An interesting nonstandard approach to the almost sure limit theorems was proposed in A. I. Martkainen's work [143].

8. CONCLUSIONS

Thus, the Leningrad–St. Petersburg school of probability and statistics has made a significant contribution to the theory of summation of independent random variables. The achievements of its representatives in other fields of probability and statistics will be outlined in subsequent articles of this series.

ACKNOWLEDGMENTS

This work was supported in part by the Russian Foundation for Basic Research (grants 16-01-00237, 16-01-00258, and 16-01-00367), St. Petersburg State University (grant SPbSU-DFG 6.65.37.2017), and Program no. 1 of the Presidium of the Russian Academy of Sciences (grant no. PRAS-18-01).

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Translated by E. Chernokozhin