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Acoustic scattering by a semi-infinite angular sector with impedance boundary conditions

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In this work we study the problem of diffraction of an acoustic plane wave by a semi-infinite angular sector with impedance boundary conditions on its surface. It is studied by means of incomplete separation of variables. With the aid of Watson–Bessel integral representation the problem is reduced to a boundary value problem on the unit sphere with an operator-impedance boundary condition on a cut of the sphere. The latter problem is further studied by means of the traditional methods of extensions of sectorial sesquilinear forms. The Sommerfeld integral representation is obtained from that of Watson–Bessel with the aim to develop the far-field asymptotics. Analytic properties of the corresponding Sommerfeld transformant are also discussed. For a narrow impedance sector, an asymptotic formula for the diffraction coefficient of the spherical wave propagating from the vertex is derived.

Keywords: diffraction by an impedance sector; integral representations; narrow cone; diffraction coefficient.

1. Introduction

1.1. *Motivation and some comments on the literature*

To our knowledge, until present time the problem of diffraction by an impedance sector has not been discussed in the literature. It seems that the reason is in the analytic difficulties arising in the study of the problem in spite of the fact that the impedance boundary conditions are more realistic in practice in comparison with those ideal ones. We modify and adapt the approach recently developed by Lyalinov (2013), for the case of the sector with Dirichlet boundary conditions, in order to describe the far-field asymptotics in the problem of diffraction by a plane angular sector with impedance boundary conditions on its surface.

Consider the unit sphere with the centre at the sector's vertex then the sector and the sphere are intersected across a segment ('cut') AB of a big circle, Fig. 1. We assume that the angular measure $2a$ of the corresponding arc satisfies the restrictions $0 < 2a < \pi$. One of the most interesting cases is the quarter-plane corresponding to $2a = \pi/2$.

In many aspects our approach has common features with those used for the problems of diffraction by cones with ideal (see e.g. Bowman *et al.*, 1987; Felsen & Marcuvitz, 1973; Borovikov, 1966; Jones, 1964; Jones, 1997; Cheeger & Taylor, 1982; Smyshlyaev, 1991; Smyshlyaev, 1990; Babich *et al.*, 2000; Bonner *et al.*, 2005) or impedance boundary conditions. Diffraction by an impedance cone is studied in the papers of Bernard (1997), Bernard & Lyalinov (2001), and Bernard *et al.* (2008). Some additional results and references can be found in Lyalinov *et al.* (2010) and Lyalinov & Zhu (2012). As regards papers on diffraction by a sector with the ideal boundary conditions reader might

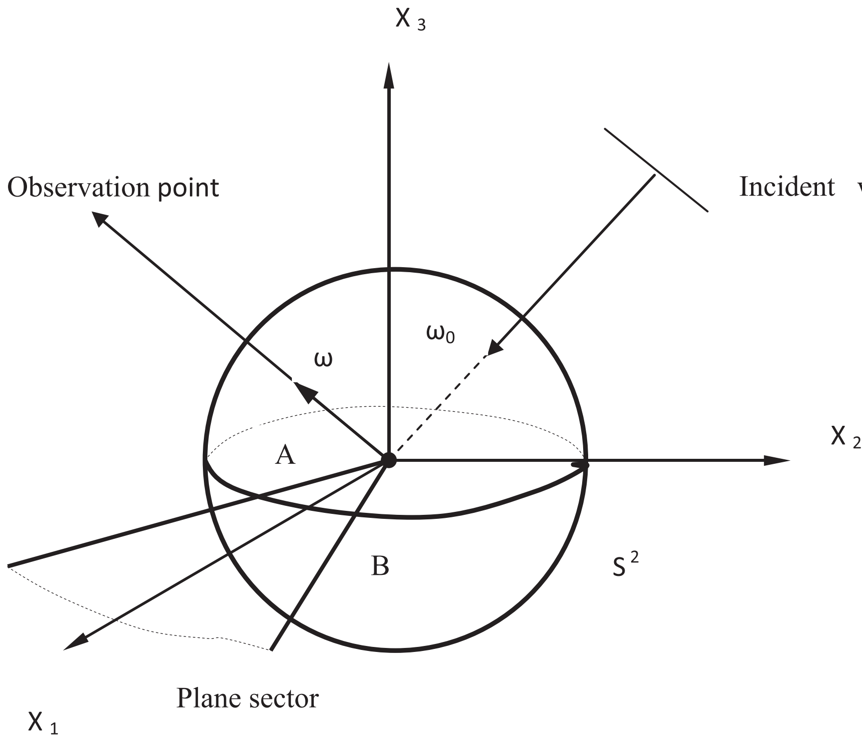

FIG. 1. Diffraction by an impedance sector.

be forwarded to discussion of the literature in our work (Lyalinov, 2013, see also Kraus & Levine, 1961; Hansen, 1991; Meister & Speck, 1988; Radlow, 1961; Albani, 2007; Valyaev & Shanin, 2012; Abawi *et al.*, 1997; Assier & Peake, 2012a; Assier & Peake, 2012b; Budaev & Bogy, 2005).

It is worth noting that the far-field asymptotics is of basic importance in numerous applications of such canonical problem like diffraction by an impedance sector. However, in order to develop the corresponding asymptotic expressions it is necessary to obtain the corresponding analytic tools, integral representations, to justify them and give a constructive way to obtain a solution and then to develop efficient numerical procedures. In this work we pay much attention to such tools implying that a detailed study of the far-field asymptotics will be discussed in a forthcoming publication. Nevertheless, we demonstrate the efficiency of the developed formulae by deducing a practically useful expression for the diffraction coefficient of the spherical wave from the vertex of a narrow impedance sector.

In the modern interpretation a diffraction problem might be considered satisfactorily solved provided the following conditions are fulfilled. First, it is demonstrated that it has a unique solution and all necessary representations (integral or others) for the solution are established. Second, the far-field asymptotics, including those in the transition zones, are derived from such representations and the expressions for the diffraction coefficients are obtained. Finally, the numerical elaboration of the solution is given and the numerical results for the far field are represented. From this point of view, for instance, the problem of diffraction by a right-circular cone with ideal boundary conditions can be considered solved. As regard the problem of diffraction by an impedance sector, the study represented in this work basically deals with the first step of the solution in the above-mentioned sense. On the other hand, a formula for the diffraction coefficient in the so-called oasis (defined below) is derived and its asymptotic expression for a narrow impedance sector is given.

1.2. *Description of the approach*

In the following sections we formulate the problem of diffraction by an impedance sector, Fig. 1. The wave field satisfies the Helmholtz equation, the impedance boundary conditions, Meixner's conditions at the edges and at the vertex. Provided the scattered wave satisfies the radiation condition at infinity, the classical solution of the problem is unique. However, in the case of incidence of a plane wave the scattered field does not satisfy the radiation condition and we then discuss asymptotic behaviour of the wave field at infinity. We introduce a set of characteristic domains of the unit sphere centred at the vertex of the sector. Each point of such a domain on the sphere is attributed to some direction of observation. In different characteristic domains the far-field asymptotics consists of different wave field components.

In order to separate the radial variable we make use of the Watson–Bessel integral representation for the solution and formulate a problem for the unknown 'spectral' function on the unit sphere with the cut $\overline{AB}$. An elliptic (Helmholtz type) equation for the spectral function depends on the variable of separation ν, whereas the boundary condition on the sides of the cut $\overline{AB}$ is nonlocal with respect to ν and has a form of the mixed boundary condition with an operator-impedance in it. We then carefully study such a problem attributing an m-sectorial operator to it. This operator is introduced by means of the corresponding sesquilinear form admitting a closed extension.[1] Then the meromorphic continuation for spectral function w.r.t. ν is discussed. The latter exploits the corresponding Green's theorem on the unit sphere with the cut and leads to an integral equation of the second kind for the spectral function on the cut. The Sommerfeld integral representation for the wave field is derived from that of Watson–Bessel and the properties of the Sommerfeld transformant (an analytic function in the integrand) are discussed. In particular, domain of regularity of the transformant is described, which enables one to point out domains on the complex plane, where singularities of the Sommerfeld transformant are located. The singularities of the integrand give rise to different far-field components provided the Sommerfeld integral is evaluated by means of the saddle point technique. Indeed, some of these singularities may be captured in the process of deformation of the Sommerfeld double loop contour into the steepest descent paths (Lyalinov, 2013). The contribution of the saddle points is responsible for the spherical wave from the vertex. Remark that, provided the observation point is in the space region, where only the spherical wave from the vertex is observed in the scattered far field (in the so-called 'oasis'), these singularities are not intersected. For a narrow sector ($2a \ll 1$) an approximate asymptotic expression for the diffraction coefficient of the spherical wave from the vertex in the oasis is deduced by means of the above-mentioned results.

2. Formulation of the problem

Let us use the spherical coordinates (r, ϑ, φ) attributed to the Cartesian ones by the correlations

$$X_1 = r\cos\varphi\sin\vartheta,\ X_2 = r\sin\varphi\sin\vartheta,\ X_3 = r\cos\vartheta.$$

[1] It is worth remarking that in our study the use of closable sectorial forms and m-sectorial operators looks natural for the problem on the unit sphere with the cut. It has been partly inspired by a very useful work by Assier *et al.* (2016), where similar but simpler spectral problems for the Laplace–Beltrami operator with ideal boundary conditions on the cut have been considered.

We consider a plane wave[2] which is incident from the direction specified by $\omega_0 = (\vartheta_0, \varphi_0)$ (Fig. 1)

$$U_i(r, \vartheta, \varphi) = \exp\{-\mathrm{i}kr\cos\theta_i(\omega, \omega_0)\}, \tag{2.1}$$

where $\omega = (\vartheta, \varphi)$ corresponds to the direction of observation and

$$\cos\theta_i(\omega, \omega_0) = \cos\vartheta\cos\vartheta_0 + \sin\vartheta\sin\vartheta_0\cos[\varphi - \varphi_0],$$

$\theta_i(\omega, \omega_0)$ coincides with the geodesic distance between two points ω and ω_0 denoted also $\theta(\omega, \omega_0)$, $\theta(\omega, \omega_0) = \theta_i(\omega, \omega_0)$.

The wave field $U(r, \vartheta, \varphi) + U_i(r, \vartheta, \varphi)$ is the sum of the scattered and incident fields, U fulfils the Helmholtz equation

$$(\triangle + k^2)U(r, \vartheta, \varphi) = 0, \tag{2.2}$$

$k > 0$ is the wave number. Let $\Sigma = S^2 \setminus \overline{AB}$ be the exterior of the cut on S^2 and $\sigma = \partial\Sigma$, $\sigma = S \cap S^2$ its boundary, $\sigma = \sigma_+ \cup \sigma_-$ and $\sigma_\pm$ are two sides of the cut AB. The impedance boundary condition

$$\frac{1}{r}\frac{\partial(U_i + U)}{\partial\mathscr{N}_\pm}\bigg|_{S_\pm} - \mathrm{i}k\eta_\pm(U_i + U)|_{S_\pm} = 0, \tag{2.3}$$

is satisfied on the sector S, $S_\pm$ are two faces of the sector $S = \{(r, \omega) : r \geqslant 0, \omega \in \overline{AB}\}$ corresponding to $\sigma_\pm$ on S^2. The vectors $\mathscr{N}_\pm$ are in the tangent plane to S^2 at the points of $\sigma_\pm$, are orthogonal to $\sigma_\pm$ and point out to the 'exterior' of Σ. The surface impedances $\eta_\pm = \varepsilon_\pm + \mathrm{i}\chi_\pm$ do not depend on k, $\varepsilon_\pm > 0$, which means absorption of the wave energy on the sector's faces.[3] The case of reactive faces ($\varepsilon_\pm = 0$) can be considered as a limiting one.

The Meixner's edge conditions are assumed near the edges ∂S_i, $i = 1, 2$ (and outside some close vicinity of the vertex)

$$U \sim C^0_{1,2}(z) + C_{1,2}(z, \phi)\rho^{1/2} + \dots, \ \rho \to 0 \tag{2.4}$$

uniformly bounded with respect to z, ϕ, assuming that ρ, ϕ, z are natural local cylindrical coordinates attributed to the edges ∂S_i, $i = 1, 2$, where the index i is omitted for the coordinates. We connect notations A and B with the edges ∂S_1 and ∂S_2 correspondingly. The conditions at the vertex of the sector take the form

$$|U| \leqslant C\,r^{-1/2+\epsilon}, \ |\nabla U| \leqslant C\,r^{-3/2+\epsilon}, \ r \to 0 \tag{2.5}$$

which are valid uniformly with respect to the angular variables for some positive ε, see Section 5.7 in Van Bladel (1991).

[2] The harmonic time-dependence $\mathrm{e}^{-\mathrm{i}\widehat{\omega}t}$ is assumed and suppressed throughout the paper.

[3] The sign of $\chi_\pm$ is not fixed; however, in the case $\chi_\pm < 0$, $\varepsilon_\pm = 0$ surface waves are excited on the impedance surfaces of the sector and propagate to infinity without attenuation.

We are looking for a classical solution of the problem, i.e. $U \in C^2_{\mathrm{loc}}(R^3 \setminus \overline{S})$ and such that $\frac{1}{r}\frac{\partial U}{\mathcal{N}_\pm}$ and U exist and are continuous on $S_\pm$. Now we turn to the behaviour of the solution at infinity.

2.1. *Uniqueness*

Let us now assume that the scattered field satisfies the radiation condition (which is not the case for the incidence of a plane wave)

$$\int_{S_R \setminus S} |\partial_r U - \mathrm{i}kU|^2 \, \mathrm{d}s \;\to\; 0 \,, \quad R \;\to\; \infty \,, \;\; \varepsilon_\pm > 0 \,, \tag{2.6}$$

where $S_R = \{(r, \omega) : \; r = R \,, \; \omega \in S^2 \setminus \overline{\sigma}\}$, $\omega = (\vartheta, \varphi)$.

THEOREM 2.1 The classical solution of the homogeneous problem (2.2)–(2.6) (i.e. with $U_i = 0$) is trivial, $U \equiv 0$.

Consider a ball B_δ of small radius δ centred at the vertex O with the boundary S_δ, $\partial B_\delta := S_\delta$, Fig. 2. Also we introduce semi-infinite cylinders C^i_δ of the radius δ having the axis ∂S_i, $i = 1, 2$. We denote d_δ the set $B_\delta \cup C^1_\delta \cup C^2_\delta$ with the external boundary ∂d_δ, see Fig 2. The domain d_δ is a δ −neighbourhood of the edges $\partial S_1 \cup \partial S_2$. Let B_R be a ball of the large radius R, $\partial B_R := S_R$, then $D_{\delta,R} = B_R \setminus (S \cup d_\delta)$ is a part of this ball being exterior to S and to d_δ. Remark that the boundary of the domain $D_{\delta,R}$ is $\partial D_{\delta,R}$ consisting of $\partial d_{\delta,R} := \partial d_\delta \cap B_R$, $S^\pm_{\delta,R} := B_R \cap (S_\pm \setminus d_\delta)$ and $S_{\delta,R} := S_R \setminus (d_\delta \cup S)$.

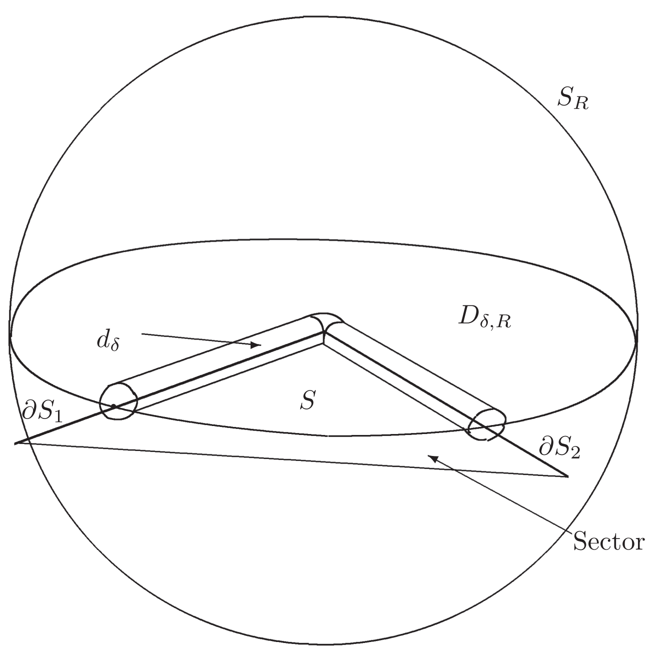

FIG. 2. Domains in the proof of uniqueness.

We apply the Green's identity

$$\int_{D_{\delta,R}} \triangle U\,\overline{U}\mathrm{d}x \;=\; \int_{\partial D_{\delta,R}} \partial_n U\,\overline{U}\,\mathrm{d}s \;-\; \int_{D_{\delta,R}} |\nabla U|^2\,\mathrm{d}x$$

to U and $\overline{U}$ in $D_{\delta,R}$, exploit the Helmholtz equation, boundary conditions and take the imaginary part. Simple calculations lead to

$$k\sum_{\pm} \Re(\eta_\pm) \int_{S^{\pm}_{\delta,R}} |U|^2\,\mathrm{d}s \;+\; \Im\left(\int_{\partial d_{\delta,R}} \partial_n U\,\overline{U}\,\mathrm{d}s\right) \;+\; \Im\left(\int_{S_{\delta,R}} \partial_r U\,\overline{U}\,\mathrm{d}s\right) \;=\; 0,$$

where ∂_n is the differentiation along the normal directed outward w.r.t. $D_{\delta,R}$. Let $\delta \to 0$, we apply the Meixner's conditions at the edges and vertex. The integral over $\partial d_{\delta,R}$ vanishes. We then verify the chain of inequalities

$$\begin{aligned} 0 \leqslant k\sum_{\pm} \varepsilon_\pm \int_{S^{\pm}_{0,R}} |U|^2\,\mathrm{d}s \;&=\; -\Im\left(\int_{S_R\setminus S} \partial_r U\,\overline{U}\,\mathrm{d}s\right) \\ &=\; -\Im\left(\int_{S_R\setminus S} (\partial_r U \;-\; \mathrm{i}kU)\,\overline{U}\,\mathrm{d}s\right) - k\int_{S_R\setminus S} |U|^2\,\mathrm{d}s \\ &\leqslant \left(\int_{S_R\setminus S} |\partial_r U \;-\; \mathrm{i}kU|^{\,2}\,\mathrm{d}s\right)^{1/2} \left(\int_{S_R\setminus S} |U|^2\,\mathrm{d}s\right)^{1/2} - k\int_{S_R\setminus S} |U|^2\,\mathrm{d}s. \end{aligned} \tag{2.7}$$

The latter expression is non-negative and from the radiation condition we have

$$k\left(\int_{S_R\setminus S} |U|^2\,\mathrm{d}s\right)^{1/2} \leqslant \left(\int_{S_R\setminus S} |\partial_r U \;-\; \mathrm{i}kU|^{\,2}\,\mathrm{d}s\right)^{1/2} \;\to\; 0$$

as $R \to \infty$. We find that

$$k\lim_{R\to\infty} \int_{S_R\setminus S} |U|^2\,\mathrm{d}s \;=\; 0$$

and from (2.7)

$$\int_{S_\pm} |U|^2\,\mathrm{d}s \;=0,$$

$$U|_{S_\pm} \;=\; 0.$$

Making use of the boundary conditions, we find

$$\frac{1}{r}\frac{\partial U}{\partial \mathcal{N}_\pm}\bigg|_{S_\pm} \;=0.$$

As a result, we deal with the homogeneous Cauchy problem for an elliptic (Helmholtz) equation satisfied by U with the trivial initial conditions on the surface $S_{\pm}$. Such a solution is known to be trivial, $U \equiv 0$.

As we have already mentioned the radiation condition (2.6) is not valid for the scattered field excited by a plane incident wave. It must also be modified appropriately as $\varepsilon_+ = 0$ or $\varepsilon_- = 0$. In such cases surface waves can be excited and propagate along the sector's surface at infinity without attenuation.

2.2. *The far-field asymptotics*

In order to describe the asymptotic behaviour of the scattered field as $r \to \infty$ it is useful to define some sub-domains on the unit sphere S^2 with the cut $\sigma = \overline{AB}$. Let Ω_{δ_0} be a close vicinity of the cut on the unit sphere

$$\Omega_{\delta_0} = \left\{\omega \in S^2 : \ \operatorname{dist}(\omega, \sigma) < \delta_0\right\},$$

where δ_0 is some small positive constant. In what follows we assume that $\omega_0 \in S^2 \setminus \Omega_{\delta_0}$, i.e. the incident plane wave arrives from the directions being not very close to $\overline{AB}$.

Consider the geodesic distance $\theta(\omega, \omega_0)$ between two points ω and ω_0 on the sphere S^2 (see, Lyalinov, 2013, Section 2 for details). In the same manner, introduce the 'broken' geodesic $\theta_r(\omega, \omega_0)$. This geodesic has simple geometrical meaning: this is a broken geodesic of the minimal length which originates at the source ω_0 then reflects on the boundary $\sigma = \overline{AB}$ in accordance with geometrical optics laws and arrives at the point ω. The 'incident' parts of such broken geodesics fill in the spherical triangle $\omega_0 AB$, whereas the 'reflected' parts fill in the spherical triangle $F_r BA$ which is further denoted $\Omega_r =: \Omega_r(\omega_0)$, (see Figs 3–5).

Specify the spherical triangular domain Ω_r^* coinciding with the triangle ABF_i in Fig. 3. The domain Ω_r^* is the mirror image of Ω_r with respect to the boundary σ for the same fixed position of ω_0.

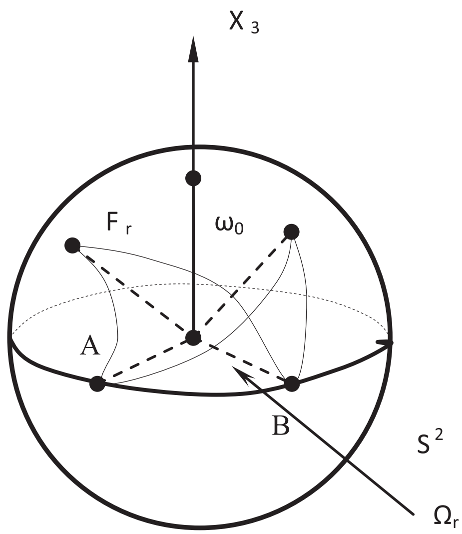

FIG. 3. The triangular domain Ω_r with the vertexes ABF_r on S^2.

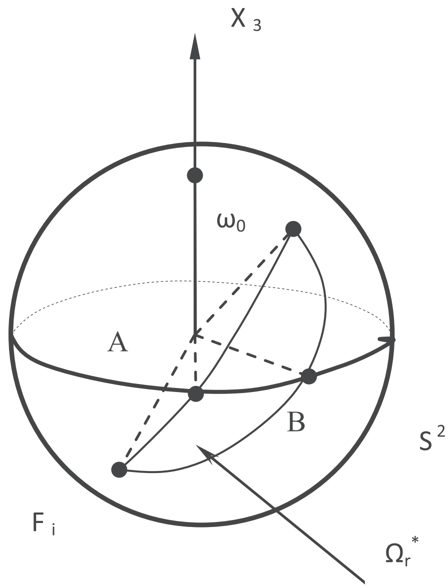

FIG. 4. The triangular domain Ω_r^* with the vertexes ABF_i on S^2.

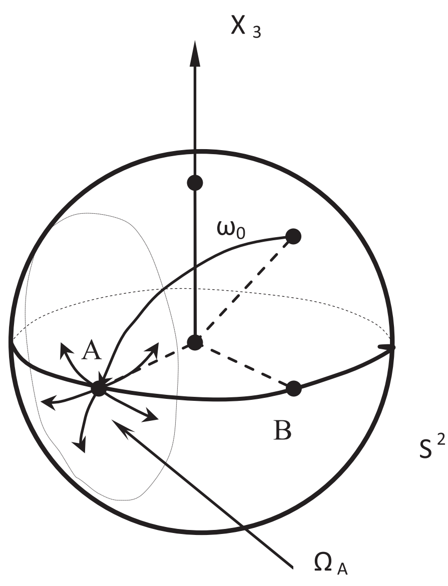

FIG. 5. The domain Ω_A on S^2.

We also make use of two other domains $\Omega_A(\omega_0)$ (see Fig. 5) and $\Omega_B(\omega_0)$. Consider the ray (geodesic) $\omega_0 A$ which arrives at the edge point A and produces a set of 'diffracted' rays (geodesics) outgoing from A in all direction. For each point ω there exists such a diffracted ray with length

$\psi_A(\omega)$, ($\psi_A(\omega) < \pi$) that arrives at this point. We define $\Omega_A(\omega_0)$ as a domain on the sphere such that $\Omega_A(\omega_0) = \{\omega \in S^2 :\ 0 \leqslant \theta_A(\omega, \omega_0) := \theta_i(A, \omega_0) + \psi_A(\omega) < \pi\}$, see also Section 2 in Lyalinov (2013).

The domains Ω_A and Ω_B on S^2 intersect with Ω_r and Ω_r^* and with each other. The domain Ω_r^* corresponds to directions in which geometrical shadow of the incident wave is observed. Ω_r forms a set of directions of propagation of the space rays reflected from the sector S. The domain Ω_A (or Ω_B) corresponds to the directions for which the diffracted cylindrical wave from the edge A (or from B) is present in the far-field asymptotics. These simple facts follow from the analysis also presented in Lyalinov (2013).

Remark It is worth noting that we can also introduce domain Ω_{AB} (and analogously Ω_{BA}), see Lyalinov (2013) as well as the domains $\Omega_{ABA}, \Omega_{BAB}, \Omega_{ABAB}\dots$, etc. which are defined and interpreted analogously. They correspond to multiple diffractions.

We introduce the domain $\Omega_0 = S^2 \setminus (\Omega_A \cup \Omega_B \cup \Omega_r)$ on the sphere S^2, which is called 'oasis'. The scattered far-field U (total minus incident) in this domain of directions consists of the spherical wave propagating from the vertex of the sector

$$U(r, \vartheta, \varphi) = D(\omega, \omega_0)\frac{\exp(\mathrm{i}kr)}{-\mathrm{i}kr}\left(1 + O\left(\frac{1}{kr}\right)\right), \quad kr \to \infty. \tag{2.8}$$

In the asymptotics (2.8), which is non-uniform with respect to $\omega \in \overline{\Omega}_0$, the diffraction coefficient $D(\omega, \omega_0)$ is one of the most important characteristics of the scattered field.[4]

In the exterior of the oasis the structure of asymptotics is more complex and contains also other wave components in the far field. Consider the directions from $(S^2 \setminus (\Omega_r^* \cup \Omega_r \cup \Omega_B)) \cap \Omega_A$ in which the spherical wave, the cylindrical wave from the edge A as well as surface waves (possibly also other multiply diffracted) are observed in the far field

$$U(r, \vartheta, \varphi) = D(\omega, \omega_0)\frac{\exp(\mathrm{i}kr)}{-\mathrm{i}kr}\left(1 + O\left(\frac{1}{kr}\right)\right)$$
$$+ d_A(\omega, \omega_0)\,\frac{\exp\left(-\mathrm{i}kr\cos\theta_A(\omega, \omega_0)\right)}{\sqrt{-\mathrm{i}kr\sin\psi_A}}\left(1 + O\left(\frac{1}{kr\sin\psi_A}\right)\right) + V_s(r, \vartheta, \varphi) + \dots, \tag{2.9}$$

where dots denote the waves multiply diffracted from the edges, provided the corresponding directions belong also to $\Omega_{BA}, \Omega_{ABA}, \dots$.[5] The yet unknown function $d_A(\omega, \omega_0)$ in (2.9) is connected with the diffraction coefficient of the cylindrical wave from the edge A. The summand $V_s(r, \vartheta, \varphi)$ is the sum of

[4] The asymptotics (2.8) fails provided the observation point approaches the boundary of Ω_0, near the boundaries of the domains $\Omega_i,\ i = 0, r, A, B, \dots$ some special transition functions apply to match the local asymptotics.

[5] Usually these waves are neglected in comparison with the first two terms because they have higher order with respect to $(kr)^{-1}$.

the surface waves generated by the interaction of the incident waves with the vertex or edges. It is only essential provided the observation point is in the close vicinity of the sector, $\omega \in \Omega_{\delta_0}$ and as $\varepsilon_\pm = 0$, $\chi_\pm < 0$,

$$V_s(r,\vartheta,\varphi) = D_s(\omega,\omega_0)\,\frac{\exp(-\mathrm{i}kr\cos\theta_s(\omega,\omega_0))}{\sqrt{-\mathrm{i}kr}}\left(1+O\left(\frac{1}{kr}\right)\right)$$
$$+ D_s^A(\omega,\omega_0)\,\exp\left(-\mathrm{i}kr\cos\theta_s^A(\omega,\omega_0)\right) + D_s^B(\omega,\omega_0)\,\exp\left(-\mathrm{i}kr\cos\theta_s^B(\omega,\omega_0)\right) + \ldots \tag{2.10}$$

with complex valued eikonals $\theta_s(\omega,\omega_0), \theta_s^{A,B}(\omega,\omega_0)$. These eikonals solve the equations

$$(\nabla_\omega\theta_s)^2 = 1, \quad \left(\nabla_\omega\theta_s^{A,B}\right)^2 = 1, \tag{2.11}$$

satisfying the conditions

$$\Im(\theta_s)|_\sigma = 0, \ \ \Im(\theta_s) > 0, \ \ 0 \leqslant \Re(\theta_s) < \pi,$$
$$\Im(\theta_s^{A,B})|_\sigma = 0, \ \ \Im(\theta_s^{A,B}) > 0 \ \ 0 \leqslant \Re(\theta_s^{A,B}) < \pi$$

as $\varepsilon_\pm = 0$, $\chi_\pm < 0$.

In the domain $(S^2 \setminus (\Omega_r^* \cup \Omega_r \cup \Omega_A)) \cap \Omega_B$ the asymptotics has the same form as in (2.9) with the change of the subscript A on to B in the second summand which describes the cylindrical wave from the edge B.

In the directions ω from $\Omega_r \cap (\Omega_B \cup \Omega_A)$ the leading terms consist of the reflected, spherical and diffracted (from the edges A and B) waves (as well as surfaces ones)

$$U(r,\vartheta,\varphi) = R(\omega,\omega_0)\exp\left(-\mathrm{i}kr\cos\theta_r(\omega,\omega_0)\right) + D(\omega,\omega_0)\frac{\exp(\mathrm{i}kr)}{-\mathrm{i}kr}\left(1+O\left(\frac{1}{kr}\right)\right)$$
$$+\ d_A(\omega,\omega_0)\,\frac{\exp\left(-\mathrm{i}kr\cos\theta_A(\omega,\omega_0)\right)}{\sqrt{-\mathrm{i}kr\sin\psi_A}}\left(1+O\left(\frac{1}{kr\sin\psi_A}\right)\right) \tag{2.12}$$
$$+\ d_B(\omega,\omega_0)\,\frac{\exp\left(-\mathrm{i}kr\cos\theta_B(\omega,\omega_0)\right)}{\sqrt{-\mathrm{i}kr\sin\psi_B}}\left(1+O\left(\frac{1}{kr\sin\psi_B}\right)\right) + V_s(r,\vartheta,\varphi)\ + \ldots,$$

where $R(\omega, \omega_0)$ is the reflection coefficient in the first summand $U_r(\omega,\omega_0) = R\exp(-\mathrm{i}kr\cos\theta_r(\omega,\omega_0))$ of (2.12) which is the reflected wave. The summand $V_s(r,\ \vartheta,\ \varphi)$ has the same meaning as in the formula (2.9).

The wave field $U + U_i$ in the shadow of the incident wave, i.e. as $\omega \in \Omega_r^* \cap (\Omega_B \cup \Omega_A)$, reads

$$\begin{aligned} U(r,\vartheta,\varphi) + U_i(r,\vartheta,\varphi) = D(\omega,\omega_0)\frac{\exp(\mathrm{i}kr)}{-\mathrm{i}kr}\left(1 + O\left(\frac{1}{kr}\right)\right) \\ + d_A(\omega,\omega_0)\,\frac{\exp\left(-\mathrm{i}kr\cos\theta_A(\omega,\omega_0)\right)}{\sqrt{-\mathrm{i}kr\sin\psi_A}}\left(1 + O\left(\frac{1}{kr\sin\psi_A}\right)\right) \\ + d_B(\omega,\omega_0)\,\frac{\exp\left(-\mathrm{i}kr\cos\theta_B(\omega,\omega_0)\right)}{\sqrt{-\mathrm{i}kr\sin\psi_B}}\left(1 + O\left(\frac{1}{kr\sin\psi_B}\right)\right) \\ + V_s(r,\vartheta,\varphi) + \ldots, \end{aligned} \tag{2.13}$$

$kr \to \infty$.

We do not describe the asymptotics of the far field in the transition domains, where the expressions depend on special transition functions, see e.g. Lyalinov (2013), Sections 6 and 7. It is worth mentioning the work of Shanin (2011), where similar asymptotics for diffraction by an ideal circular cone were established.

3. Watson–Bessel integral representation of the solution and reduction to the problem for the spectral function

We turn to the separation of the radial variable of the solution which exploits the Watson–Bessel integral representation. We begin with that for the incident wave.

3.1. *Watson–Bessel integral representation for the incident wave*

We make use of the known Watson–Bessel integral representation for the incident wave (2.1)

$$U_i(r,\vartheta,\varphi) = 4\mathrm{i}\sqrt{\frac{\pi}{2}}\int_{C_\phi} \nu\,\mathrm{e}^{-\mathrm{i}\nu\pi/2}u_\nu^i(\omega,\omega_0)\frac{J_\nu(kr)}{\sqrt{-\mathrm{i}kr}}\,d\nu, \tag{3.1}$$

where $u_\nu^i(\omega,\omega_0) = -\frac{P_{\nu-1/2}(-\cos\theta_i(\omega,\omega_0))}{4\cos\pi\nu}$,

$$\left(\triangle_\omega + (\nu^2 - 1/4)\right)u_\nu^i(\omega,\omega_0) = \delta(\omega-\omega_0), \tag{3.2}$$

$P_{\nu-1/2}(x)$ is the Legendre function and C_ϕ with $\phi \in [0, \pi/2)$ is shown in Fig. 6. The contour C_ϕ is traditionally taken for $\phi = 0$; however, for some reductions it is possible also to use C_ϕ with $\phi \in (0, \pi/2)$. The 'spectral' function $u_\nu^i(\omega,\omega_0)$ corresponding to the incident plane wave admits the estimate

$$\left|u_\nu^i(\omega,\omega_0)\right| \,<\, C\,\frac{1}{\sqrt{|\nu|}}\exp\{-|\nu|\,|\sin\phi|\,\theta(\omega,\omega_0)\}, \tag{3.3}$$

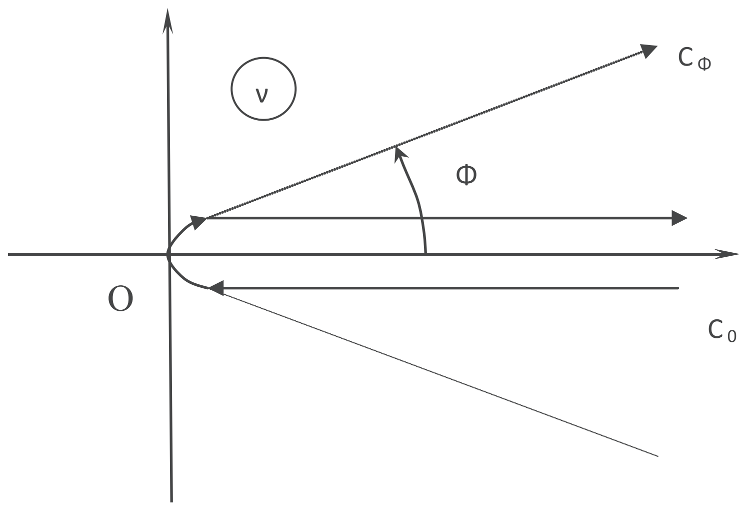

FIG. 6. The contours C_0 and C_ϕ.

as $|\nu| \to \infty$ and $\nu \in C_\phi$. Remark that $J_\nu(kr) \sim \frac{[kr/2]^\nu}{\Gamma(\nu+1)}$, $\Re\,\nu \to +\infty$, $|\arg \nu| < \pi/2$. The estimate of the integrand in (3.1) on C_ϕ as $|\nu| \to \infty$, $\nu = |\nu| e^{i\phi}$, $0 \leqslant \phi < \pi/2$ is given by[6]

$$\left|\nu\, e^{-i\nu\pi/2} u^i_\nu(\omega,\omega_0) J_\nu(kr)\right| < C \exp\{-|\nu|(\log|\nu| - 1)\cos\phi - |\nu|(\sin\phi[\arg(kr/2) - \pi/2 - \phi] + |\sin\phi|\theta(\omega,\omega_0) - \cos\phi \log|kr/2|)\}. \tag{3.4}$$

It is assumed in (3.4) that k may be complex with $|\arg k| < \pi/2$ although we consider $\arg k = 0$ in this work. Actually the constants in the estimates (3.4) and (3.3) are different; however, for convenience we take the maximal one and denote it by C. It is also useful to have these estimates not only for real $k > 0$ but for $|\arg k| < \pi/2$ as well. The integral in (3.1) then rapidly converges. It is worth noting that the function $u^i_\nu(\omega,\omega_0)$ is even with respect to ν and is holomorphic in the strip $\Pi_{1/2}$,

$$\Pi_\delta = \{\nu \in \mathbb{C} : \ |\Im(\nu)| < \delta\}.$$

Its simple poles are located at zeros of $\cos\pi\nu$.

3.2. *Watson–Bessel integral representation for the scattered field and separation of the radial variable*

In order to separate the radial variable for the problem at hand we look for the solution in the integral form

$$U(r,\vartheta,\varphi) = 4i\sqrt{\frac{\pi}{2}} \int_{C_0} \nu\, e^{-i\nu\pi/2}\, u_\nu(\omega,\omega_0) \frac{J_\nu(kr)}{\sqrt{-ikr}}\, d\nu \tag{3.5}$$

with unknown ‘spectral’ function $u_\nu(\omega,\omega_0)$.

[6] The Stirling asymptotics for the complex argument of the gamma-function is also exploited.

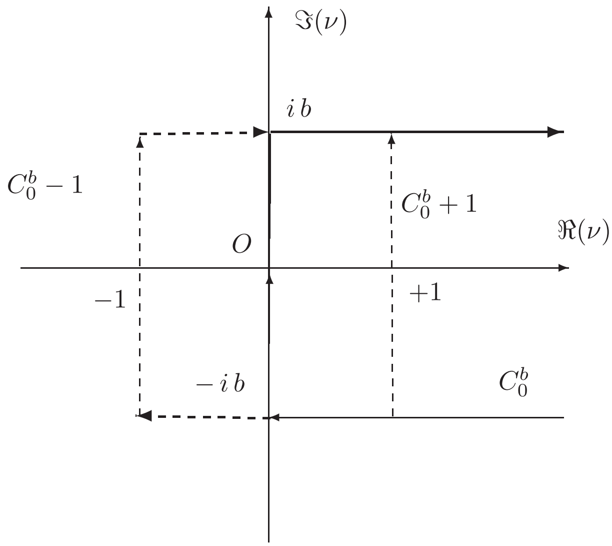

FIG. 7. Integration contour C_0^b and its deformation.

We begin with some formal reductions enabling us to motivate our approach and to make reasonable assumptions about the basic properties of the spectral function to be constructed. Let $u_\nu(\omega, \omega_0)$ satisfy the equation

$$\left(\triangle_\omega + (\nu^2 - 1/4)\right) u_\nu(\omega, \omega_0) = 0 \tag{3.6}$$

in the classical sense, i.e. $u_\nu(\cdot, \omega_0) \in C^2(\Sigma)$, $\Sigma = S^2 \setminus \sigma$ then, as is well known (see e.g. Smyshlyaev, 1991 and others), the scattered field $U(r, \vartheta, \varphi)$ in (3.5) solves the Helmholtz equation (2.2).

Now we turn to the boundary condition (2.3)

$$\frac{1}{r}\frac{\partial(U_i + U)}{\partial \mathcal{N}_\pm}\bigg|_{\omega\in\sigma_\pm} - \mathrm{i}k\eta_\pm\,(U_i + U)|_{\omega\in\sigma_\pm}$$
$$= 4\mathrm{i}\sqrt{\frac{\pi}{2}}\int\limits_{C_0^b} \nu\,\mathrm{e}^{-\mathrm{i}\nu\pi/2}\left(\frac{\partial \widehat{u}_\nu(\omega,\omega_0)}{\partial \mathcal{N}_\pm}\bigg|_{\sigma_\pm}\frac{(-\mathrm{i})J_\nu(kr)}{(-\mathrm{i}kr)^{3/2}} - \mathrm{i}\eta_\pm\,\widehat{u}_\nu(\omega,\omega_0)|_{\sigma_\pm}\frac{J_\nu(kr)}{(-\mathrm{i}kr)^{1/2}}\right)d\nu = 0,$$

where C_0^b is shown in Fig. 7, $\widehat{u}_\nu(\omega,\omega_0) = u_\nu(\omega,\omega_0) + u_\nu^i(\omega,\omega_0)$. We assumed that u_ν is holomorphic[7] and even w.r.t. ν in some vicinity of the imaginary axis, meromorphic in C and some of its possible sigularities are located inside the contour C_0^b.

[7] More accurate formulations will be given in the following section.

We take into account the identity $zJ_{\nu-1}(z)+zJ_{\nu+1}(z)=2\nu J_\nu(z)$. After some reductions one has

$$4\sqrt{\frac{\pi}{2}}\int\limits_{C_0^b} d\nu\ \mathrm{e}^{-\mathrm{i}\nu\pi/2}\left(\frac{\partial\widehat{u}_\nu(\omega,\omega_0)}{\partial\mathcal{N}_\pm}\bigg|_{\sigma_\pm}\frac{\mathrm{i}(J_{\nu-1}(kr)+J_{\nu+1}(kr))}{2\sqrt{-\mathrm{i}kr}}+\eta_\pm\,\nu\,\widehat{u}_\nu(\omega,\omega_0)\Big|_{\sigma_\pm}\frac{J_\nu(kr)}{\sqrt{-\mathrm{i}kr}}\right)=0,$$

then appropriately change the variables of integration so that

$$2\mathrm{i}\sqrt{\frac{\pi}{2}}\left(\int\limits_{C_0^b-1} d\nu\ \mathrm{e}^{-\mathrm{i}(\nu+1)\pi/2}\ \frac{\partial\widehat{u}_{\nu+1}(\omega,\omega_0)}{\partial\mathcal{N}_\pm}\bigg|_{\sigma_\pm}\frac{J_\nu(kr)}{\sqrt{-\mathrm{i}kr}}+\int\limits_{C_0^b+1} d\nu\ \mathrm{e}^{-\mathrm{i}(\nu-1)\pi/2}\ \frac{\partial\widehat{u}_{\nu-1}(\omega,\omega_0)}{\partial\mathcal{N}_\pm}\bigg|_{\sigma_\pm}\frac{J_\nu(kr)}{\sqrt{-\mathrm{i}kr}}\right.$$

$$\left.-\,2\mathrm{i}\eta_\pm\int\limits_{C_0^b} d\nu\ \nu\,\mathrm{e}^{-\mathrm{i}\nu\pi/2}\widehat{u}_\nu(\omega,\omega_0)\Big|_{\sigma_\pm}\frac{J_\nu(kr)}{\sqrt{-\mathrm{i}kr}}\right)=0,$$

where the contours of integration C_0^b+1 and C_0^b-1 are those in Fig. 7 obtained by shifting C_0^b along the real axis to the right- or left-hand side accordingly and C_0^b consists of the parts $(\infty-\mathrm{i}b,-\mathrm{i}b]$, $[-\mathrm{i}b,\,\mathrm{i}b]$ and $[\mathrm{i}b,\infty+\mathrm{i}b)$. The contours C_0^b+1 and C_0^b-1 are explicitly described by $\{C_0^b+1\}=(\infty-\mathrm{i}b+1,-\mathrm{i}b+1]\cup[-\mathrm{i}b+1,\mathrm{i}b+1]\cup[\mathrm{i}b+1,\infty+\mathrm{i}b+1)$ and $\{C_0^b-1\}=(\infty-\mathrm{i}b-1,-\mathrm{i}b-1]\cup[-\mathrm{i}b-1,\mathrm{i}b-1]\cup[\mathrm{i}b-1,\infty+\mathrm{i}b-1)$.

Let us now assume that $u_\nu(\omega,\omega_0)$ is taken meromorphic w.r.t. $\nu\in C$ with the poles inside some strip $|I(\nu)|<b$, holomorphic in Π_δ for some small $\delta>0$ and such that $\frac{\partial\widehat{u}_\nu(\omega,\omega_0)}{\partial\mathcal{N}_\pm}\Big|_{\sigma_\pm}$ is regular (holomorphic) in the strip $\Pi_{\delta+1}$. We are now able to deform the contours C_0^b+1 and C_0^b-1 into that C_0^b in Fig. 7 thus having

$$2\sqrt{\frac{\pi}{2}}\int\limits_{C_0^b} d\nu\,\mathrm{e}^{-\mathrm{i}\nu\pi/2}\left(\frac{\partial\widehat{u}_{\nu+1}(\omega,\omega_0)}{\partial\mathcal{N}_\pm}\bigg|_{\sigma_\pm}-\frac{\partial\widehat{u}_{\nu-1}(\omega,\omega_0)}{\partial\mathcal{N}_\pm}\bigg|_{\sigma_\pm}+2\eta_\pm\nu\,\widehat{u}_\nu(\omega,\omega_0)|_{\sigma_\pm}\right)\frac{J_\nu(kr)}{\sqrt{-\mathrm{i}kr}}=0.$$

We conclude that, provided $\widehat{u}_\nu(\omega,\omega_0)=u_\nu(\omega,\omega_0)+u^i_\nu(\omega,\omega_0)$ satisfies the boundary condition on $\sigma=\sigma_+\cup\sigma_-$

$$\frac{\partial\widehat{u}_{\nu+1}(\omega,\omega_0)}{\partial\mathcal{N}_\pm}\bigg|_{\sigma_\pm}-\frac{\partial\widehat{u}_{\nu-1}(\omega,\omega_0)}{\partial\mathcal{N}_\pm}\bigg|_{\sigma_\pm}+2\nu\,\eta_\pm\,\widehat{u}_\nu(\omega,\omega_0)|_{\sigma_\pm}=0, \tag{3.7}$$

the wave field $U+U_i$ fulfils the boundary condition (2.3).

Remark The condition (3.7) is nonlocal with respect to the spectral variable ν, which means that, contrary to the Helmholtz equation, the radial variable in the mixed boundary condition (2.3) is not separable in a traditional meaning.

In order to transform the boundary condition to an alternative form we take into account a simple Lemma which actually follows from the known technique developed for a class of functional equations, Babic *et al.* (2008), Chapter 7.

LEMMA 3.1 Let $H(\nu)$ be holomorphic as $\nu \in \Pi_\delta$ and $|H(\nu)| \leqslant C\mathrm{e}^{-\kappa|\nu|}$, $|\nu| \to \infty$, $\kappa > 0$ in this strip, $H(\nu) = -H(-\nu)$. Then an even solution $s(\nu)$ of the equation

$$s(\nu+1) - s(\nu-1) = -2\mathrm{i}H(\nu),$$

which is regular (holomorphic) in the strip $\nu \in \Pi_{1+\delta}$ and exponentially vanishes as $|\nu| \to \infty$ there, is given by

$$s(\nu) = \frac{1}{4}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\tau\, H(\tau)\left(\frac{\sin(\pi\tau/2)}{\cos(\pi\tau/2) - \sin(\pi\nu/2)} + \frac{\sin(\pi\tau/2)}{\cos(\pi\tau/2) + \sin(\pi\nu/2)}\right)$$

$$= \frac{1}{4}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\tau\, H(\tau)\,\frac{\sin\pi\tau}{\cos\pi\tau + \cos\pi\nu}, \quad \nu \in \Pi_{1+\delta}.$$

Assuming exponential decreasing of $\widehat{u}_\nu$ as $|\nu| \to \infty$, $\nu \in \Pi_\delta$ and making use of this Lemma, we arrive at a new form of the boundary condition (3.7)

$$\left.\frac{\partial \widehat{u}_\nu(\omega,\omega_0)}{\partial \mathscr{N}_\pm}\right|_{\sigma_\pm} = \frac{\eta_\pm}{2\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\tau\, \frac{\tau\, \sin\pi\tau\, \widehat{u}_\tau(\omega,\omega_0)|_{\sigma_\pm}}{\cos\pi\tau + \cos\pi\nu}, \quad \nu \in \Pi_\delta. \tag{3.8}$$

REMARK Provided $\widehat{u}_\nu(\omega,\omega_0)|_{\sigma_\pm}$ is holomorphic as $\nu \in \Pi_\delta$, then from the Lemma and condition (3.8) $\left.\frac{\partial \widehat{u}_\nu(\omega,\omega_0)}{\partial \mathscr{N}_\pm}\right|_{\sigma_\pm}$ is holomorphic as $\nu \in \Pi_{1+\delta}$.

In order to formulate a basic statement of this Section we first postulate a set of conditions specifying a desired class for solution of the problem (3.6), (3.8).

3.3. *Conditions specifying a class of solutions for spectral function $u_\nu(\omega, \omega_0)$*

We make use of some known definitions (Dieudonne, 1960; Chapter 7) dealing with a meromorphic in the complex plane C (or holomorphic in a domain $D \subset C$) mapping f_ν from C (or from D) into a Banach space B, i.e. with such meromorphic (or holomorphic) function of ν that $f_\nu : \nu \mapsto f_\nu(\cdot)$, where $f_\nu(\cdot) \in B$.

Let the spectral function $u_\nu(\omega, \omega_0)$ satisfy the conditions

1. The spectral function is such a mapping that $u_\nu : \nu \mapsto u_\nu(\cdot, \omega_0)$, $(u_\nu(\cdot,\omega_0) \in C^2_{loc}(\Sigma))$ is meromorphic in the complex plane C for all $\omega_0 \notin \Omega_{\delta_0}$, where ω_0 is a parameter. For all regular ν it admits the estimate (Meixner's condition on the unit sphere) $u_\nu(\omega,\omega_0) = c_0 + c_1(\chi_A)\,\psi_A^{1/2} + \ldots$, $\psi_A \to 0$, where ψ_A is the geodesic distance from A to $\omega = (\psi_A, \chi_A)$. A similar condition is valid for the point B.

2. The trace of the spectral function on σ is holomorphic in Π_δ, i.e. the mapping $u_\nu|_\sigma : \nu \mapsto u_\nu(\cdot|_\sigma, \omega_0)$ is holomorphic in $\nu \in \Pi_\delta u_\nu(\cdot|_\sigma, \omega_0) \in C(\sigma)$ for all $\omega_0 \notin \Omega_{\delta_0}$.

3. The set Q of poles of the meromorphic function u_ν is contained in the strip $|I(\nu)| < b$ for some $b > 0$.
4. The spectral function is even with respect to ν, $u_\nu(\omega, \omega_0) = u_{-\nu}(\omega, \omega_0)$.
5. There exist the traces $\frac{\partial \widehat{u}_\nu(\omega,\omega_0)}{\partial \mathcal{N}_\pm}\Big|_{\sigma_\pm}$ on σ which are holomorphic functions of $\nu \in \Pi_{1+\delta}$ with the values in $C(\sigma)$, $\omega_0 \notin \Omega_{\delta_0}$. Notice that, in view of the boundary conditions (3.7), (3.8), these traces admit meromorphic continuation on the complex plane.
6. The spectral function satisfies the estimate ($|\nu| \to \infty$, $\nu = |\nu| e^{i\phi}$, $-\pi/2 < \phi < \pi/2$)

$$|u_\nu(\omega, \omega_0)| \leqslant C \frac{1}{\sqrt{|\nu|}} \exp\{-|\nu|\ (|\sin\phi|\ \tau_0(\omega,\omega_0) - \cos\phi\ |\tau_1(\omega,\omega_0)|\} \tag{3.9}$$

for some τ_0, τ_1, where $\tau_0 > 0$. This estimate is valid on the contour C_ϕ in Fig. 6.

Remark that analogous properties may be verified for $u^i_\nu(\omega, \omega_0)$. It is worth commenting on the origin of the estimate (3.9). The equation (3.6) has high-frequency (or quasi-classical) structure as $\nu \to \infty$. The solution of the equation in this case may be determined as a sum of the 'ray' expansions

$$u_\nu(\omega) \asymp \sum_j U^j(\nu, \omega)\,,$$

$$U^j(\nu, \omega) = e^{i\nu\tau_j(\omega)} \sum_{m=1}^{\infty} v_m(\omega)/\nu^{1/2+m}.$$

The boundary condition for $u_\nu(\omega)$ also depends on the large parameter and is not self-adjoint because, in particular, it depends on complex $\eta_\pm$. As a result, amongst solutions $\tau_j(\omega)$ of the eikonal equations $(\nabla_\omega \tau_j(\omega))^2 = 1$ there are those complex valued having the structure $\tau_j(\omega) = \tau^j_0(\omega) + i\tau^j_1(\omega)$ with the real valued $\tau^j_0(\omega) > 0$ and $\tau^j_1(\omega)$. Assuming that the ray solutions admit continuation to $\nu = |\nu| e^{i\phi}$, $|\nu| \to \infty$, $0 < \phi < \pi/2$, we may expect

$$|u_\nu(\omega)| \leqslant \mathrm{Const}\, |\nu|^{-1/2} \left| e^{i\nu(\tau_0(\omega)+i\tau_1(\omega))} \right|,$$

where $\tau_0(\omega) = \min_j(\tau^j_0(\omega))$ and $|\tau_1(\omega)| = \max_j(|\tau^j_1(\omega)|)$. This accounts for the condition 6.

It is worth noting that rapid convergence of the Watson–Bessel integral representation (3.5) follows from the estimate (see also (3.4)) given by

$$\left|\nu\, e^{-i\nu\pi/2} u_\nu(\omega, \omega_0) J_\nu(kr)\right| < C \exp\{-|\nu|(\log|\nu| - 1)\cos\phi - -|\nu|(\sin\phi[\arg(kr/2) - \pi/2 - \phi] + |\sin\phi|\tau_0(\omega,\omega_0)) + |\nu|\cos\phi[|\tau_1(\omega,\omega_0)| + \log|kr/2|]\} \tag{3.10}$$

on C_ϕ as $|\nu| \to \infty$, $\nu = |\nu| e^{i\phi}$, $-\pi/2 < \phi < \pi/2$.

Taking into account the discussion in this section, we arrive at

THEOREM 3.1 Let $u_\nu(\omega, \omega_0)$ be a solution of the problem (3.6), (3.8) from the class of functions described by the conditions 1–6. Then Watson–Bessel integral representation (3.5) for $U(r, \theta, \varphi)$ is the desired classical solution of the problem .

We mentioned that the radiation condition (2.6) is not valid for the solution in the case of the plane wave incidence. On the other hand, we should emphasize that description of the far-field asymptotics for the diffraction problem at hand will not be exhaustively considered in this work, however, we shall discuss the far-field expression in the oasis Ω_0. Nevertheless, we shall develop an efficient formalism based on the Sommerfeld integral representation which is well adapted for such a description (Lyalinov, 2013).

In the following section we shall consider an approach that enables one to show a way of construction of the spectral function solving the problem (3.6), (3.8).

4. Study of the problem for the spectral function

In this section we consider the problem (3.6), (3.8) and restrict it on $\nu \in [0, \mathrm{i}\infty)$ exploiting also that $u_\nu(\omega, \omega_0)$ is even. Then it is useful to study the inhomogeneous equation instead of (3.6) and, vice versa, homogeneous boundary condition so that

$$\Big(\triangle_\omega + (\nu^2 - 1/4)\Big)\, w_\nu(\omega) = F(\nu, \omega)\,, \quad \omega \in \Sigma$$

$$\left.\frac{\partial w_\nu(\omega)}{\partial \mathscr{N}_\pm}\right|_{\sigma_\pm} = \frac{\eta_\pm}{\mathrm{i}} \int_0^{\mathrm{i}\infty} d\tau\, \frac{\tau\, \sin \pi\tau\, w_\tau(\omega)|_{\sigma_\pm}}{\cos \pi\tau + \cos \pi\nu}\,. \tag{4.1}$$

This problem is not traditional because of the non-local dependence on the parameter of separation ν. It should be noticed that the problems (3.6), (3.8) and (4.1) are actually connected by a simple change of the unknown function $u_\nu(\omega, \omega_0) = w_\nu(\omega, \omega_0) + v(\nu, \omega, \omega_0)$, where v is uniquely defined. We shall write $w_\nu(\omega)$ instead of $w_\nu(\omega, \omega_0)$ omitting dependence on the parameter ω_0.

We are looking for solution $w_\nu(\omega)$ of (4.1) which is from $C([0, \mathrm{i}\infty), C^2(\Sigma))$, having also continuous value of $\frac{\partial w_\nu(\omega)}{\partial \mathscr{N}_\pm}$ on σ such that the boundary condition in (4.1) is satisfied. Having such a solution, we shall continue it appropriately onto the whole complex plane w.r.t ν and also ensure the conditions 1–6. for the spectral function $u_\nu(\omega, \omega_0)$. It is convenient to make use of the variable x instead of ν and the new unknown function $\mathscr{U}(x, \omega)$ defined by the expressions

$$x = 1/\cos \pi\nu\,, \quad \nu = \mathrm{i}d(x), \quad d(x) := \frac{1}{\pi}\, \mathrm{arccosh}(1/x) = \frac{1}{\pi}\, \log\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right), \quad x \in [0, 1]\,,$$

$$\mathscr{U}(x, \omega) = \sqrt{d(x)}\, \cos\left(\mathrm{i}\pi d(x)\right)\, w_{\mathrm{i}d(x)}(\omega).$$

Remark that

$$0 \leqslant \frac{1}{\pi}\, \log(1/x) \leqslant d(x) \leqslant d_0(x)$$

with $d_0(x) = \frac{1}{\pi}\, \log(2/x)$.

As a result, we write the problem (4.1) in the form

$$\Big(-\triangle_\omega + d^2(x) + 1/4\Big)\,\mathscr{U}(x,\omega) = f(x,\omega)\,, \quad \omega \in \Sigma$$

$$\left.\frac{\partial \mathscr{U}(x,\omega)}{\partial \mathscr{N}_\pm}\right|_{\sigma_\pm} + \eta_\pm\,\mathscr{A}\,\mathscr{U}(x,\omega|_{\sigma_\pm}) = 0, \tag{4.2}$$

where $f(x,\omega) = -\sqrt{d(x)}\cosh(\pi d(x)F(\mathrm{i}d(x),\omega)$ and the operator-impedance $\mathscr{A}$ is given by

$$\mathscr{A}\,\mathscr{U}(x,\omega|_{\sigma_\pm}) = \frac{1}{\pi}\int_0^1 \frac{\sqrt{d(x)d(y)}}{x+y}\,\mathscr{U}(y,\omega|_{\sigma_\pm})\,\mathrm{d}y. \tag{4.3}$$

The operator $\mathscr{A}$ in (4.3) is formally symmetric.

It is worth commenting on the reduction of the problem (4.1) to that (4.2). We are looking for the solution of the problem (4.1) which meromorphically depends on the variable ν. However, the study of the meromorphic operator-function

$$\mathscr{M}_\nu = \left\{\triangle_\omega + (\nu^2 - 1/4)\,,\ \left.\left(\frac{\partial}{\partial\,\mathscr{N}_\pm} - \eta_\pm\,\mathscr{A}_0\right)\right|_{\sigma_\pm}\right\},$$

and of the equation

$$\mathscr{M}_\nu\, w_\nu(\omega) = \mathscr{F}_\nu(\omega)$$

is not a simple task. In the latter problem the operator $\mathscr{A}_0$ is defined by the right-hand side of the boundary condition in (4.1) and $\mathscr{F}_\nu(\omega) = \{F_\nu(\omega), 0\}$.

Instead, we consider the reduced problem (4.2). The solution of the latter problem $\mathscr{U}(x,\omega)$ is defined on the segment $x \in (0,\,1)$ and is then analytically continued as a meromorphic function $w_\nu(\omega)$ onto the complex plane $\nu \in C$ taking into account the change of the variable $x \mapsto \nu$ and of $\mathscr{U}(x,\omega) \mapsto w_\nu(\omega)$.

Our further goal is to study unique solvability of the problem (4.2) in an appropriate functional space. To this end, we introduce a Hilbert space and attribute an operator, acting in this space, with the problem (4.2). It is performed by use of the traditional technique based on a sesquilinear form connected with the problem (4.2). The corresponding sectorial form is taken densely defined and proved to be closable, see Kato (1972), Chapter 6. The latter circumstance enables one to define an m-sectorial operator A attributed to (4.2), see also Assier *et al.* (2016). It turns out that this operator is boundedly invertible, $A^{-1} = (A - \Lambda)^{-1}|_{\Lambda=0}$ is correctly defined because $\Lambda = 0$ belongs to the resolvent set of the operator A, where Λ is the spectral parameter.

4.1. *Definition of the sectorial form attributed to the problem (4.2)*

Consider the Hilbert space $H = L_2((0,\,1);L_2(\Sigma))$ of such functions u of $x \in (0,\,1)$ with the values in $L_2(\Sigma)$ denoting them $u = u(x,\,\omega)$. The Hilbert norm $\|v\|_H = \sqrt{< v, v >}$ is specified by the scalar product

$$< u, v >= \int_0^1 d\,x \left(\int_\Sigma u(x,\omega)\overline{v(x,\omega)}\,\mathrm{d}\,\omega \right) .$$

In order to give a motivated expression for the sesquilinear form t_A attributed to the problem (4.2) we apply the Green's identity to the differential operator and make use of the boundary condition for u

$$\begin{aligned}
< (-\triangle_\omega + d^2 + 1/4)u, v >= & \int_0^1 \mathrm{d}\,x \left(\int_\Sigma (-\triangle_\omega + d^2(x) + 1/4)u(x,\omega)\overline{v(x,\omega)}\mathrm{d}\,\omega \right) \\
= & \int_0^1 \mathrm{d}\,x \left(\int_\Sigma (\nabla_\omega u(x,\omega)\cdot\nabla_\omega \overline{v(x,\omega)} + (d^2(x) + 1/4)u(x,\omega)\overline{v(x,\omega)})\mathrm{d}\,\omega \right) - \\
& - \int_0^1 \mathrm{d}\,x \int_{\sigma_+} \mathrm{d}\sigma\, \frac{\partial u(x,\sigma)}{\partial \mathcal{N}_+}\overline{v(x,\sigma)} - \int_0^1 \mathrm{d}\,x \int_{\sigma_-} \mathrm{d}\sigma\, \frac{\partial u(x,\sigma)}{\partial \mathcal{N}_-}\overline{v(x,\sigma)} \\
= & \int_0^1 \mathrm{d}\,x \left(\int_\Sigma \mathrm{d}\omega (\nabla_\omega u(x,\omega)\cdot\nabla_\omega \overline{v(x,\omega)} + (d^2(x) + 1/4)u(x,\omega)\overline{v(x,\omega)}) \right) \\
& + \frac{\eta_+}{\pi} \int_0^1 \mathrm{d}\,x \int_{\sigma_+} \mathrm{d}\sigma \int_0^1 \mathrm{d}y\, \frac{\sqrt{d(x)d(y)}}{x+y}\, u(y,\sigma)\overline{v(x,\sigma)} \\
& + \frac{\eta_-}{\pi} \int_0^1 \mathrm{d}\,x \int_{\sigma_-} \mathrm{d}\sigma \int_0^1 \mathrm{d}y\, \frac{\sqrt{d(x)d(y)}}{x+y}\, u(y,\sigma)\overline{v(x,\sigma)} ,
\end{aligned}$$

where u and v are taken such that the reductions above are justified.

Let us now define a sesquilinear form

$$t_A[u,v] \;=\; t_A^1[u,v] + t_A^2[u,v] + t_A^3[u,v] , \tag{4.4}$$

where

$$t_A^1[u,v] = \int_0^1 \mathrm{d}x \left(\int_\Sigma \left(\nabla_\omega u(x,\omega) \cdot \nabla_\omega \overline{v(x,\omega)} + \frac{1}{4} u(x,\omega)\overline{v(x,\omega)} \right) \mathrm{d}\omega \right),$$

$$t_A^2[u,v] = \int_0^1 \mathrm{d}x \left(\int_\Sigma d^2(x) u(x,\omega)\overline{v(x,\omega)} \mathrm{d}\,\omega \right),$$

$$t_A^3[u,v] = \sum_\pm \frac{\eta_\pm}{\pi} \int_0^1 \mathrm{d}x \int_{\sigma_\pm} \mathrm{d}\sigma \int_0^1 \mathrm{d}y \, \frac{\sqrt{d(x)d(y)}}{x+y} \, u(y,\sigma)\overline{v(x,\sigma)},$$

with the domains

$$D\left(t_A^1\right) = \left\{ v: \ v \in L_2\left((0,1); H^1(\Sigma)\right) \subset H \right\},$$

$$D\left(t_A^2\right) = \left\{ v: \ v \in H = L_2((0,1); L_2(\Sigma)) \text{ such that } d\,v \in L_2((0,1); L_2(\Sigma)) \right\},$$

$$D\left(t_A^3\right) = \left\{ w: \ w \in H, \ w|_\sigma \in L_2((0,1); L_2(\sigma)) \text{ such that } \sqrt{d}\,w|_\sigma \in L_2((0,1); L_2(\sigma)) \right\}$$

correspondingly. Then the domain of t_A is

$$D(t_A) = \bigcap_{i=1}^{3} D\left(t_A^i\right) \tag{4.5}$$

and t_A is densely defined. It is worth mentioning that $H^1(\Sigma)$ is the usual Sobolev space and is boundedly embedded into $L_2(\sigma)$. The norm in $L_2((0,1); H^1(\Sigma))$ is given by

$$\|v\|_1^2 = \int_0^1 \mathrm{d}x \left(\int_\Sigma (|\nabla_\omega u(x,\omega)|^2 + |u(x,\omega)|^2 \mathrm{d}\,\omega \right)$$

and this space is boundedly embedded into $L_2((0,1); L_2(\sigma))$.

The form t_A^3 may be also written as

$$t_A^3[u,v] = \sum_\pm \eta_\pm \int_0^1 \mathrm{d}x \int_{\sigma_\pm} \mathrm{d}\sigma \, (\mathscr{A}u)(x,\sigma)\overline{v(x,\sigma)} = \sum_\pm \eta_\pm (\mathscr{A}u, v)_\pm,$$

where the operator-impedance $\mathscr{A} \geqslant 0$ is an integral operator, has the kernel $\frac{\sqrt{d(x)d(y)}}{x+y}$ and is formally symmetric.

We aim to show that the form t_A is sectorial (Kato, 1972, Chapter 6) with the vertex Γ which is a real number and with the half-angle Θ. We verify that the range $R(t_A)$ of the form, which is the set of values of the quadratic form $t_A[u, u]$ provided $u \in D(t_A)$ and $\|u\|_H = 1$, is located in the sector $|\arg(\Lambda - \Gamma)| \leqslant \Theta, 0 \leqslant \Theta < \pi/2$ on the complex plane of the variable Λ. To that end, we represent the form as

$$t_A = \Re(t_A) + \mathrm{i}\Im(t_A),$$

where (see Kato, 1972, Chapter 6) real part $\Re(t_A)$ of the form is defined by

$$\Re(t_A)[u, v] = \frac{t_A[u, v] + \overline{t_A[v, u]}}{2}$$

and similarly the imaginary part

$$\Im(t_A)[u, v] = \frac{t_A[u, v] - \overline{t_A[v, u]}}{2\mathrm{i}}$$

for any $u, v \in D(t_A)$. Simple calculations lead to the expressions

$$\Re(t_A)[u, v] = \int_0^1 \mathrm{d}x \left(\int_\Sigma \nabla_\omega u(x, \omega) \cdot \nabla_\omega \overline{v(x, \omega)} + \left(d^2(x) + \frac{1}{4} \right) u(x, \omega)\overline{v(x, \omega)} \mathrm{d}\omega \right)$$
$$+ \sum_\pm \varepsilon_\pm (\mathscr{A} u, v)_\pm, \Im(t_A)[u, v] = \sum_\pm \chi_\pm (\mathscr{A} u, v)_\pm$$

recalling that $\eta_\pm = \varepsilon_\pm + \mathrm{i}\chi_\pm$,

$$(\mathscr{A} u, v)_\pm = \frac{1}{\pi} \int_0^1 \mathrm{d}x \int_{\sigma_\pm} \mathrm{d}\sigma \int_0^1 \mathrm{d}y \, \frac{\sqrt{d(x)d(y)}}{x + y} u(y, \sigma)\overline{v(x, \sigma)}\,.$$

It is obvious that

$$\Re(t_A)[u, u] \geqslant \frac{1}{4}\|u\|_H^2$$

so that the vertex $\Gamma \geqslant \frac{1}{4}$ and the form $\Re(t_A)$ is positive definite. For the imaginary part we find

$$|\Im(t_A)[u, u]| \leqslant \sum_\pm |\chi_\pm|(\mathscr{A} u, u)_\pm \leqslant \sum_\pm \frac{|\chi_\pm|}{\varepsilon_\pm} \left(\varepsilon_\pm (\mathscr{A} u, u)_\pm + \|\nabla_\omega u\|_H^2 + \|d\, u\|_H^2 \right)$$
$$\leqslant \left(\frac{|\chi_+|}{\varepsilon_+} + \frac{|\chi_-|}{\varepsilon_-} \right) \left(\Re(t_A)[u, u] - \frac{1}{4}\|u\|_H^2 \right)$$

for any $u \in D(t_A)$. As a result, we have the estimate for the half-angle Θ of the sectorial form t_A

$$0 \leqslant \tan\Theta \leqslant \left(\frac{|\chi_+|}{\varepsilon_+} + \frac{|\chi_-|}{\varepsilon_-}\right).$$

It is crucial to prove that the sesquilinear sectorial form t_A is closable, i.e. admits a closed extension, which is equivalent to that for $\Re(t_A)$.

4.2. *The form t_A is closable*

In order to prove that t_A admits a closure it is sufficient to show that its summands $t_A^i,\ i = 1,2,3$ are closable (i.e. admit closed extensions, see Theorem 1.31 in Chapter 6 of Kato, 1972).

To do this we shall use the following simple statement (see, Birman & Solomyak, 1987, Section 10.1(4))

LEMMA 4.1 Let h be a densely defined sesquilinear form and G be an h −dense in $D(h)$ set. Also let the conditions $h[u_n - u_m, u_n - u_m] \to 0$ and $\|u_n\|_H \to 0$ (as $n, m \to \infty$) be followed by

$$h[u_n, g] \to 0, \quad n \to \infty$$

for any $g \in G$ then the form h is closable.

Introduce the set G by

$$G = \left\{ g \in D(t_A) : \quad d_0^2 g \in H = L_2((0,1); L_2(\Sigma)), \ \ d_0^2 g\big|_\sigma \in L_2((0,1); L_2(\sigma)) \right\},$$

$d_0(x) = \frac{1}{\pi}\log(2/x)$.

Remark that the form t_A^1 is closable because it is directly connected with the form

$$\tau^1[u,v] = \int_\Sigma \left(\nabla_\omega u(\omega)\cdot\nabla_\omega \overline{v(\omega)} + \frac{1}{4}u(\omega)\overline{v(\omega)}\right)\mathrm{d}\omega$$

which is known to be closable in $H^1(\Sigma)$, see e.g. Assier *et al.* (2016). That the form t_A^1 is closable implies the following: from $\|u_n\| \to 0$ and $t_A^1[u_n - u_m, u_n - u_m] \to 0$ $(n, m \to \infty)$ it follows that (see Theorem 1.17 in Chapter 6, Kato, 1972)

$$t_A^1[u_n, u_n] \to 0, \quad \text{as } n \to \infty. \tag{4.6}$$

The form t_A^2 is closable because

$$\left|t_A^2[u_n,g]\right| \leqslant \int_0^1 \mathrm{d}x \left(\int_\Sigma \left|d^2(x)u(x,\omega)\right| |g(x,\omega)|\mathrm{d}\omega\right)$$

$$\leqslant \int_0^1 \mathrm{d}x \left(\int_\Sigma |u(x,\omega)| \left|d_0^2(x)g(x,\omega)\right| \mathrm{d}\omega\right) \leqslant \|u_n\|_H \, \|d_0^2 g\|_H \to 0,$$

as $\|u_n\|_H , \to 0$, $n \to \infty$ for any $g \in G$.

The corresponding estimate for t_A^3 requires a bit more work

$$\left|t_A^3[u_n,g]\right| \leqslant \sum_\pm \frac{|\eta_\pm|}{\pi} \int_0^1 \mathrm{d}x \int_{\sigma_\pm} \mathrm{d}\sigma \int_0^1 \mathrm{d}y \frac{\sqrt{d(x)d(y)}}{d_0^2(x)(x+y)} |u_n(y,\sigma)| \left|d_0^2(x)g(x,\sigma)\right|$$

$$\leqslant \sum_\pm \frac{|\eta_\pm|}{\pi} \left\{\int_0^1 \mathrm{d}x \int_{\sigma_\pm} \mathrm{d}\sigma \left[\int_0^1 \mathrm{d}y \frac{\sqrt{d(y)}}{\sqrt{d_0(x)}^3(x+y)} |u_n(y,\sigma)|\right]^2\right\}^{1/2}$$

$$\left\{\int_0^1 \mathrm{d}x \int_{\sigma_\pm} \mathrm{d}\sigma \, |d_0^2(x)g(x,\sigma)|^2\right\}^{1/2}$$

$$\leqslant \sum_\pm C \frac{|\eta_\pm|}{\pi} \left\{\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \frac{d(y)}{d_0(x)^3(x+y)^2}\right\}^{1/2}$$

$$\times \left\{\int_0^1 \mathrm{d}x \int_{\sigma_\pm} \mathrm{d}\sigma \, |u_n(x,\sigma)|^2\right\}^{1/2} \left\{\int_0^1 \mathrm{d}x \int_{\sigma_\pm} \mathrm{d}\sigma \left|d_0^2(x)g(x,\sigma)\right|^2\right\}^{1/2},$$

where we have multiply used the Cauchy–Schwarz inequality. From (4.6) and that $L_2((0, 1), H^1(\Sigma))$ is boundedly embedded into $L_2((0, 1), L_2(\sigma))$ ($u_n \in L_2((0, 1), H^1(\Sigma))$ we have

$$\left\{\int_0^1 \mathrm{d}x \int_{\sigma_\pm} \mathrm{d}\sigma \, |u_n(x,\sigma)|^2\right\}^{1/2} \leqslant C\|u_n\|_1 \leqslant C_1 \, t_A^1[u_n,u_n] \to 0, \text{ as } n \to \infty.$$

It remains to verify that

$$\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \, \frac{d(y)}{d_0(x)^3 (x+y)^2} \leqslant \text{Const},$$

i.e. it is bounded. The latter double integral over the square $S = (0, 1) \times (0, 1)$ is represented as a sum of two integrals, one is over $S_1 = \{(x, y) : \sqrt{x^2 + y^2} < 1\} \cap S$ and that over $S \setminus S_1$. The integral over $\overline{S \setminus S_1}$ is obviously bounded because the integrand is continuous in this domain. The integral over the sector S_1 is estimated by making use of the polar coordinates ρ, ϕ in its evaluation and is estimated, which is a simple exercise, by a constant C_1,

$$\int_0^{\pi/2} \mathrm{d}\phi \int_0^1 \frac{\mathrm{d}\rho}{\rho} \frac{|\log \rho + \log(\sin \phi)|}{|\log(2/\rho) - \log(\cos \phi)|^3 (\cos \phi + \sin \phi)^2} \leqslant \mathrm{C}_1.$$

Taking into account the estimates above we find that $|t_A^3[u_n, g]| \to 0$ as $\|u_n\|_H \to 0$, $n \to \infty$ for any $g \in G$. As a result, we can assert that the form t_A is closable.

4.3. *An m-sectorial operator A corresponding to the problem (4.2). Solvability of the problem*

Let us denote T_A the closure of t_A with $D(t_A) \subset D(T_A)$.[8] In accordance with the theorem on representation for the closed sectorial forms (see Kato, 1972, Section 2, Chapter 6, Theorem 2.1) we obtain

THEOREM 4.1 There exists a unique m-sectorial operator A such that $Dom(A) \subset D(T_A)$ and

$$< Au, v > = T_A[u, v]$$

with $u \in Dom(A)$, $v \in D(T_A)$.

It is important to notice that this operator is the desired operator attributed to the problem (4.2). Let $u \in D(T_A)$ and for any $v \in D(T_A)$

$$T_A[u, v] = < f, v >, \quad f \in H,$$

then u is called *weak* solution of the equation

$$Au = f. \tag{4.7}$$

This solution is also a *weak* solution of the problem (4.2) by definition. From the theorem on representation it also follows that, provided $u \in D(T_A)$, $f \in H$ and the equality

$$T_A[u, v] = < f, v >$$

is valid for any v from the core of the form T_A (i.e. from $Dom(A)$) then $u \in Dom(A)$ and u is a solution of the equation (4.7).

[8] It is possible to describe the domain of T_A more efficiently.

The range $\Theta(A)$ of the m-sectorial operator A is a dense subset of the range $\Theta(T_A)$, which implies that $\Lambda = 0$ belongs to the resolvent set of the operator A. The inverse operator $A^{-1} = (A-\Lambda)^{-1}|_{\Lambda=0}$ is bounded and the problem (4.2) has a *weak* solution $\mathscr{U}$.

Recall that

$$F(\nu,\omega)|_{x=1/\cos\pi\nu} = -\frac{f(x,\omega)}{\sqrt{d(x)}\cosh(\pi d(x))}$$

in (4.1) and, provided f is from $C([0,1];C(\sigma)) \subset H$ the function $F(\nu,\omega)$ is continuous with respect to ν and is from $C(\sigma)$ w.r.t. ω, exponentially vanishing as $\nu \to \mathrm{i}\infty$.

Similar simple arguments enable us to conclude that solution of the problem (3.6), (3.8) exists and is unique, $u_\nu(\omega,\omega_0)$ is from $C([0,\mathrm{i}\infty); C^2(\Sigma))$ which admits an estimate

$$|u_\nu(\omega,\omega_0)| \leqslant C\left|\frac{\mathrm{e}^{\mathrm{i}\nu\tau_*}}{\sqrt{\nu}}\right|, \quad \nu \to \mathrm{i}\infty,$$

$0 < \tau_* \leqslant \pi/2$ as $dist(\omega_0,\sigma) > \pi/2$. This estimate follows from the fact that $\mathscr{U} \in L_2((0.1); H^1(\Sigma))$. Remark that $\mathscr{U}_i(x,\omega) = \sqrt{d(x)}\cos(\mathrm{i}\pi d(x))\, u^i_{\mathrm{i}d(x)}(\omega,\omega_0)$ is from $H = L_2((0,1);L_2(\Sigma))$ provided $dist(\omega_0,\sigma) > \pi/2$, which is implied.

5. Meromorphic continuation of $u_\nu(\omega,\omega_0)$

In order to continue $u_\nu(\omega,\omega_0)$ specified on $\nu \in [0,\mathrm{i}\infty)$, as demonstrated in the previous section, onto the complex plane we need to develop some technical tools. First, we continue $u_\nu(\omega,\omega_0)$ from $[0,\mathrm{i}\infty)$ onto $\mathrm{i}R$ making use of evenness w.r.t. ν.

Then, consider $g_\nu(\omega,\omega')$ the Green's function of the operator $\triangle_\omega + (\nu^2 - 1/4)$ on the unit sphere

$$g_\nu(\omega,\omega') = -\frac{P_{\nu-1/2}\left(-\cos\theta(\omega,\omega')\right)}{4\cos\pi\nu},$$

satisfying

$$\left(\triangle_\omega + (\nu^2 - 1/4)\right) g_\nu(\omega,\omega') = \delta(\omega-\omega').$$

We apply Green's identity to $u_\nu(\omega,\omega_0)$ and $g_\nu(\omega,\omega')$ and, omitting some technical details, obtain

$$\int_\Sigma \mathrm{d}\omega' \left(\triangle_{\omega'} g_\nu(\omega,\omega')\, u_\nu(\omega',\omega_0)\right) - g_\nu(\omega,\omega')\,\triangle_{\omega'} u_\nu(\omega',\omega_0))$$
$$= \sum_\pm \int_{\sigma_\pm} \mathrm{d}\sigma \left(\frac{\partial g_\nu(\omega,\sigma)}{\partial\mathscr{N}_\pm} u_\nu(\sigma,\omega_0) - \frac{\partial u_\nu(\sigma,\omega_0)}{\partial\mathscr{N}_\pm} g_\nu(\omega,\sigma)\right).$$

Making use of the equations for $g_\nu(\omega, \omega')$ and $u_\nu(\omega, \omega')$ and the boundary condition (see (3.8)), we arrive at the representation

$$u_\nu(\omega,\omega_0) = \sum_{\pm}\int_{\sigma_\pm} \mathrm{d}\sigma \left(\frac{\partial g_\nu(\omega,\sigma)}{\partial \mathscr{N}_\pm} u_\nu(\sigma,\omega_0) - \eta_\pm \mathscr{A}_0 u_\nu(\sigma,\omega_0)\, g_\nu(\omega,\sigma)\right) + \Psi_\nu(\omega,\omega_0), \tag{5.1}$$

where

$$\mathscr{A}_0 u_\nu(\sigma,\omega_0) := \frac{1}{2\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} d\tau\, \frac{\tau\, \sin\pi\tau\, u_\tau(\sigma,\omega_0)}{\cos\pi\tau + \cos\pi\nu},$$

$$\Psi_\nu(\omega,\omega_0) = \sum_{\pm}\int_{\sigma_\pm} \mathrm{d}\sigma \left(\frac{\partial u^i_\nu(\omega,\sigma)}{\partial \mathscr{N}_\pm} g_\nu(\sigma,\omega_0) - \eta_\pm \mathscr{A}_0 u^i_\nu(\sigma,\omega_0)\, g_\nu(\omega,\sigma)\right).$$

The integral equation for $u_\nu(s, \omega_0)$ as s belongs to the boundary $\sigma_+ \cup \sigma_-$ is then derived and takes the form

$$\alpha_s u_\nu(s,\omega_0) = \sum_{\pm}\int_{\sigma_\pm} \mathrm{d}\sigma \left(\frac{\partial g_\nu(s,\sigma)}{\partial \mathscr{N}_\pm} u_\nu(\sigma,\omega_0) - \eta_\pm \mathscr{A}_0 u_\nu(\sigma,\omega_0)\, g_\nu(s,\sigma)\right) + \Psi_\nu(s,\omega_0), \tag{5.2}$$

where $\alpha_s = \frac{1}{2}$ as $s \in (\sigma_+ \cup \sigma_-)$, $s \notin \{A, B\}$ and $\alpha_s = 1$ as $s \in \{A, B\}$. The properties of the single and double layer potentials have been used in (5.2). It is not difficult to show that $\Psi_\nu(\omega, \omega_0)$ in (5.2), (5.1) admits meromorphic continuation as $\nu \in \mathrm{C}$ such that it is holomorphic as $\nu \in \Pi_\delta$ for some $\delta > 0$.

Remark that the integral equation really has solution, because it is derived from the uniquely solvable problem (3.6), (3.8). It is easily verified that Ψ_ν is a holomorphic mapping $\nu \mapsto L_2(\sigma)$ as $\nu \in \Pi_\delta$, $\Psi_\nu(\cdot, \omega_0) \in L_2(\sigma)$.

Now we explain that the solution $u_\nu(s, \omega_0)$ of the equation (5.2) obeys the same property. To that end, we write the equation (5.2) in an equivalent form

$$(I - \mathscr{K}_0)u_\nu = 2\chi_{\nu,0}, \tag{5.3}$$

where

$$\mathscr{K}_0 u_\nu(s,\omega_0) = \sum_{\pm}\int_{\sigma_\pm} \mathrm{d}\sigma \frac{\partial g_\nu(s,\sigma)}{\partial \mathscr{N}_\pm} u_\nu(\sigma,\omega_0),\ s \in \sigma \setminus \{A,B\}$$

$$2\chi_{\nu,0}(s,\omega_0) = 2\sum_{\pm}\eta_\pm \int_{\sigma_\pm} \mathrm{d}\sigma\ g_\nu(s,\sigma)\mathscr{A}_0 u_\nu(\sigma,\omega_0) + 2\Psi_\nu(s,\omega_0),$$

so that the operator $(I - \mathscr{K}_0)$ is boundedly invertible for any $\nu \in \Pi_\delta$ and $(I - \mathscr{K}_0)^{-1}$ is holomorphic with respect to $\nu \in \Pi_\delta$. Because the right-hand side $\chi_{\nu,0}$ in (5.3) is a holomorphic mapping $\nu \mapsto L_2(\sigma)$ as $\nu \in \Pi_\delta$, $\chi_{\nu,0}(\cdot, \omega_0) \in L_2(\sigma)$, we conclude that the solution u_ν of the equation (5.2) is a holomorphic mapping $\nu \mapsto L_2(\sigma)$ as $\nu \in \Pi_\delta$, $u_\nu(\cdot, \omega_0)|_\sigma \in L_2(\sigma)$ (see condition 2 in Section 3.3).

The representation (5.1) enables one to assert that u_ν is also a holomorphic mapping $\nu \mapsto L_2(\Sigma)$ as $\nu \in \Pi_\delta$, $u_\nu(\cdot, \omega_0) \in L_2(\Sigma)$. Indeed, the integrand in the right-hand side of (5.1) is holomorphic as $\nu \in \Pi_\delta$ for all $\omega \in \Sigma$ because $g_\nu(\omega, \cdot)|\sigma$ obeys this property as well as $u_\nu(\cdot, \omega_0)|_\sigma$ in the integrand, $g_\nu(\omega, \cdot)$ is continuous w.r.t. $\omega \in \Sigma$. $u_\nu(s, \omega_0)|_\sigma$ is continuous w.r.t. s assuming that $\omega_0 \notin \Omega_{\delta_0}$. The same regularity is valid for $\mathscr{A}_0 u_\nu(\sigma, \omega_0)$, which follows from the explicit formula for $\mathscr{A}_0 u_\nu(\sigma, \omega_0)$. After integration w.r.t. σ we obtain the desired, where we also make use of the regularity of $\Psi_\nu(\omega, \omega_0)$ declared above.

Making use of the regularity of the mapping $(u_\nu + u_\nu^i)|_\sigma$ as $\nu \in \Pi_\delta$, from the boundary condition (3.8) we have that $\left.\frac{\partial \widehat{u}_\nu(\omega,\omega_0)}{\partial \mathscr{N}_\pm}\right|_{\sigma_\pm}$ specifies a holomorphic mapping as $\nu \in \Pi_{1+\delta}$. The latter directly follows from the fact that the denominator $\cos \pi\tau + \cos \pi\nu \neq 0$ as $\tau \in \mathrm{i}R$ and $\nu \in \Pi_1$ in the integrand of (3.8), whereas $(u_\nu + u_\nu^i)|_\sigma$ is holomorphic as $\nu \in \Pi_\delta$.

Further use of the integral equation (5.2) enables one to continue analytically u_ν onto the strip $\nu \in \Pi_{1+\delta}$ as a meromorphic mapping with the poles located in the strip $|I\nu| < b$, $b > 0$. Indeed, we make use of the same argumentation, exploiting the equation written in the form (5.3) and the fact that $(I - \mathscr{K}_0)$ is boundedly invertible, in order to verify that $u_\nu(s, \omega_0)|_\sigma$ has only polar singularities as $\nu \in \Pi_{1+\delta}$ because $\mathscr{A}_0 u_\nu(\sigma, \omega_0)$ is regular and $g_\nu(s, \sigma)$, $\Psi_\nu(s, \omega_0)$ are meromorphic in this strip with the poles located on the real axis.

The condition (3.7) shows that having the meromorphic functions in the strip $\Pi_{1+\delta}$ one can continue $\left.\frac{\partial \widehat{u}_\nu(\omega,\omega_0)}{\partial \mathscr{N}_\pm}\right|_{\sigma_\pm}$ from this strip onto a neighbouring strip on the left- or right-hand sides. Because the shift of the argument ν is performed along the real axis, the corresponding poles appear only in some strip $|I\nu| < b$, $b > 0$. On this way we obtain that $\left.\frac{\partial \widehat{u}_\nu(\omega,\omega_0)}{\partial \mathscr{N}_\pm}\right|_{\sigma_\pm}$ and $u_\nu(s, \omega_0)|_\sigma$ are meromorphic in the strip $\Pi_{2+\delta}$.

The corresponding procedure can be iterated and one can prove meromorphic continuation of the mapping u_ν (as well as of $\left.\frac{\partial \widehat{u}_\nu}{\partial \mathscr{N}_\pm}\right|_{\sigma_\pm}$) from the strip $\Pi_{2+\delta}$ onto the complex plane with the singularities located in some strip $|I\nu| < b$, $b > 0$. The representation (5.1) enables one to assert that $u_\nu(s, \omega_0)|_\sigma$ is a meromorphic mapping $\nu \mapsto L_2(\Sigma)$ as $\nu \in C$, with the values $u_\nu(\cdot, \omega_0) \in L_2(\Sigma)$.

6. Sommerfeld integral representation for the wave field

In order to study the far-field asymptotics of the scattered wave field $U(r, \varphi)$ it is profitable to reduce the Watson-Bessel representation to the Sommerfeld integral because, contrary to that of the Watson-Bessel, it is well adapted to this goal.

6.1. *Sommerfeld representation for the incident wave*

We transform the Watson–Bessel representation (3.1) for the incident plane wave making use of the Sommerfeld formula for the Bessel function

$$J_\nu(kr) = \frac{1}{2\pi} \int\limits_{\gamma_-} \mathrm{e}^{-\mathrm{i}kr\cos\alpha}\, \mathrm{e}^{\mathrm{i}\nu\pi/2 - \mathrm{i}\nu\alpha}\, d\alpha,$$

where γ_- is the lower part of the double-loop Sommerfeld contour (Fig. 8).

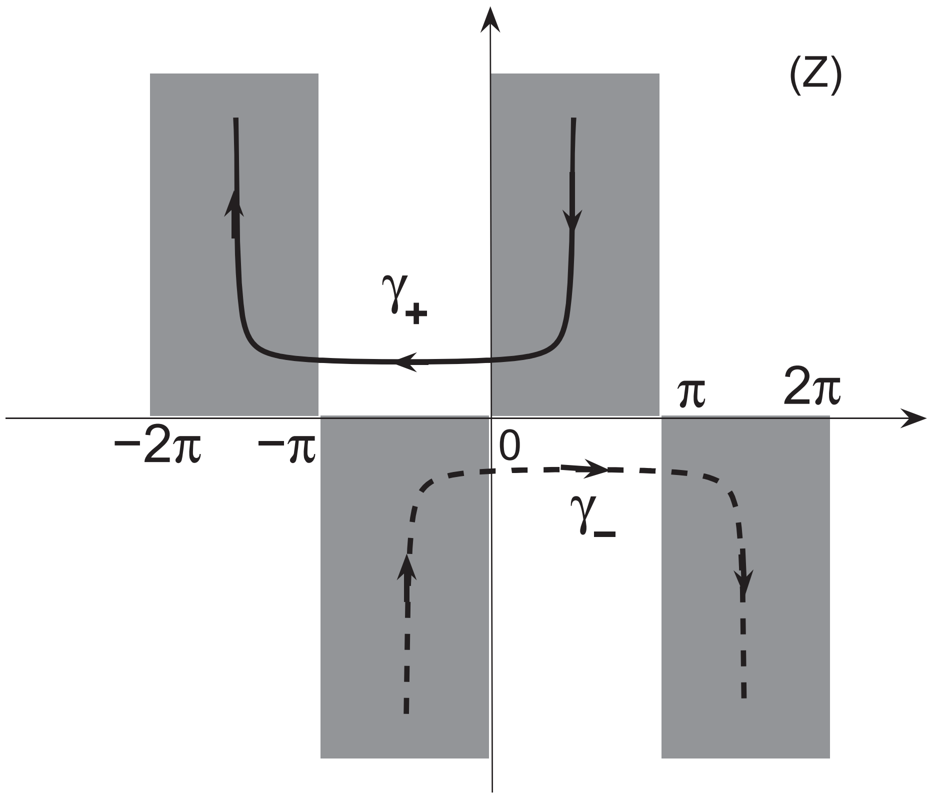

FIG. 8. Sommerfeld double-loop contour, $\gamma = \gamma_+ \cup \gamma_-$.

Substitute the latter into (3.1) and change the orders of integration, which is justified,

$$U_i(r,\omega,\omega_0) = 4\mathrm{i}\sqrt{\frac{\pi}{2}}\int\limits_{C_0} d\nu\, u^i_\nu(\omega,\omega_0)\frac{\nu\,\mathrm{e}^{-\mathrm{i}\nu\pi/2}}{\sqrt{-\mathrm{i}kr}}\left(\frac{1}{2\pi}\int\limits_{\gamma_-}\mathrm{e}^{-\mathrm{i}kr\cos\alpha}\,\mathrm{e}^{\mathrm{i}\nu\pi/2-\mathrm{i}\nu\alpha}\,d\alpha\right)$$

$$= \frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma_-} d\alpha\,\frac{\mathrm{e}^{-\mathrm{i}kr\cos\alpha}}{\sqrt{-\mathrm{i}kr}}\left(-2\sqrt{2\pi}\int\limits_{C_0}\nu\,\mathrm{e}^{-\mathrm{i}\nu\alpha}u^i_\nu(\omega,\omega_0)\,d\nu\right)$$

$$= \frac{1}{2\pi\mathrm{i}}\int\limits_{\gamma_-}\frac{\mathrm{e}^{-\mathrm{i}kr\cos\alpha}}{\sqrt{-\mathrm{i}kr}}\Psi_i(\alpha,\omega,\omega_0)\,d\alpha, \qquad (6.1)$$

where we introduced notation

$$\Psi_i(\alpha,\omega,\omega_0) = -2\sqrt{2\pi}\int\limits_{C_0}\nu\,\mathrm{e}^{-\mathrm{i}\nu\alpha}u^i_\nu(\omega,\omega_0)\,d\nu,$$

$\Psi_i(\alpha,\omega,\omega_0)$ is regular in $D_- = \{\alpha \in C : \Im\alpha < 0\}$ because $u^i_\nu(\omega,\omega_0)$ is bounded on C_0 for any $\omega \in S^2$.

In view of the estimate (3.3) on the contour C_ϕ one can verify that $\Psi_i(\cdot, \omega, \omega_0)$ is regular in the domain

$$D_\theta^\phi = \{\alpha \in C: \ \sin\phi\,\Re\alpha + \cos\phi\,\Im\alpha < \theta|\sin\phi|\} \cap \{\alpha \in C: \ -\sin\phi\,\Re\alpha + \cos\phi\,\Im\alpha < \theta|\sin\phi|\}$$

because

$$\left|\nu\,\mathrm{e}^{-\mathrm{i}\nu\alpha} u_\nu^i(\omega,\omega_0)\right| < C|\nu|^{1/2}\exp\left\{|\nu|\big(|\sin\phi|\,\Re(\alpha) + \cos\phi\,\Im(\alpha) - \theta|\sin\phi|\big)\right\},$$

$\theta = \theta(\omega, \omega_0)$. It is useful to notice that the regularity domain varies from the halfplane D_θ^0 to the strip $D_\theta^{\pi/2}$ (see also Fig. 7 in Lyalinov, 2013).

Consider $\Psi_i(\cdot, \omega, \omega_0)$ in $D_\theta^{\pi/2}$ and introduce a regular function $\Phi_i(\cdot, \omega, \omega_0)$ by the equality

$$\Phi_i(\alpha,\omega,\omega_0) := \frac{1}{2}\Psi_i(\alpha,\omega,\omega_0) = \mathrm{i}\sqrt{2\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \nu\,\sin(\nu\alpha) u_\nu^i(\omega,\omega_0)\,d\nu,$$

which is obviously odd in α, $\Phi_i(\alpha, \omega, \omega_0) = -\Phi_i(-\alpha, \omega, \omega_0)$. We easily verify that $\Phi_i(\cdot, \omega, \omega_0)$ is analytically continued as a holomorphic function into $C\backslash l_\pm$, $l_\pm = \{\alpha \in C: \pm\Re(\alpha) \geqslant \theta(\omega, \omega_0), \Im(\alpha) = 0\}$, i.e. it is regular in the complex plane with the cuts along the lines $l_\pm$. Remark that $\Phi_i(\alpha, \omega, \omega_0)$ can be computed in a closed form Lyalinov (2013)

$$\Phi_i(\alpha,\omega,\omega_0) = \frac{\partial\widetilde{\Phi}_i(\alpha,\omega,\omega_0)}{\partial\alpha}$$

with

$$\widetilde{\Phi}_i(\alpha,\omega,\omega_0) = -\frac{\sqrt{\pi}}{2}\big[\cos\alpha - \cos\theta(\omega,\omega_0)\big]^{-1/2},$$

the branch is fixed by the condition $\sqrt{\cos\alpha - \cos\theta(\omega,\omega_0)} > 0$ as $-\theta < \alpha < \theta$. This expression is obtained by means of the formula (see Gradstein & Ryzhik, 1980, 7.216)

$$\frac{1}{\sqrt{2}}[\cos\alpha - \cos\theta]^{-1/2} = \int_0^\infty \mathrm{d}\tau\,\frac{P_{\mathrm{i}\tau-1/2}\big(\cos[\pi-\theta]\big)}{\cos\mathrm{i}\pi\tau}\cos\mathrm{i}\alpha\tau\,, \quad (\mathrm{i}\alpha > 0)\,.$$

The Sommerfeld representation for the incident wave takes the form

$$U_i(r,\vartheta,\varphi) = \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\mathrm{e}^{-\mathrm{i}kr\cos\alpha}}{\sqrt{-\mathrm{i}kr}}\,\Phi_i(\alpha,\omega,\omega_0)\,d\alpha, \tag{6.2}$$

and also

$$U_i(r,\vartheta,\varphi) = \frac{\sqrt{-\mathrm{i}kr}}{2\pi\mathrm{i}}\int_\gamma \mathrm{e}^{-\mathrm{i}kr\cos\alpha}\,\sin\alpha\,\widetilde{\Phi}_i(\alpha,\omega,\omega_0)\,d\alpha, \tag{6.3}$$

where $\gamma = \gamma_+ \cup \gamma_-$ is shown in Fig. 8.

6.2. *Sommerfeld representation for the scattered field*

The derivations given in the previous section motivate appearance of the analogous representations for the scattered field. We have

$$\begin{aligned} U(r,\vartheta,\varphi) &= 4\mathrm{i}\sqrt{\frac{\pi}{2}}\int_{C_0} d\nu\, u_\nu(\omega,\omega_0)\frac{\nu\,\mathrm{e}^{-\mathrm{i}\nu\pi/2}}{\sqrt{-\mathrm{i}kr}}\left(\frac{1}{2\pi}\int_{\gamma_-}\mathrm{e}^{-\mathrm{i}kr\cos\alpha}\,\mathrm{e}^{\mathrm{i}\nu\pi/2-\mathrm{i}\nu\alpha}\,d\alpha\right)\\ &= \frac{1}{2\pi\mathrm{i}}\int_{\gamma_-} d\alpha\,\frac{\mathrm{e}^{-\mathrm{i}kr\cos\alpha}}{\sqrt{-\mathrm{i}kr}}\left(-2\sqrt{2\pi}\int_{C_0}\nu\,\mathrm{e}^{-\mathrm{i}\nu\alpha}u_\nu(\omega,\omega_0)\,d\nu\right)\\ &= \frac{1}{2\pi\mathrm{i}}\int_{\gamma_-}\frac{\mathrm{e}^{-\mathrm{i}kr\cos\alpha}}{\sqrt{-\mathrm{i}kr}}\Psi(\alpha,\omega,\omega_0)\,d\alpha, \end{aligned} \tag{6.4}$$

where we introduced

$$\Psi(\alpha,\omega,\omega_0) = -2\sqrt{2\pi}\int_{C_0}\nu\,\mathrm{e}^{-\mathrm{i}\nu\alpha}u_\nu(\omega,\omega_0)\,d\nu,$$

$\Psi(\cdot,\omega,\omega_0)$ is regular in $D_{\tau_1} = \{\alpha\in C:\ \Im(\alpha) < -|\tau_1(\omega,\omega_0)|\}$ because $u_\nu(\omega,\omega_0)$ satisfies the estimate (3.9) on C_0 (see Fig. 7) for any $\omega\in S^2$. We take into account the estimate

$$\begin{aligned}\left|\nu\,\mathrm{e}^{-\mathrm{i}\nu\alpha}u_\nu(\omega,\omega_0)\right| < C|\nu|^{1/2}\exp\{|\nu|\,(|\sin\phi|\Re(\alpha)+\cos\phi\,(\Im\alpha)\\ -\tau_0(\omega,\omega_0)|\sin\phi| + \operatorname{sign}(\phi)\,\cos\phi|\tau_1(\omega,\omega_0)|\},\end{aligned}$$

and conclude that the regularity domain for $\Psi(\cdot,\omega,\omega_0)$ is

$$\begin{aligned} D^{\phi}_{\tau_0\tau_1} = \{\alpha\in C:\ \sin\phi\Re(\alpha)+\cos\phi\Im(\alpha) < \tau_0|\sin\phi| - \cos\phi\,\tau_1\}\cap\\ \{\alpha\in C:\ -\sin\phi\Re(\alpha)+\cos\phi\Im(\alpha) < \tau_0|\sin\phi| + \cos\phi\,\tau_1\}, \end{aligned}$$

which is verified from

$$\Psi(\alpha,\omega,\omega_0) = -2\sqrt{2\pi}\int_{C_\phi}\nu\,\mathrm{e}^{-\mathrm{i}\nu\alpha}u_\nu(\omega,\omega_0)\,d\nu,$$

where $\phi\in(0,\pi/2)$.

REMARK Provided the contour C_ϕ varies from C_0 into $C_{\pi/2}$ the domain of regularity $D^{\phi}_{\tau_0\tau_1}$ of $\Psi(\cdot,\omega,\omega_0)$ deforms from the halfplane $\Im(\alpha) < -|\tau_1(\omega,\omega_0)|$ onto the strip $|I(\alpha)| < \tau_0(\omega,\omega_0)$.

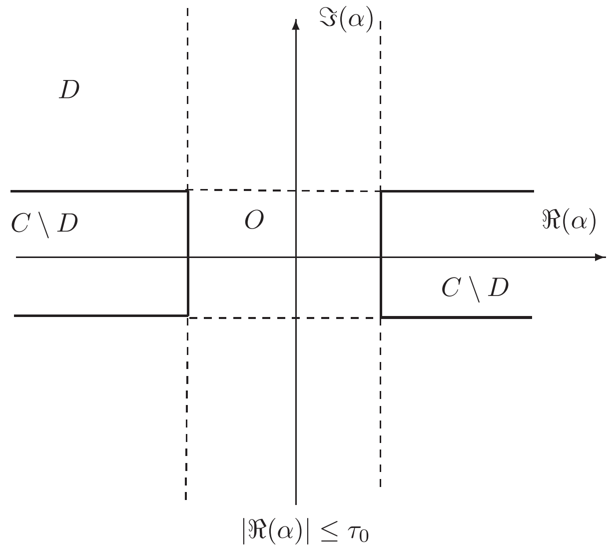

FIG. 9. Domain of regularity D, singularities are in $C \setminus D$.

We consider $\Psi(\cdot, \omega, \omega_0)$ in the strip $D^{\pi/2}_{\tau_0\tau_1}$ and introduce there a regular function

$$\Phi(\alpha, \omega, \omega_0) := \frac{1}{2}\Psi(\alpha, \omega, \omega_0) = \mathrm{i}\sqrt{2\pi} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \nu \, \sin(\nu\alpha) u_\nu(\omega, \omega_0)\, d\nu,$$

which is odd in α, $\Phi(\alpha, \omega, \omega_0) = -\Phi(-\alpha, \omega, \omega_0)$.

Simple analysis enables us to conclude that $\Phi(\cdot, \omega, \omega_0)$ is continued as a regular function into the domain

$$D = D^{\pi/2}_{\tau_0\tau_1} \cup D^{0}_{\tau_0\tau_1} \cup \left(D^{\pi/2}_{\tau_0\tau_1}\right)^*,$$

where $\left(D^{\pi/2}_{\tau_0\tau_1}\right)^*$ is symmetric to $D^{\pi/2}_{\tau_0\tau_1}$ with respect to the origin, (Fig. 9). In other words, we can assert that all singularities of $\Phi(\cdot, \omega, \omega_0)$ are located in $C \setminus D$. For the diffraction by a sector with Dirichlet boundary conditions such singularities are symmetrically distributed on the real axis Lyalinov, 2013. For the sector with the impedance boundary conditions there are some additional complex singularities which are responsible for the surface waves propagating along the surface of the sector.

The Sommerfeld representations for the scattered field take the form

$$U(r, \vartheta, \varphi) = \frac{1}{2\pi\mathrm{i}} \int_{\gamma} \frac{\mathrm{e}^{-\mathrm{i}kr\cos\alpha}}{\sqrt{-\mathrm{i}kr}} \, \Phi(\alpha, \omega, \omega_0)\, d\alpha, \tag{6.5}$$

or

$$U(r,\vartheta,\varphi) = \frac{\sqrt{-\mathrm{i}kr}}{2\pi\mathrm{i}} 1 \int\limits_{\gamma} \mathrm{e}^{-\mathrm{i}kr\cos\alpha}\, \sin\alpha\, \widetilde{\Phi}(\alpha,\omega,\omega_0)\, d\alpha \tag{6.6}$$

with

$$\Phi(\alpha,\omega,\omega_0) = \frac{\partial \widetilde{\Phi}(\alpha,\omega,\omega_0)}{\partial\alpha}.$$

As we remarked the singularities of $\Phi(\alpha,\omega,\omega_0)$, $\widetilde{\Phi}(\alpha,\omega,\omega_0)$ play a crucial role in studying the far-field asymptotics by use of the Sommerfeld integral representations. In order to describe the singularities of the Sommerfeld transformants $\Phi(\alpha,\omega,\omega_0)$, $\widetilde{\Phi}(\alpha,\omega,\omega_0)$ we turn to formulation of the problems for them, although the singularities themselves will be studied elsewhere.

6.3. *Problems for the Sommerfeld transformants $\Phi(\alpha,\omega,\omega_0)$, $\widetilde{\Phi}(\alpha,\omega,\omega_0)$*

It is obvious that the Fourier transform of (3.6) with integration along the imaginary axis, enables us to write down the equation for the Sommerfeld transformant (Lyalinov, 2013),

$$\left(\triangle_\omega - \partial_\alpha^2 - 1/4\right)\Phi(\alpha,\omega,\omega_0) = 0 \tag{6.7}$$

as $\omega \in S^2 \setminus \sigma$.

Let us turn to the boundary conditions

$$r^{-1}\frac{\partial\left(U+U^{\mathrm{i}}\right)}{\partial\mathscr{N}_\pm}\bigg|_{\sigma_\pm} = \frac{1}{\sqrt{-\mathrm{i}kr}}\frac{1}{2\mathrm{i}\pi}\int\limits_\gamma \mathrm{e}^{-\mathrm{i}kr\cos\alpha}(-\mathrm{i}k\sin\alpha)\,\frac{\partial}{\partial\mathscr{N}_\pm}\left(\widetilde{\Phi}+\widetilde{\Phi}_i\right)(\alpha,\omega,\omega_0)\bigg|_{\sigma_\pm}\mathrm{d}\alpha$$
$$= \mathrm{i}k\eta_\pm\left(U+U^{\mathrm{i}}\right)\Big|_S = \frac{\mathrm{i}k\eta_\pm}{\sqrt{-\mathrm{i}kr}}\frac{1}{2\mathrm{i}\pi}\int\limits_\gamma \mathrm{e}^{-\mathrm{i}kr\cos\alpha}\,\frac{\partial}{\partial\alpha}\left(\widetilde{\Phi}+\widetilde{\Phi}_i\right)(\alpha,\omega,\omega_0)\bigg|_{\sigma_\pm}\mathrm{d}\alpha, \tag{6.8}$$

then, exploiting the Malyuzhinets theorem (see Lyalinov & Zhu, 2012, Chapter 1), we obtain

$$\sin\alpha\,\frac{\partial\left(\widetilde{\Phi}+\widetilde{\Phi}_{\mathrm{i}}\right)}{\partial\mathscr{N}_\pm}(\alpha,\omega,\omega_0)\bigg|_{\sigma_\pm} = -\eta_\pm\frac{\partial\left(\widetilde{\Phi}+\widetilde{\Phi}_i\right)}{\partial\alpha}(\alpha,\omega,\omega_0)\bigg|_{\sigma_\pm},$$

which is followed by

$$\frac{\partial(\Phi+\Phi_i)}{\partial\mathscr{N}_\pm}(\alpha,\omega,\omega_0)\bigg|_{\sigma_\pm} = -\eta_\pm\frac{\partial}{\partial\alpha}\frac{(\Phi+\Phi_i)(\alpha,\omega,\omega_0)}{\sin\alpha}\bigg|_{\sigma_\pm}, \tag{6.9}$$

where

$$\Phi_i(\alpha,\omega,\omega_0) = \frac{\partial}{\partial\alpha}\widetilde{\Phi}_i(\alpha,\omega,\omega_0).$$

The equation (6.7) and the condition (6.9) are proved to be valid in the domain D by means of the analytic continuation. For real α from this strip the equation (6.7) is of hyperbolic type. In order to have a correct problem for this equation on $S^2 \setminus \sigma$ with the boundary condition (6.9) as $\alpha > 0$ we should add initial conditions at $\alpha = 0$.

For any $\alpha \in [0, \tau_0)$ we have

$$\Phi(\alpha, \omega, \omega_0) + \Phi_i(\alpha, \omega, \omega_0) = \mathrm{i}\sqrt{2\pi} \int\limits_{\mathrm{i}R} \nu \, \sin(\nu\alpha) u_\nu(\omega, \omega_0)\, d\nu \; - \frac{\sqrt{\pi}}{4} \frac{\sin\alpha}{(\cos\alpha - \cos\theta(\omega, \omega_0))^{3/2}} \tag{6.10}$$

which is a regular function. Therefore, assuming that $u_\nu(\omega, \omega_0)$ is known in the integrand, we arrive at the initial conditions

$$\Phi(0, \omega, \omega_0) = 0, \quad \left.\frac{\partial \Phi(\alpha, \omega, \omega_0)}{\partial \alpha}\right|_{\alpha=0} = \mathrm{i}\sqrt{2\pi} \int\limits_{\mathrm{i}R} \nu^2 \, u_\nu(\omega, \omega_0)\, d\nu. \tag{6.11}$$

The Cauchy boundary value problem (6.7), (6.9), (6.11) can be used in order to determine singularities of $\Phi(\alpha, \omega, \omega_0)$ as $\alpha \in C \setminus D$. However, instead of the conditions (6.11) one can exploit the equality (6.10) for any point $\alpha \in [0, \tau_0)$ together with (6.7), (6.9).

The analogous problem is valid for $\widetilde{\Phi}(\alpha, \omega, \omega_0)$

$$\left(\triangle_\omega - \partial_\alpha^2 - 1/4\right) \widetilde{\Phi}(\alpha, \omega, \omega_0) = 0 \tag{6.12}$$

as $\omega \in S^2 \setminus \sigma$ and

$$\sin\alpha \left.\frac{\partial\left(\widetilde{\Phi} + \widetilde{\Phi}_{\mathrm{i}}\right)}{\partial N_\pm}\right|_{\sigma_\pm} = -\eta_\pm \left.\frac{\partial\left(\widetilde{\Phi} + \widetilde{\Phi}_i\right)}{\partial \alpha}\right|_{\sigma_\pm}, \tag{6.13}$$

and

$$\widetilde{\Phi}(0, \omega, \omega_0) = -\mathrm{i}\sqrt{2\pi} \int\limits_{\mathrm{i}R} u_\nu(\omega, \omega_0)\, d\nu, \quad \left.\frac{\partial \widetilde{\Phi}(\alpha, \omega, \omega_0)}{\partial \alpha}\right|_{\alpha=0} = 0. \tag{6.14}$$

The discussion above enables us to prove

THEOREM 6.1 There exists an analytic function $\Phi(\alpha, \omega, \omega_0)$ which is holomorphic in the domain D. For real $\alpha \in [0, \tau_0)$ this function solves the problem (6.7), (6.9), (6.11). For the strip $|\Re(\alpha)| < \tau_0(\omega, \omega_0)$ it is specified by the equality (see (6.10))

$$\Phi(\alpha, \omega, \omega_0) = \mathrm{i}\sqrt{2\pi} \int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \nu \, \sin(\nu\alpha) u_\nu(\omega, \omega_0)\, \mathrm{d}\nu.$$

A similar statement is valid for $\widetilde{\Phi}(\alpha, \omega, \omega_0)$.

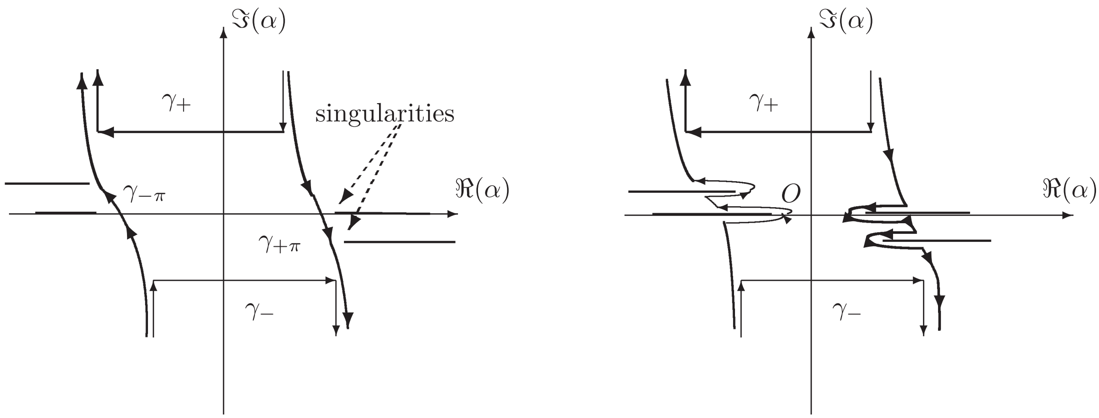

FIG. 10. Deformation of the Sommerfeld contour into the steepest descent paths $\gamma_+ \cup \gamma_-$: (left) singularities are not captured, $\omega \in \Omega_0$; (right) singularities are captured $\omega \in S^2 \setminus \Omega_0$.

7. Comments on the derivations of the far-field asymptotics. Diffraction coefficient for narrow sector in 'oasis'

In this section we give some comments dealing with the derivations of the far-field asymptotics described in Section 2.2 although the detailed exposition of the results will appear in a future work.

Solution $\Phi(\alpha, \omega, \omega_0)$ of the problems for the hyperbolic equation (6.7) as $\alpha \in [0, \tau_0)$ is unique. This solution is represented by the Fourier type integral (see, Theorem 6.1) as $|\Re(\alpha)| < \tau_0(\omega, \omega_0)$. However, one needs to have analytic continuation of the Sommerfeld transformant into $C \setminus D$, in particular, into vicinities of the singularities[9] located in $C \setminus D$. As we mentioned, these singularities are of crucial importance for studying the far-field asymptotics (Lyalinov, 2013). Indeed, in order to evaluate the Sommerfeld integral representation (6.5) asymptotically as $kr \to \infty$ we ought to deform the Sommerfeld double-loop contour γ into the steepest descent paths $\gamma_{\pm\pi}$ (Fig. 10). In the process of such deformation some singularities of the transformant can be captured. Contributions of these singularities (Fig. 10, right) give rise to the corresponding components of the far field such as reflected wave, edge waves (see Lyalinov, 2013 for the Dirichlet boundary conditions) as well as the surface waves which may be excited near the impedance surface of the sector (2.9), (2.10). These surface waves are governed by complex singularities which do not exist in the case of the ideal boundary conditions. It is worth mentioning that the singularities migrate (Fig. 10) when the observation point varies its position. The saddle point $\pm\pi$ are responsible for the spherical wave from the vertex of the sector. The diffraction coefficient of this wave is easily calculated by use of the direct application of the saddle point technique provided the observation point (r, ω) is located in the oasis, $\omega \in \Omega_0$. In this case the singularities are not captured (Fig. 10, left) and the asymptotics of the scattered far-field is given by (see also (2.8))

$$U(r, \vartheta, \varphi) = D(\omega, \omega_0)\frac{\exp(\mathrm{i}kr)}{-\mathrm{i}kr}\left(1 + O\left(\frac{1}{kr}\right)\right), \quad kr \to \infty.$$

[9] These singularities are of the branch point type, see also Lyalinov (2013).

with

$$D(\omega,\omega_0) = -\sqrt{\frac{2}{\pi}}\,\Phi(\pi,\omega,\omega_0) = \frac{2}{\mathrm{i}}\int\limits_{\mathrm{i}R} \nu\,\sin(\pi\nu)u_\nu(\omega,\omega_0)\,\mathrm{d}\nu \tag{7.1}$$

or[10]

$$D(\omega,\omega_0) = 2\int\limits_{\mathrm{i}R} \nu\,\exp(-\mathrm{i}\pi\nu)u_\nu(\omega,\omega_0)\,\mathrm{d}\nu,$$

where the integrals converge exponentially as $\omega \in \Omega_0$. It is obvious that the numerical determination of the spectral function $u_\nu(\omega,\omega_0)$ is necessary for the calculation of the diffraction coefficient.

In order to demonstrate also some practical issues from the results developed above we consider derivation of the closed form of the asymptotic expression for the diffraction coefficient in the case of a narrow impedance sector, $\beta = 2a \ll 1$. In this case a closed asymptotic formula for the spectral function $u_\nu(\omega,\omega_0)$ can be given and the integral for the diffraction coefficient (7.1) can be computed explicitly leading to a simple expression in the high order approximation as $\beta \ll 1$.

7.1. *Expression for $D(\omega,\omega_0)$ as $\beta = 2a \ll 1$*

There are, at least, two ways to determine such an expression. The first one is in asymptotic solution of the boundary value problem (3.6), (3.8) for the spectral function $u_\nu(\omega,\omega_0)$. It is based on matching local asymptotic expansions (see e.g. Babich, 1997; Lyalinov & Zhu, 2012, pp. 122–124) for the problem on the unit sphere S^2 with the narrow cut $\sigma = \sigma_+ \cup \sigma_-$, $mes(\sigma) = 2mes(\sigma_\pm) = 2\beta \ll 1$.

In this work we exploit an alternative approach, based on approximate solution of the integral equation (5.2) for $u_\nu(\sigma,\omega_0)$ as $mes(\sigma_\pm) = \beta \ll 1$, Bernard & Lyalinov (2001). It leads to the same desired result and also makes use of the integral representation (5.1) of the solution $u_\nu(\omega,\omega_0)$, $\omega \in \Omega_0$.

Simple analysis of the integral equation (5.2) enables us to assert that $u_\nu(\omega|_{\sigma_\pm},\omega_0)$ is of $O(\beta)$, therefore, taking into account that $mes(\sigma_+) = \beta$ and $mes(\sigma_-) = \beta$, from the integral representation (5.1) we find

$$\begin{aligned} u_\nu(\omega,\omega_0) = {}& \Psi_\nu(\omega,\omega_0)(1+O(\beta\log\beta)) \\ = {}& -\frac{1}{2\mathrm{i}}\sum_\pm \eta_\pm \int\limits_{\sigma_\pm} \mathrm{d}\sigma\, g_\nu(\omega,\sigma)\left(\int\limits_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\tau\,\frac{\tau\,\sin\pi\tau\,u^i_\tau(\sigma,\omega_0)}{\cos\pi\tau+\cos\pi\nu}\right)(1+O(\beta\log\beta)), \quad \omega\in\Omega_0, \end{aligned} \tag{7.2}$$

where we took into account the estimate

$$\sum_\pm \int\limits_{\sigma_\pm} \mathrm{d}\sigma\, g_\nu(\omega,\sigma)\,\frac{\partial u^i_\nu}{\partial \mathcal{N}_\pm}\bigg|_\sigma = O(\beta^2\log\beta).$$

[10] For the first time this kind of formulae for the ideal circular cones has been obtained by Smyshlyaev, see e.g. Smyshlyaev (1990).

Substituting the integral expression from (7.2) into the formula for the diffraction coefficient (7.1) in oasis, in the leading approximation we arrive at

$$D(\omega,\omega_0) = \sum_{\pm} \frac{\eta_\pm}{16} \int_{\sigma_\pm} \mathrm{d}\sigma$$

$$\left(\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\tau \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{d}\nu \, \frac{\tau \, \tan\pi\tau \;\; \nu \, \tan\pi\nu}{\cos\pi\tau + \cos\pi\nu} P_{\nu-1/2}(\cos\theta(\omega,\sigma)) P_{\tau-1/2}(-\cos\theta(\sigma,\omega_0)) \right)$$

The iterated integral in the brackets was computed in the work (Bernard & Lyalinov (1999) so that one has

$$D(\omega,\omega_0) = -\sum_{\pm} \frac{\eta_\pm}{4\pi} \int_{\sigma_\pm} \frac{\mathrm{d}\sigma}{[\cos\theta(\omega,\sigma) + \cos\theta\;(\sigma,\omega_0)]^2} (1 + O(\beta\log\beta))$$

as $\theta(\omega,\sigma) + \theta(\sigma,\omega_0) > \pi$.

Integration in the latter formula is conducted along asymptotically small arcs $\sigma_\pm$ then asymptotically equivalent version of this formula takes the form

$$D(\omega,\omega_0) = -\frac{\eta_+ + \eta_-}{2\pi} \frac{2a}{[\cos\theta(\omega,M) + \cos\theta\;(M,\omega_0)]^2} \;(1 + O(\beta\log\beta)),$$

$\theta(\omega, M) + \theta(M, \omega_0) > \pi$ and $\omega \in \Omega_0$, M is the middle point of the arc AB. The advantage of the latter formula is that it is quite elementary and can be easily used in the engineering applications.[11]

It is instructive to compare this result with that obtained by Babich (1997), where he made use of matching asymptotic series in the case of Dirichlet boundary condition on the surface of a cone,

$$D_d(\omega,\omega_0) = -4\pi \int_{-\infty}^{\infty} \mathrm{d}\tau \;\tau\, \mathrm{e}^{\pi\tau} \frac{g_\tau(\omega,M)\, g_\tau(M,\omega_0)}{W_\beta - \Re\psi(\mathrm{i}\tau - 1/2) - \mathscr{C} - \log 2} \;(1 + O(\beta\log\beta))\,,$$

$\theta(\omega, M) + \theta(M, \omega_0) > \pi$, where $W_\beta = -\log(a/2) = \mathrm{O}(\log\beta)$, W_β is the Wiener capacity, an integral characteristic of the segment of the length $2a$, $\mathscr{C}$ is Euler constant, $g_\tau(\omega,M) = -\frac{P_{\mathrm{i}\tau-1/2}(-\cos\theta(\omega,M))}{4\cosh\pi\tau}$. As mentioned in Babich (1997) the latter expression for the diffraction coefficient $D_d(\omega, \omega_0)$ integral may be simplified by formally neglecting $-\Re\psi(\mathrm{i}\tau - 1/2) - \mathscr{C} - \log 2$ in comparison with W_β then computed, which leads to

$$D_d(\omega,\omega_0) = -\frac{1}{2\log\frac{a}{2}} \frac{1}{[\cos\theta(\omega,M) + \cos\theta(M,\omega_0))]} \;(1 + O(1/\log\beta)),$$

as $\theta(\omega, M) + \theta(M, \omega_0) > \pi$.

[11] It is worth remarking that the identical result can be obtained by use of the matching the local asymptotic series, which is shown in Lyalinov & Zhu (2012), Chapter 5, for a convex impedance cone.

It is obvious that the diffraction coefficient $D_d(\omega, \omega_0)$ for the Dirichlet case is of $O(1/\log(a))$ in the leading approximation, whereas $D(\omega, \omega_0) = O(a)$ as $a \to 0$ for a narrow sector with the impedance boundary conditions.

8. Conclusion

In this work we developed a motivated procedure to study the problem of diffraction by a semi-infinite sector with impedance boundary conditions. Such kind of the boundary conditions thwarts complete separation of variables. The separation of the radial variable in the boundary conditions leads to a condition which is non-local with respect to the parameter of separation. Nevertheless, after separation of the radial variable the problem on the unit sphere with the non-local condition on the cut AB admits an efficient study. To that end, the traditional theory of extension of sectorial sesquilinear forms has been exploited. The Watson–Bessel integral representation for the solution is not efficient for the derivation of the far-field asymptotics that is why reduction to the Sommerfeld integral representations was used. Analytic properties of the Sommerfeld transformants were considered. In particular, domains, where singularities of the transformants are localized, were indicated. Although a complete study of the singularities and of their contributions to the far-field asymptotics is postponed to a future work, we obtained a practically useful simple formula for the diffraction coefficient of the spherical wave from the vertex in 'oasis' in the case of scattering by a narrow impedance sector.

One of the goals for the further studies is to develop a reliable numerical procedure in order to compute the spectral function $u_\nu(\omega, \omega_0)$. Such a development could be based on our study of the problem for the spectral function given in this work. The situation with the impedance boundary conditions is more complex than that for the ideal ones. In particular, there is no reason to assert that the spectrum of the operator A from the Section 4.3 is discrete. Contrary to the case of the Laplace–Beltrami operator on the unit sphere with usual mixed (Robin) boundary conditions with complex impedance parameter (see Assier *et al.*, 2016, Theorem 4.10) on the cut the argumentation based on the compact resolvent does not work in our spectral problem. However, there are some hopes that the study of the spectral properties of the non-local operator $\mathscr{A}$ in the boundary condition of the problem (4.2) may be exploited in order to create an efficient numerical procedure to compute the spectral function $u_\nu(\omega, \omega_0)$. This might be used in order to reduce the spectral problem with non-local operator in the boundary condition to that with simpler conditions like those complex Robin in Assier *et al.* (2016).

It is also important to give a complete description of the far-field asymptotics. Some natural extensions of the ideas proposed in Lyalinov (2013) might be useful in this case.

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