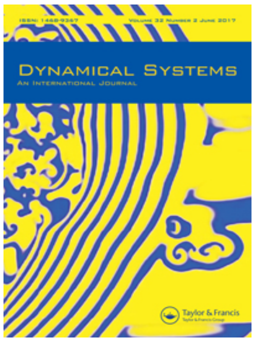

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Inverse shadowing in group actions

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ABSTRACT

We study the inverse shadowing property for actions of some finitely generated groups. A tube condition for such actions is introduced and analysed. We prove a reductive inverse shadowing theorem for actions of virtually nilpotent groups.

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1. Introduction

The theory of shadowing of pseudotrajectories (approximate trajectories) is now a well-developed branch of the theory of dynamical systems (see, for example, the monographs [1,2] and the recent survey [3]).

A classical dynamical system can be considered as an action of the group $\mathbb{Z}$. The first shadowing results for actions of groups more general than $\mathbb{Z}$ were obtained in [4] (we describe these results and their generalizations in Section 2).

The main shadowing result obtained in [4] for actions of the group $\mathbb{Z}^p$, $p > 1$, was a kind of reductive shadowing theorem (RST) – it was shown that if the action of a one-dimensional subgroup of $\mathbb{Z}^p$ is topologically Anosov (i.e. it has the shadowing property and is expansive), then the action of $\mathbb{Z}^p$ is topologically Anosov as well.

In parallel with the theory of shadowing, the theory of inverse shadowing was developed (the corresponding definitions first appeared in [5,6]). The difference between the two theories is as follows. The shadowing property means that, given an approximate trajectory, we can find an exact trajectory close to it. The inverse shadowing property means that, given a family of approximate trajectories (generated by a so-called approximate method), we can find a member of this family that is close to any chosen exact trajectory.

In this paper, we study the property of inverse shadowing for actions of some finitely generated groups. Our main result is the so-called reductive inverse shadowing theorem (RIST); it gives sufficient conditions under which an action of a virtually nilpotent group has

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the inverse shadowing property. This result is motivated by a similar result for shadowing proved in [7] (and its proof uses an approach close to the approach of [7]).

The main novelty of our approach compared to that of [4,7] is the introduction of the so-called tube condition (TC) which replaces the notion of a topologically Anosov action.

In a sense, the TC means that near any exact trajectory there is a narrow 'tube' formed by approximate trajectories that are close to the exact one and, in addition, any approximate trajectory that belongs to a neighbourhood of a fixed size of the exact trajectory belongs to the narrow tube.

We show that the TC is natural in the theory of inverse shadowing (it allows us to prove the RIST, Theorem 2.1).

At the same time, the TC indicates the essential difference between the shadowing and inverse shadowing properties. We show this in Section 3, where we treat the simplest case of a hyperbolic linear automorphism. Such a system is topologically Anosov, it has the shadowing property, and (in the case of a small enough one step error) the shadowing trajectory is unique. We construct an example which shows that for the case of inverse shadowing, where the TC is satisfied, the situation may be essentially different, and arbitrarily narrow tubes around an exact trajectory may contain infinite sets of different approximate trajectories generated by an approximate method.

The structure of the paper is as follows. In Section 2, we define shadowing and inverse shadowing for classical dynamical systems (actions of $\mathbb{Z}$) and for actions of finitely generated groups. We describe results of [4,7]. The TC for group actions is introduced. In Section 3, we analyse the TC in the case of a hyperbolic linear automorphism. Section 4 is devoted to the proof of the main result (RIST for a virtually nilpotent group).

2. Definitions and main result

Let us first recall the definition of the standard shadowing property for homeomorphisms of metric spaces.

Let f be a homeomorphism of a metric space (M, dist). Let $d > 0$. A sequence of points $\{x_n \in M\}$ is called a d-pseudotrajectory of f if

$$\mathrm{dist}(x_{n+1}, f(x_n)) < d, \quad n \in \mathbb{Z}. \tag{1}$$

We say that f has the (standard) shadowing property if for any $\varepsilon > 0$ there exists a $d > 0$ such that for any d-pseudotrajectory $\{x_n\}$ there exists an exact trajectory $\{f^n(p)\}$ satisfying the inequalities

$$\mathrm{dist}(f^n(p), x_n) < \varepsilon, \quad n \in \mathbb{Z}. \tag{2}$$

Now, let us pass to a similar property for actions of finitely generated groups. Let Φ be an action of a finitely generated group G on (M, dist). We consider G in multiplicative notation with unit element e (unless otherwise explicitly stated; for example, it is more convenient to treat a standard dynamical system with discrete time as an action of the group $\mathbb{Z}$ with additive notation).

Let S be a finite symmetric generating set of G (everywhere below, the term 'generating set' means a finite symmetric generating set).

We say that a family $\{x_g: g \in G\}$ is a d-pseudotrajectory of the action Φ with respect to the generating set S if

$$\mathrm{dist}(x_{sg}, \Phi(s, x_g)) < d, \quad g \in G, \ s \in S. \tag{3}$$

Remark 2.1: Let us note that the exact analogue of condition (3) for a standard dynamical system generated by a homeomorphism f (and considered as an action of $\mathbb{Z}$ with symmetric generating set $S = \{\pm 1\}$) is not condition (1) but the condition

$$\mathrm{dist}(x_{n\pm 1}, f^{\pm 1}(x_n)) < d, \quad n \in \mathbb{Z}. \tag{4}$$

Let us assume that f^{-1} is uniformly continuous and find a function $\delta(d) \to 0$ as $d \to 0$ such that the inequality

$$\mathrm{dist}(x, y) < d, \quad d > 0, \tag{5}$$

implies the inequality

$$\mathrm{dist}(f^{-1}(x), f^{-1}(y)) < \delta(d).$$

Clearly, in this case, condition (1) implies the analogue of condition (4) with d replaced by $\max(d, \delta(d))$.

We say that a group action Φ has the shadowing property with respect to a generating set S if for any $\varepsilon > 0$ there exists a $d > 0$ such that for any d-pseudotrajectory $\{x_g\}$ with respect to the generating set S there exists an exact trajectory $\{p_g = \Phi(g, p_e): g \in G\}$ of Φ satisfying the inequalities

$$\mathrm{dist}(p_g, x_g) < \varepsilon, \quad g \in G. \tag{6}$$

Let us say that Φ is uniformly continuous if, for some generating set S, any $\Phi(s, .)$, $s \in S$, is uniformly continuous. Clearly, if Σ is a different generating set, then any $\Phi(\sigma, \cdot)$, $\sigma \in \Sigma$, is uniformly continuous as well (thus, we may speak about the uniform continuity of Φ).

Main technical assumption (MTA).Everywhere below, we assume that all the considered actions Φ are uniformly continuous.

Remark 2.2: It is easy to show (see [7] and compare with the proof of Lemma 2.2 below) that if S and Σ are two generating sets, then Φ has (or does not have) the shadowing property with respect to S and Σ simultaneously (thus, the shadowing property does not depend on the choice of S).

Generalizing the well-known property of discrete dynamical systems, we say that an action Φ is expansive if there exists an $a > 0$ such that if

$$\mathrm{dist}(\Phi(g, x), \Phi(g, y)) \le a, \quad g \in G,$$

then $x = y$.

The following property is also an analogue of a known property: an action Φ is called topologically Anosov if

- Φ has the shadowing property;
- Φ is expansive.

Let us call the next statement the RST for a group G: if there exists an element $g \in G$ such that the homeomorphism $f(\cdot) = \Phi(g, \cdot)$ is topologically Anosov, then Φ is topologically Anosov (and hence, it has the shadowing property).

The first RST was proved in [4]: the RST is valid for $G = \mathbb{Z}^p,\ p > 1$.

At present, the most general result known to the author is published in [7]: the RST is valid for virtually nilpotent groups (we give a formal definition of a virtually nilpotent group below, in Section 4).

At the same time, the RST is not valid for some solvable groups (see an example in [7]: the RST is not valid for the Baumslag–Solitar group $BS(1, n) = \langle a, b | ba = a^n b \rangle, n > 1$).

Passing to inverse shadowing, we first note that this property depends not only on the system but also on the family of approximate trajectories (usually, such families are generated by so-called (approximate) methods). In this paper, we select one class of methods (similar to methods of class Θ_s introduced in [8]).

Let $f: M \to M$ be a homeomorphism. We say that a family of continuous mappings

$$\Gamma = \{\gamma_n \in C(M, M) : \ n \in \mathbb{Z}\}$$

is a d-method for f if

$$\operatorname{dist}(\gamma_{n\pm1}(x),\ f^{\pm1}(\gamma_n(x))) < d, \quad n \in \mathbb{Z},\ x \in M; \tag{7}$$

we assume that $\gamma_0 = \mathrm{Id}$.

A sequence $\xi = \{x_n \in M\}$ is called a trajectory of a d-method Γ if

$$x_n = \gamma_n(x_0), \quad n \in \mathbb{Z}.$$

Clearly, a trajectory of a d-method is a d-pseudotrajectory of f in the sense of condition (4).

We say that f has the inverse shadowing property if for any $\varepsilon > 0$ there exists a $d > 0$ such that for any $x \in M$ and any d-method Γ there exists a trajectory $\xi = \{x_n \in M\}$ of Γ satisfying the inequalities

$$\operatorname{dist}(f^n(x), x_n) < \varepsilon, \quad n \in \mathbb{Z}.$$

Now, we define a similar property for group actions.

Let Φ be an action of a finitely generated group G; a d-method for Φ with respect to a generating set S of G is a family

$$\Gamma = \{\gamma_g \in C(M, M) : \ g \in G\}$$

such that

$$\operatorname{dist}(\gamma_{sg}(x),\ \Phi(s, \gamma_g(x))) < d, \quad g \in G,\ s \in S,\ x \in M; \tag{8}$$

we assume that $\gamma_e = \mathrm{Id}$.

A family

$$\{x_g = \gamma_g(x_e) \in M : \ g \in G\}$$

is called a trajectory of the method Γ.

We say that Φ has the inverse shadowing property with respect to the generating set S if for any $\varepsilon > 0$ there exists a $d > 0$ such that for any $p \in M$ and any d-method Γ with respect to the generating set S there exists a trajectory $\{x_g\}$ of Γ satisfying inequalities (6), where $p_g = \Phi(g, p)$.

Let us fix two generating sets of G, S and Σ. Clearly, there exists a constant $C > 0$ such that any $s \in S$ can be represented as follows:

$$s = \sigma'_1 \cdots \sigma'_c, \quad \sigma'_1, \ldots, \sigma'_c \in \Sigma, \ c \le C. \tag{9}$$

Lemma 2.1: *For any $d_0 > 0$ there exists a $d^* = d^*(d_0, C)$ such that if $\Gamma = \{\gamma_g\}$ is a d^*-method with respect to Σ, then Γ is a d_0-method with respect to S.*

Proof: Fix a function $\delta(d) \to 0$ as $d \to 0$ such that inequality (5) implies the inequalities

$$\mathrm{dist}(\Psi(\sigma, x), \Phi(\sigma, y)) < \delta(d), \quad \sigma \in \Sigma.$$

Let $\Gamma = \{\gamma_g\}$ be a d^*-method with respect to Σ. Fix arbitrary $g \in G$ and $x \in M$.

Since Γ is a d^*-method with respect to Σ,

$$\mathrm{dist}(\gamma_{\sigma'_c g}(x), \Phi(\sigma'_c, \gamma_g(x))) < d^*, \tag{10}$$

where σ'_c is from representation (9).

Now, we estimate

$$\begin{aligned}
&\mathrm{dist}(\gamma_{\sigma'_{c-1}\sigma'_c g}(x), \Phi(\sigma'_{c-1}\sigma'_c, \gamma_g(x))) \\
&\quad \le \mathrm{dist}(\gamma_{\sigma'_{c-1}\sigma'_c g}(x), \Phi(\sigma'_{c-1}, \gamma_{\sigma'_c g}(x))) + \mathrm{dist}(\Phi(\sigma'_{c-1}, \gamma_{\sigma'_c g}(x)), \Phi(\sigma'_{c-1}, \Phi(\sigma'_c, \gamma_g(x)))) \\
&\quad\quad < d^* + \delta(\mathrm{dist}(\gamma_{\sigma_c g}(x), \Phi(\sigma'_c, \gamma_g(x)))) \le d^* + \delta(d^*),
\end{aligned}$$

where we take into account inequality (10), its analogue with σ'_c replaced by σ'_{c-1} and g replaced by $\sigma'_c g$, and the equality

$$\Phi(\sigma'_{c-1}\sigma'_c, \gamma_g(x)) = \Phi(\sigma'_{c-1}, \Phi(\sigma'_c, \gamma_g(x))).$$

Similarly,

$$\mathrm{dist}(\gamma_{\sigma'_{c-2}\sigma'_{c-1}\sigma'_c g}(x), \Phi(\sigma'_{c-2}\sigma'_{c-1}\sigma'_c, \gamma_g(x))) < d^* + \delta(d^* + \delta(d^*)),$$

and so on.

This obviously proves our lemma since the number of steps of this process does not exceed C. □

Lemma 2.2: *If S and Σ are two generating sets, then Φ has (or does not have) the inverse shadowing property with respect to S and Σ simultaneously (thus, the inverse shadowing property does not depend on the choice of S).*

Proof: Let us assume that Φ has the inverse shadowing property with respect to S.

Take an $\varepsilon > 0$ and find the corresponding $d = d_0$. Apply Lemma 2.1 to find a $d^* = d^*(d_0, C)$ such that if $\Gamma = \{\gamma_g\}$ is a d^*-method with respect to Σ, then Γ is a d_0-method with respect to S.

In this case, for any $p \in M$ there exists a trajectory $\{x_g\}$ of Γ satisfying inequalities (6). Thus, d^* corresponds to the chosen ε for the generating set Σ. □

In the study of inverse shadowing for group actions, we replace the condition 'the action is topologically Anosov' by a different condition (TC below).

Let H be a finitely generated group, let S be a generating set of H, and let Ψ be an action of H.

Let $\Gamma = \{\gamma_h : h \in H\}$ be a d-method for Ψ with respect to the generating set S.

Fix a number $\varepsilon > 0$ and a point $p \in M$.

We define the 'tube' $\mathcal{T}(\varepsilon, p, H, \Gamma)$ as the set of families

$$Z = \{z_h \in M : \ h \in H\}$$

such that

$$z_h = \gamma_h(z_e), \quad h \in H$$

(i.e. Z is a trajectory of Γ), and

$$\operatorname{dist}(z_h, \Psi(h, p)) < \varepsilon, \quad h \in H.$$

We say that Ψ satisfies the TC if there exists a constant $a > 0$ with the following property: for any $\varepsilon \in (0, a)$, there exists a constant $d = d(\varepsilon) > 0$ such that if Γ is a d-method for Ψ with respect to S, then

$$\mathcal{T}(\varepsilon, p, H, \Gamma) \neq \emptyset$$

and

$$\mathcal{T}(a, p, H, \Gamma) = \mathcal{T}(\varepsilon, p, H, \Gamma)$$

for any $p \in M$.

Remark 2.3: (1) In the definition of the TC, we do not indicate the dependence of this property on the chosen generating set S of H (though S is mentioned in the definition of the TC) since Lemma 2.1 implies that if Ψ satisfies the TC for some generating set S, then it satisfies the TC for any generating set Σ of H (with the same a, but, possibly, with a different $d(\varepsilon)$).

(2) Standard reasoning shows that if Ψ is an action of $\mathbb{Z}$ generated by a homeomorphism h, both h and h^{-1} are uniformly continuous, k is a fixed natural number, and Ψ^k is the action of h^k, then Ψ and Ψ^k satisfy the TC simultaneously.

The main result of this paper is the following statement (which we call the RIST).

Theorem 2.1: *If Φ is a uniformly continuous action of a virtually nilpotent group G and there exists an element $g \in G$ such that $\Phi|_{\langle g\rangle}$ satisfies the TC, where $\langle g\rangle$ is the subgroup generated by g, then Φ has the inverse shadowing property.*

We prove this theorem in Section 4.

3. Tubes for hyperbolic linear automorphisms

In this section, we prove that a dynamical system generated by a hyperbolic linear automorphism satisfies the TC (of course, the reasoning used in the proofs of Lemmas 3.1 and 3.2 is more or less standard for the hyperbolic theory). We restrict ourselves to the linear case because in this case, the reasoning is 'transparent'; clearly, a similar reasoning can be applied in a neighbourhood of a hyperbolic set of a diffeomorphism.

Consider a hyperbolic linear automorphism of m-dimensional vector space $X = \mathbb{R}^m$ or $X = \mathbb{C}^m$,

$$f(x) = Ax, \quad x \in X.$$

The reasoning in the cases $X = \mathbb{R}^m$ and $X = \mathbb{C}^m$ is similar, and we do not distinguish these cases.

Since the matrix A is hyperbolic, we may assume that it has a block-diagonal form,

$$A = \operatorname{diag}(B, C), \tag{11}$$

and there exists a number $\lambda \in (0, 1)$ such that

$$\|B\|, \|C^{-1}\| \le \lambda \tag{12}$$

(see, for example, Lemma 4.1 of [9]).

Let us represent any $x \in X$ in the form $x = (x^s, x^u)$ according to representation (11) of the matrix A.

Let $\Gamma = \{\gamma_k,\ k \in \mathbb{Z}\}$ be a d-method for f.

Fix a point $p \in X$ and let $p_k = f^k(p),\ k \in \mathbb{Z}$.

Lemma 3.1: *For any d-method Γ, there exists its trajectory $\{z_k\}$ such that*

$$|(z_k - p_k)^s| \le d/(1-\lambda), \quad |(z_k - p_k)^u| \le d\lambda/(1-\lambda). \tag{13}$$

Proof: We search for z_k in the form

$$z_k = p_k + v_k.$$

A sequence of that form is a trajectory of Γ if and only if

$$v_{k+1} = \left(Bv_k^s, Cv_k^u\right) + w_{k+1}(v_0), \tag{14}$$

where

$$w_{k+1}(v_0) = \gamma_{k+1}(p_0 + v_0) - f(\gamma_k(p_0 + v_0)),$$

so that

$$|w_{k+1}^s(v_0)|, |w_{k+1}^u(v_0)| < d. \tag{15}$$

Consider the space H of sequences $V = \{v_k \in X\}$ such that

$$|v_k^s| \leq d/(1-\lambda), \quad |v_k^u| \leq d\lambda/(1-\lambda)$$

with the Tikhonov product topology. Then H is convex and compact.

Define an operator R on H as follows. A sequence $V = \{v_k\}$ is mapped to the sequence $T = \{t_k\}$ such that

$$t_k^s = \sum_{i=-\infty}^{k} B^{k-i} w_i^s(v_0)$$

and

$$t_k^u = -\sum_{i=k+1}^{\infty} C^{k-i} w_i^u(v_0).$$

The convergence of the series is obvious; the continuity of the operator R with respect to the product topology is proved in Section 12.3 of [9].

It follows from (12) and (15) that

$$|t_k^s| \leq d(1+\lambda+\cdots) = d/(1-\lambda)$$

and

$$|t_k^u| \leq d(\lambda+\lambda^2+\cdots) = d\lambda/(1-\lambda);$$

hence, R maps H into itself. By the Schauder fixed point theorem, R has a fixed point V in H.

Easy calculation shows that V satisfies equalities (14). □

Lemma 3.2: *If $\{z_k\}$ is a trajectory of a d-method Γ such that*

$$\sup_{k\in\mathbb{Z}} |z_k - p_k| < \infty, \tag{16}$$

then this trajectory satisfies inequalities (13).

Proof: Fix a trajectory $\{z_k\}$ of a d-method Γ for which inequality (16) holds.

A sequence V such that $z_k = p_k + v_k$ satisfies Equation (14). Fix an index k and write the equality for the 'unstable' component in the representation $z_k = p_k + v_k$:

$$v_{k+1}^u = Cv_k^u + w_{k+1}^u(v_0).$$

It follows from the equality

$$v_k^u = C^{-1} v_{k+1}^u - C^{-1} w_{k+1}^u(v_0),$$

from the estimate of $\|C^{-1}\|$ in (12), and from (15) that

$$|v_{k+1}^u| \geq \lambda^{-1} |v_k^u| - d.$$

Iterating this inequality, we see that

$$|v_{k+2}^u| \geq \lambda^{-2} |v_k^u| - \lambda^{-1} d - d = \lambda^{-2}(|v_k^u| - d\lambda - d\lambda^2),$$
$$\dots,$$
$$|v_{k+m}^u| \geq \lambda^{-m}(|v_k^u| - d(\lambda + \lambda^2 + \cdots)) \geq \lambda^{-m}\left(|v_k^u| - d\lambda/(1-\lambda)\right), \quad m > 0.$$

Since $\lambda^{-m} \to \infty$ as $m \to \infty$, it follows from inequality (16) that

$$|v_k^u| - d\lambda/(1-\lambda) \leq 0,$$

which implies the second estimate in (13). The first estimate is proved similarly. □

Lemmas 3.1 and 3.2 imply that f satisfies the TC with any fixed $a > 0$ (for definiteness, let us take $a = 1$); if Γ is a d-method for f, then

$$\mathcal{T}(\varepsilon, p, \mathbb{Z}, \Gamma) \neq \emptyset$$

and

$$\mathcal{T}(1, p, \mathbb{Z}, \Gamma) = \mathcal{T}(\varepsilon, p, \mathbb{Z}, \Gamma)$$

for any $p \in M$ and any

$$\varepsilon \in \left(d\frac{1+\lambda}{1-\lambda}, 1 \right)$$

(of course, we assume that $d(1+\lambda) < \varepsilon(1-\lambda)$).

Example 3.1: Consider the hyperbolic mapping $f(x) = 2x$ on $X = \mathbb{R}$. Obviously, the results similar to Lemmas 3.1 and 3.2 are valid in this case.

Fix a small $\delta > 0$ and an index $k \geq 1$ and define a continuous function $\gamma_k(x)$ on $\mathbb{R}$ as follows:

$$\gamma_k(x) = 2^k x, \quad x \in (-\infty, 1) \cup (2, \infty),$$

$$\gamma_k(x) \equiv 2^k, \quad x \in I_\delta := [1, 1+\delta],$$

and γ_k is linear on $[1+\delta, 2]$. Thus, $\gamma_{k+1}(x) = 2\gamma_k(x)$ for $k \geq 1$ and $x \in \mathbb{R}$.

Set $\gamma_k(x) = 2^k x$ for $k < 0$ (and recall that $\gamma_0(x) = x$). Let us estimate the values

$$\Delta_{k,\pm 1}(x) := |\gamma_{k\pm 1}(x) - f^{\pm 1}(\gamma_k(x))|, \quad x \in \mathbb{R}.$$

Clearly, only the functions

$$\Delta_{0,1}(x) = |\gamma_1(x) - f(\gamma_0(x))|$$

and

$$\Delta_{1,-1}(x) = |\gamma_0(x) - f^{-1}(\gamma_1(x))|$$

may take non-zero values.

It is easily seen that

$$\Delta_{0,1}(x) \leq 2\delta \text{ and } \Delta_{1,-1}(x) \leq \delta.$$

Hence, $\Gamma = \{\gamma_k\}$ is a 3δ-method for f.

Take $p = p_0 = 1$, then $p_k = 2^k$. If $z_0 \in I_\delta$, then

$$z_k = \gamma_k(z_0) = 2^k = p_k, \quad k \geq 0.$$

Otherwise, if $z_0 \notin I_\delta$, then

$$|z_k - p_k| \to \infty, \quad k \to \infty.$$

On the other hand, if we take any $z_0 \in I_\delta$, then $z_k = 2^k z_0$ for $k < 0$; this gives us a trajectory of Γ that δ-shadows $\{p_k\}$ since $|z_k - p_k| \leq 2^k \delta$ for $k \leq 0$.

Hence, for any $\varepsilon > \delta$, the tube $\mathcal{T}(\varepsilon, p, \Gamma, f)$ consists of all the sequences $\{z_k\}$ such that

$$z_k = 2^k z_0, \quad k \leq 0, \ z_0 \in I_\delta,$$

and

$$z_k = 2^k, \quad k > 0.$$

4. Inverse shadowing in actions of virtually nilpotent groups

In this section, we prove Theorem 2.1.

Let us give the definition of a virtually nilpotent group.

First, let us recall that H is a normal subgroup of G (we write $H \lhd G$) if $gH = Hg, g \in G$.

The commutator of $g, h \in G$ is defined as follows:

$$[g, h] = ghg^{-1}h^{-1}, \quad g, h \in G.$$

If G_1, G_2 are subgroups, then $[G_1, G_2]$ is generated by $[g_1, g_2]$, where $g_1 \in G_1, g_2 \in G_2$.

We say that an Abelian group is nilpotent of class 1.

We say that G is a nilpotent group of class n if there exist subgroups $G_1, \dots, G_{n+1}$ such that

$$G = G_1 \rhd \cdots \rhd G_{n+1} = e, \;\; G_n \neq e, \;\; G_{i+1} = [G_i, G].$$

At last, a finitely generated group G is called virtually nilpotent if there exists a normal nilpotent subgroup G' of G having finite index (this means that the factor group G/G' is finite).

Remark 4.1: Let us note that virtually nilpotent groups play an important role in group theory; Gromov showed in [10] that a group of polynomial growth is virtually nilpotent.

The main technical statement in this section is the following lemma.

Let Φ be an action of a finitely generated group G and let H be a finitely generated normal subgroup of G. Fix a symmetric generating set S_H of H and extend it to a symmetric generating set $S = S_G$ of G.

Lemma 4.1: *If Φ satisfies the MTA (Φ is uniformly continuous) and $\Phi|_H$ satisfies the TC, then Φ satisfies the TC.*

Proof: Let $a > 0$ be the constant from the TC for $\Phi|_H$. Apply the MTA to find an $\varepsilon < a$ such that

$$\mathrm{dist}(\Phi(s, x), \Phi(s, y)) < a/2 \text{ for } s \in S \text{ and } x, y \in M \text{ with } \mathrm{dist}(x, y) < \varepsilon. \tag{17}$$

Let $d = d(\varepsilon) < a/2$ for this ε from the TC for $\Phi|_H$. Let Γ be a d-method for Φ.

Fix an arbitrary point $p \in M$ and an element $g \in G$. Inequalities (8) imply that the inequalities

$$\mathrm{dist}(\gamma_{shg}(x), \Phi(s, \gamma_{hg}(x)) < d$$

are satisfied for any $s \in S$ and $h \in H$, which means that the set

$$\Gamma|_H(g) = \{\gamma_{hg} : \; h \in H\}$$

is a d-method for $\Phi|_H$ (and let us note that the above inequalities are satisfied not only for $h \in H$ and $s \in S_H$ but also for $h \in H$ and $s \in S$, which we apply below).

By the TC for $\Phi|_H$, there exists a non-empty set $Z(g, p) \subset M$ such that

$$\mathcal{T}(\varepsilon, \Phi(g, p), \Gamma|_H(g), \Phi|_H) = \{\gamma_{hg}(z) : \; h \in H, \;\; z \in Z(g, p)\}$$

(we shadow the exact trajectory of $\Phi(g, p)$ with respect to the action of H), which means that

$$\mathrm{dist}(\gamma_{hg}(z), \Phi(h, \Phi(g, p))) < \varepsilon, \quad h \in H, \;\; z \in Z(g, p). \tag{18}$$

Now, we take an arbitrary $s \in S$. Let us take $h \in H$ and $z \in Z(g, p)$ and estimate

$$\mathrm{dist}(\gamma_{hsg}(z), \Phi(hsg, p)).$$

Since H is a normal subgroup of G, there is an element $h' \in H$ such that

$$sh' = hs.$$

Then,

$$\mathrm{dist}(\gamma_{hsg}(z), \Phi(hsg, p)) = \mathrm{dist}(\gamma_{sh'g}(z), \Phi(hsg, p)) \leq$$

$$\leq \mathrm{dist}(\gamma_{sh'g}(z), \Phi(s, \gamma_{h'g}(z))) + \mathrm{dist}(\Phi(s, \gamma_{h'g}(z))), \Phi(s, \Phi(h'g, p))) \leq$$

$$\text{(we note that } \Phi(s, (h'g, p)) = \Phi(sh'g, p) = \Phi(hsg, p))$$

$$\leq d + a/2 < a.$$

The first term in the second line is estimated by d since γ_{hg} is a d-method, and the second term is estimated using (18) (with $h = h'$) and (17).

By the TC for $\Phi|_H$,

$$\mathrm{dist}(\gamma_{hsg}(z), \Phi(hsg, p)) < \varepsilon, \quad h \in H.$$

This shows that $Z(g, p) \subset Z(sg, p)$. But the above reasoning is symmetric, and we conclude that $Z(sg, p) \subset Z(g, p)$. Hence, the set $Z(g, p)$ is, in fact, independent of g (let us denote this set by $Z(p)$).

Then, we can take $h = e$ in (18) to conclude that

$$\mathrm{dist}(\gamma_g(z), \Phi(g, p)) < \varepsilon$$

for any $g \in G$, which gives us the first part of the TC for Φ.

The second part is obvious. □

First, we prove the RIST in the case of a nilpotent group G of class n. In fact, we prove the following auxiliary statement.

Lemma 4.2: *If Φ is a uniformly continuous action of a nilpotent group G of class n, $g \in G$, and $\Phi|_{\langle g \rangle}$ satisfies the TC, then Φ satisfies the TC as well (which obviously implies the inverse shadowing property of Φ).*

Proof: We use induction on n.

If $n = 1$, then G is Abelian; in this case, $\langle g \rangle$ is a normal subgroup of G, and our statement follows from Lemma 4.1 (with $H = \langle g \rangle$).

Now let $n > 1$ and assume that we have proved our statement for nilpotent groups of class not exceeding $n - 1$.

In what follows, we refer to an auxiliary statement proved in [7] (Lemma 4.3 below).

Let G be a nilpotent group of class n, let $Q = [G, G]$, let $g \in G$, and let $P = \langle Q, g \rangle$ (i.e. P is the minimal subgroup of G containing Q and g). □

Lemma 4.3: *The following statements hold.*

(1) *The group P is a normal subgroup of G.*
(2) *The group P is a nilpotent group of class not exceeding $n - 1$.*

Let us continue the proof of Lemma 4.2 for a nilpotent group G of class n.

Any normal subgroup of a finitely generated nilpotent group is finitely generated (see, for example, [11]). Hence, P is a finitely generated nilpotent group of class not exceeding $n-1$, $g \in P$, and $\Phi|_{\langle g \rangle}$ satisfies the TC. Now our induction assumption implies that Φ_P satisfies the TC. Statement (1) of Lemmas 4.3 and 4.1 imply that Φ satisfies the TC.

Now let us complete the proof of the RIST in the case where G is a virtually nilpotent group. The group G has a normal nilpotent subgroup H of finite index. Let $g \in G$ be an element of G that satisfies the TC. Since H is a subgroup of finite index, there exists a $k > 0$ such that $g^k \in H$. The action $\Phi|_{\langle g \rangle}$ satisfies the TC; hence, the action $\Phi|_{\langle g^k \rangle}$ satisfies the TC as well (see Remark 2.5).

By Lemma 4.2, $\Phi|_H$ satisfies the TC. Thus, to complete the proof, it remains to apply Lemma 4.1 to G and H.

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