

Dynamic Model of Venture Capital Investment

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Abstract. In this paper, we consider the problem of making investment decisions in a situation where the investor has limited information about the market. Venture capital investment is a good example of this problem. In such cases, the decision to invest in a project depends on the investor's belief about the potential profitability of the project, the stage at which the decision is made, and their personal risk propensity. Using mathematical formalization and game theory, we create a model that represents an incomplete information situation. The game involves two players: the Investor and the Project. The market states and possible types of projects are considered scenarios or alternatives. Based on these assumptions, profit functions and strategies for both players can be defined. First, a one-shot game was considered, where a Bayesian Nash equilibrium was found. Then, the model was extended to a multi-stage game, where perfect Bayesian equilibria were identified. Using this constructed game, analysis was conducted to determine conditions for project profitability and investment effectiveness. All the results were illustrated with numerical examples.

Keywords: Venture capital investments · Game theoretic model · Incomplete information · Bayesian game · Bayesian Nash Equilibrium (BNE) · Multi-stage game.

1 Introduction

In modern risk management, one of the most important and significant areas is the analysis and assessment of investment risks [3]. This is particularly evident in the field of innovative and high-technology projects, where the level of uncertainty is traditionally higher than average [16]. Under these conditions, venture capital investment [4] becomes a key instrument for implementing an approach in which risk is consciously treated not as a threat, but as a potentially profitable resource. Originating in the mid-20th century and actively developing over recent decades, venture capital investment serves as a prime example of the application of the concept of “risk as a resource” [5].

The introduction of this idea represented a significant conceptual shift in risk theory, allowing us to approach the classic “risk-return paradox” in a more systematic way. This paradox, which is the contradiction between investors’ desire

to avoid risk while seeking increased returns [7], has been discussed extensively in fundamental works on portfolio choice theory, such as Markowitz’s 1952 classic paper [12], as well as more recent research on financial risk management and asset pricing theory [3, 4]. Originally proposed by M.A. Greenfield in 1998 [5], this approach views risk management as a form of resource management that interprets risk costs and benefits in economic terms.

According to Greenfield [5], resource-like risk is characterized by several specific properties that make it a potentially valuable element of strategic decision-making. Firstly, this type of risk must contain positive aspects, meaning that its growth within certain bounds is potentially linked to an increase in expected benefits. It is these positive factors that make the risk significant for decision-makers and give it a resource-like character. Secondly, an important aspect of the concept is its voluntary nature: the risk manager has the right to choose whether or not to take on the risk. This means that the concept does not require mandatory acceptance of the risk, but rather acknowledges the possibility of purposeful utilization. Thirdly, it’s crucial to understand that increasing the level of resource-like risk can be effective only up to a certain point, beyond which additional positive factors no longer have an impact. At the same time, we can discuss the existence of an optimal level of risk, where all available benefits have been realized and further increases lead to decreased management efficiency. Analyzing these features, we can conclude that the risk management strategy similar to a resource is reduced to finding and maintaining a saturation point where a balance between marginal benefits and costs is achieved.

Practical examples such as financing innovative startups, investing in new technology development and implementation, and entering international markets during globalization can illustrate this approach to risk as a resource. Previously, [10] studied the phenomenon of venture capital investment, which integrates all these types of risk scenarios, and showed that this activity fully aligns with the concept of risk as a resource.

To study the described problems and to model such processes, a particularly suitable framework is the integration of mathematical game theory [6] and decision-making under uncertainty, often formalized as a “game – nature” model [9]. In this framework, uncertainty about the environment is treated as a “game against nature”, where the decision-maker chooses a strategy without knowledge of the realized state of nature. The combination of these two approaches can be represented through the formalism of Bayesian games, also known as games of incomplete information [8, 1, 2, 17], in which beliefs about the states of nature are incorporated directly into the players’ strategic choices.

In [10], a Bayesian game framework was employed to develop a game-theoretic model of venture capital investment. The present study extends that one-shot model by treating venture investment as a dynamic, multi-stage process and representing it as a multi-stage game. Within this extended framework, the model explicitly incorporates varying levels of investor risk tolerance [7], which correspond to the particular stage of the project at which investors decide to enter.

The paper has the following structure. The introduction (section 1) provides an overview of the historical development of venture capital investment and a review of existing literature on its modeling. In section 2 the main ideas and approaches for creating suitable models are outlined. The rationale for using Bayesian games in this context is explained. Section 3 focuses on the development of a game-theoretical model with incomplete information within the framework of one-shot games. Section 4 is dedicated to the development and analysis of a multi-stage Bayesian game that describes a two-phase venture investment process. In section 5 all the results obtained for the model were illustrated with numerical examples. Finally, section 6 lists and summarizes all the results obtained, and discusses possible prospects for the further development of the model.

2 Model Formulation

Based on the model proposed in [10], let's formulate and study non-antagonistic game [11, 14], in which players are the Investor and the Project.

2.1 The Process of Interaction Between the Investor and the Project

It is worth making a few comments. When we talk about a player Investor, at this point in the research, we are not focused on the specific type of investor. These can be state-owned companies, private venture funds, institutional investors, business angels, or other capital owners who are willing to invest in a risky but potentially lucrative project. But it is important to understand that one of the main differences between a venture capitalist and other types of investors is their goal. Instead of seeking a percentage of profits, as with loans or debt financing, a venture capitalist aims for a share in a project or company. The investor's income is not derived from interest payments, but from the increase in the value of their share and the eventual exit from the investment.

When we talk about a player Project, we also don't specify what it is: it may have great innovation potential, significant at the national level in fields such as IT, AI, medicine, or biotechnology. Alternatively, it could be a local startup that is very profitable but has a short and memorable life span. Under the term Project we mean a project team or management with a well-developed concept, an implementation plan, and a strategy for occupying a niche in the market.

Formally, the Investor makes the decision choosing the strategy A (to invest) or $\bar{A}$ (not to invest). The Project's strategies are B (to be implemented) and $\bar{B}$ (not to be implemented). Following the traditional approach in Bayesian games [8, 17], it is assumed that the players make decisions simultaneously, independently of each other.

It is worth noting that the actual profitability of a project is information that is not known with certainty at the start of a project, as it depends on various

external factors (the competitors' behavior, changes in market conditions, consumer purchasing power, etc.) and not just the internal choices of the project team. In the framework of decision theory [9] these factors are known as **states of nature**, or, **scenarios**. Thus, the Chance (Nature) defines a set of types, which the player Project can possess: to be profitable (t_1) or to be unprofitable (t_2). At the start of the game the Investor does not know the type exactly, but she can assume that the Project has type t_1 with probability p and type t_2 with probability $1 - p$ correspondingly. These probabilities determine the Investor's beliefs and she chooses a strategy A or $\bar{A}$ based on them [17].

The set of strategies of player Project $\{B, \bar{B}\}$ can be seen as the ability of the project manager and her team, with sufficient funding, to launch, develop and implement a project, taking reasonable actions that are the result of rational and thoughtful decisions.

2.2 Restrictions on Parameters

Now we need to analyse players' payoff functions, taking into account that for both of them the aims are to maximise their payoffs. To do this let's make the following denotations. I is initial investment amount, without it the Project can not be implemented. R is the Project's revenue. Its value is quite high compared to the initial amount I , which is a typical feature of venture capital investments [4]. C is the total cost of launching and maintaining the project at the considered stage. These may include product design and development, marketing, promotion, sales, team, infrastructure, and other additional costs. β is Investor's share in the project, $\beta \in (0, 1)$.

At the same time, the Project may actually turn out to be profitable (the type t_1) or unprofitable (the type t_2). We formalize this by considering that in each of these cases, the Project generates profit R , but in the first case the net revenue $NR = R - C$ is positive, and in the second case it is negative [15]. From a practical point of view, it is much more important to understand that the project is profitable for the investor and for the project team itself. For an Investor, it makes sense to invest only when

$$I < \beta R, \quad (1)$$

i.e. her investments will pay off due to the high profitability of the future project. The second requirement can be expressed in the form of inequality

$$(1 - \beta) R > C. \quad (2)$$

In addition, C should not exceed the amount I invested in the project ($C \leq I$).

Accordingly, there are three possible cases.

The first case:

$$R > \max \left\{ \frac{I}{\beta}; \frac{C}{1 - \beta} \right\}.$$

This condition means that the project is profitable for both players. Let's assume that in this case R takes value R^p , which satisfies (1) and (2) simultaneously.

In the second case, when

$$R < \min \left\{ \frac{I}{\beta}; \frac{C}{1-\beta} \right\},$$

the project is unprofitable for both players. In such case its value will be denoted by R^u .

One more case is also possible:

$$R \in \left[\min \left\{ \frac{I}{\beta}; \frac{C}{1-\beta} \right\}; \max \left\{ \frac{I}{\beta}; \frac{C}{1-\beta} \right\} \right].$$

From a mathematical perspective, it is essential to conduct a parametric analysis in order to determine the conditions for precisely identifying the boundaries of this range and to examine the payoffs for players within it. However, we will sacrifice mathematical precision in favor of the economic meaning of the model under consideration. Since it was previously assumed that C should not exceed I , with approximately equal shares of players, it will therefore be the case that

$$\frac{I}{\beta} \geq \frac{C}{1-\beta}.$$

Next, we will assume that, within the context of our model, this condition holds true, regardless of the value of β . Therefore, in further analyzing the interaction between the Investor and the Project, it would be sufficient for us to focus on the two cases mentioned above.

Additionally, let's mention that in the case when the Project is not implemented, $R = 0$ as the Project generates no revenue despite the total cost C spent on launching the project.

2.3 Generalization: Bayesian Game

In the literature devoted to Bayesian games ([1, 2, 8, 17]), a game with incomplete information in normal form is usually defined as the set

$$\langle N, S, T, p, u \rangle \tag{3}$$

which consists of the following elements:

1. N is a set of players;
2. S is a set of strategies: $S = s_1 \times \dots \times s_N$, where s_i is the set of possible strategies for player i ;
3. T is a type set: $T = t_1 \times \dots \times t_N$, where t_i is the (finite) set of possible types for player i ;
4. Prior p : a joint probability distribution $p(t_1, \dots, t_N)$ over all possible type profiles: the probability that Player i has type t_i , where $i = \overline{1, N}$;
5. Payoff functions $u_i : S \times T \rightarrow R$ assign a payoff to a player given her type and the strategy.

Above, the text outlines the first four steps necessary for creating a Bayesian game with two players: the Investor and the Project. A variety of strategies for each player have been identified, along with the types of players and their prior probabilities. Further discussion focuses on constructing payoff functions that depend on the specific type of project and the strategies chosen by each player.

3 Static Model: One-Shot Game with Incomplete Information

First, let's design a one-shot game with incomplete information which is a logical extension of the model proposed in [10]. According to the classical formulation of Bayesian game [6, 8, 17], the players can possess various types.

For example, the behavior of the Investor depends on the stage of project: early stage (pre-seed and seed rounds), middle stage (start-up and expansion) and late stage (ramp-up, pre-IPO). Various stages define the availability of information about the Project, which influences on the Investor's decision. Moreover, Investor can be an economic agent with different risk propensities: risk averse, risk neutral or risk loving. First, let's restrict and simplify the model by considering only risk neutral (rational) type of the Investor.

Talking about the player Project, she could possess one of two types: to be *profitable* (t_1) or *unprofitable* (t_2), as it was said earlier.

3.1 The Game in Extensive Form

If we take into account the incompleteness of information, we obtain the following game. Together with the strategies of the players Investor and Project, we also need to consider the move by nature that determines the prior distribution of the players' types. In our model, this move will determine the prior distribution of the Project's type. We will assume that this distribution corresponds to the Investor's beliefs about the type of the second player. The game tree [11, 13, 14], that represents the game in extensive form considering the move of nature [6, 9], is shown in figure 1.

The dotted line in the figure shows the information sets of the players and the branches of the game, which relate to situations where the game conditionally continues without the actual interaction of the Investor and the Project.

To analyze this bimatrix game [11, 14], we need to study two types of Project, and, correspondingly, the Investor's beliefs.

First, let's consider the case when the Project is profitable (has type t_1). It corresponds to the positive scenario which realizes with probability p , represented by the left branch of the game tree in figure 1.

If the Investor decides to invest (chooses the strategy A) and the Project is implemented with the Investor's funding (the strategy B), both players get their positive payoffs (strategy profile (A, B)).

If, oppositely, the Project is not implemented, the Investor only loses her money I (profile $(A, \bar{B})$).

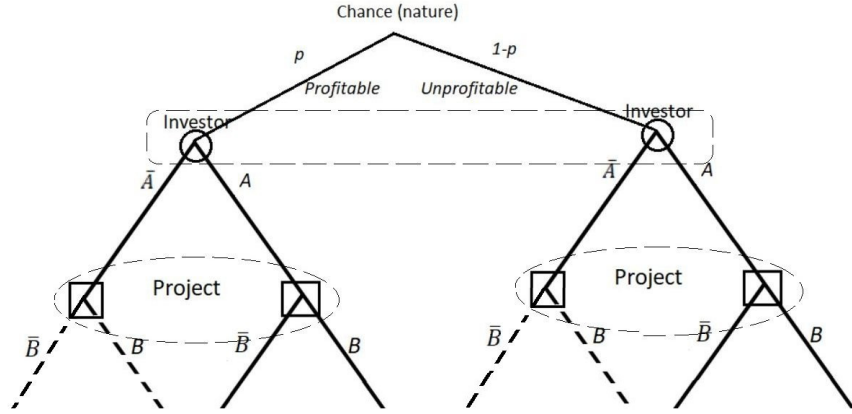


Fig. 1. The tree of the game with incomplete information

Let's make one significant note. Unlike the Investor, who does not know for sure the type of the second player, the Project has this knowledge. Therefore, it is always obvious to her whether the inequality (2) is fulfilled or not. When the Project is ready to be implemented (as she knows her type t_1), but her opponent chooses $\bar{A}$, the project team, wanting to make high profits R in the future, will look for another source of financing for this. It can be a loan that the player Project will be forced to repay with interest. Or it could be getting investments elsewhere (let's assume that they are less profitable for the project). In any case, a decrease in the initial investment volume or the amount of interest payments on the loan will mean certain financial losses for the project, which we will designate later as L . Thus, in this profile the player Project is implemented without initial investment, borrowing money or searching for another source of investments with losses L . At the same time, the Investor loses unearned income from highly profitable project (but saves the investment I). This value has the meaning of regret in the sense of the Savage's criterion [7]) or the penalty for her incorrect decision. This case corresponds to the profile $(\bar{A}, B)$.

Finally, if the Investor chooses $\bar{A}$, and the Project chooses $\bar{B}$, both players win 0. Such representation of the game corresponds to the following payoff matrix:

	B	$\bar{B}$
A	$(-I + \beta R^p, I - C + (1 - \beta) R^p)$	$(-I, I - C)$
$\bar{A}$	$(I - \beta R^p, -C - L + (1 - \beta) R^p)$	$(0, 0)$

In this game, the Nash equilibrium [6,8] is in profile (A, B) , due to the limitations (1) and (2) imposed on the variables I , C and R by the profitability condition of the project (subsection 2.2). For the player Investor the dominant strategy is "to invest" and for the player Project it is "to be implemented".

Now let's consider the case when the Project is unprofitable (has type t_2). It corresponds to the positive scenario which realizes with probability $1 - p$, represented by the right branch of the game tree 1. For this case the profit matrix will have the following form: Such representation of the game corresponds to the following payoff matrix:

$$\begin{array}{c|cc} & B & \bar{B} \\ \hline A & (-I + \beta R^u, I - C + (1 - \beta) R^u) & (-I, I - C) \\ \bar{A} & (0, -C - L + (1 - \beta) R^u) & (0, 0) \end{array}$$

In contrast to the first case, profile (A, B) here brings a negative payoff to the Investor, and the payoff that is close to zero, even if it is positive, to the Project. In profile $(\bar{A} B)$, the investor now has no lost profits or regrets, even if the project is launched through financing from the loan with interest L and generates only negative profits. In profile $(A, \bar{B})$ everything is similar to the previous case. Profile $(\bar{A}, \bar{B})$ gives zero payoffs to both players.

In this game, the Nash equilibrium is represented by profile $(\bar{A}, \bar{B})$. The dominant strategies for both players are “do not invest” and “do not implement the project”, which means refusing to interact with each other.

The final representation of the Venture Capital Investments Game with incomplete information in extensive form can be seen in figure 2. To formulate the

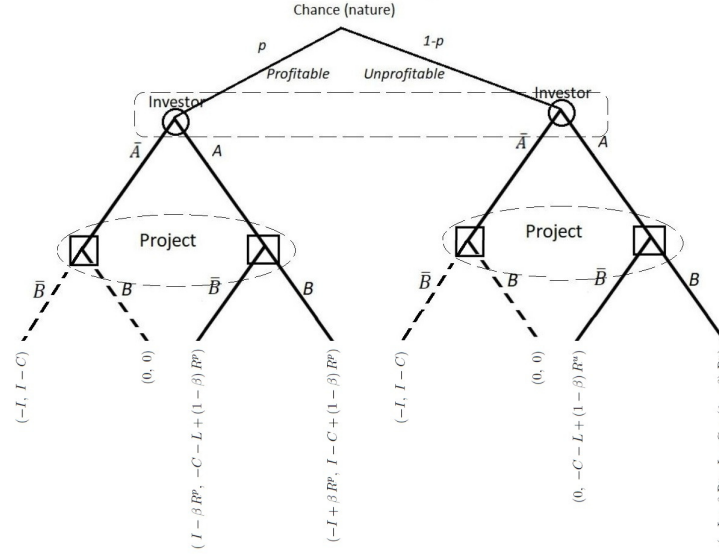


Fig. 2. The game with incomplete information in extensive form

results obtained for the studied game we need to search for the Bayesian Nash

equilibrium (BNE) [1, 2, 17]. First of all, we need to analyse how the Investor makes her decision when the Project chooses dominant strategies with different types.

3.2 The Analysis of the Game

Previously in subsection 3.1 the Project's dominant strategies were found for both types of the player; alternative strictly dominated strategy can thus be removed. Given this, if the Investor chooses A , her profit is

$$u_I(A) = p(\beta R^p - I) + (1 - p)(-I) = p\beta R^p - I.$$

If she chooses $\bar{A}$, then

$$u_I(\bar{A}) = p(I - \beta R^p) + (1 - p) \cdot 0 = pI - p\beta R^p.$$

Since in the current study we restricted ourselves to considering only the risk-neutral (rational) Investor, she will prefer a strategy *to invest* when $u_I(A) \geq u_I(\bar{A})$. Thus, the decision to invest is based on the condition:

$$p \geq \frac{I}{2\beta R^p - I}. \quad (4)$$

Remark 1. The assumption underlying the static model that the parameter p is known to both participants simplifies analytical work. In practice, investors usually do not know the value of probability p and use only its estimation based on historical data, market research, expert opinions or due diligence processes. In this case, following the tradition of Bayesian games, what is meant is not the probability itself that the project is profitable, but the assessment of this probability, or the investor's belief.

Taking into account the mathematical meaning of probability (4), we obtain that the following double inequality must be fulfilled:

$$0 \leq \frac{I}{2\beta R^p - I} \leq 1.$$

Therefore, we get that the investment amount and the project revenue should satisfy $I \leq \beta R$. The last inequality corresponds to the a non-strict requirement of profitability of investments.

But what is the relation between R and I ? To answer this question we can assume that a rational economic agent invests in funds expecting to receive a certain amount of profit from the project in the future. If we are talking about venture capital, then we assume that in the future the profit will be many times higher than the initial investment. In other words, the minimum value of amount I can be found as an Investor's discounted profit in the current investment period:

$$\underline{I} = \beta R e^{-r\tau}, \quad (5)$$

where τ is an investment period, r is a discount rate.

Summarising all the preceding considerations and the analysis of the game, we can formulate the proposition which consists of the following statements.

Proposition 1. *In the considered game (3) of Venture Capital Investments with incomplete information*

1. *Bayesian Nash Equilibrium (BNE) is achieved for the profiles (A, B) for Project type t_1 with probability p and $(\bar{A}, \bar{B})$ for Project type t_2 with probability $1 - p$;*
2. *The investments start when prior probability p satisfies (4).*

4 Dynamic Model: Multi-stage Game

In section 3, we considered a one-shot investment game in which the venture investor initially did not have information about whether the project was profitable, i.e. whether her investments would pay off in the future. But in practice, venture capital investment is not a one-time event, but rather a series of successive stages. Thus we want to apply game-theoretical approach to study the interaction between the players Investor and Project in dynamic. To do this we need to consider a multi-stage game [13], based on the Bayesian game described in section 3.

4.1 Expansion of a Bayesian game

To begin the analysis, we must determine when the game in Fig. 2 can continue and the next stages can begin. Only the branches of the game tree that correspond to the positive decisions of both players (respectively, “invest” and “to be implemented”) can have further continuation. In all other cases, the game can be considered finished. But one of these branches corresponds to the case when the Project turned out to be profitable, and the other is the case of unprofitable Project. Is the investor generally ready to continue investing in a project that turned out to be unprofitable after the first stage?

If we consider the condition that is crucial in rational deciding whether or not to invest at the initial stages, it was formulated earlier as inequality (5). Strict fulfilling this inequality is equivalent to ensuring that the project is profitable for the Investor (1).

As a general rule, venture capital investments do not start to generate profit for the venture capitalist until later stages of the project when it is time to exit the investment [4]. During early stages, the company is typically unprofitable or operates with zero profit, and the entire potential economic effect of the investor is of a capitalization nature, reflected in an increase in the value of his share.

On this basis, first of all, we will combine the mentioned stages under the general name “Initial stage” and assume that it is described by the game studied in the previous section 3 and illustrated in the Fig. 2.

Secondly, we will consider both types (t_1 and t_2) of player Project as possible for further multi-stage game, and the fulfillment of condition (1) as not mandatory in this context. In this context, the types t_1 and t_2 mean the profitability or non-profitability of the Initial stage, rather than the entire project as a whole.

When discussing venture capital investments, it's important to understand that each stage of the process involves a different level of risk. The initial stages (Pre-seed, Seed, Start-up and Early stage [4]) involve a high level of risk and only economic agents with high risk propensity will invest during these periods [9]. The following stages, such as Expansion and Late stage (pre-IPO), involve a lower level of risk and are suitable for risk-neutral investors. Risk-averse investors may only consider the latest stages of an IPO for potential participation.

Therefore, we will also combine the pre-IPO stages (Expansion and Late stage) into a single, "Middle stage" and study it from the position of a risk-neutral Investor, whose rational interaction with the player Project can lead to the final stage such as IPO.

Let's return to the Bayesian game in extensive form (Fig. 2). For both scenarios ("Project is profitable" (type t_1) and "Project is unprofitable" (type t_2)) the Middle stage of the game can start at the nodes of the graph corresponding to the profile (A, B) . Fig. 3 shows that the game between the Investor and the Project will be repeated again.

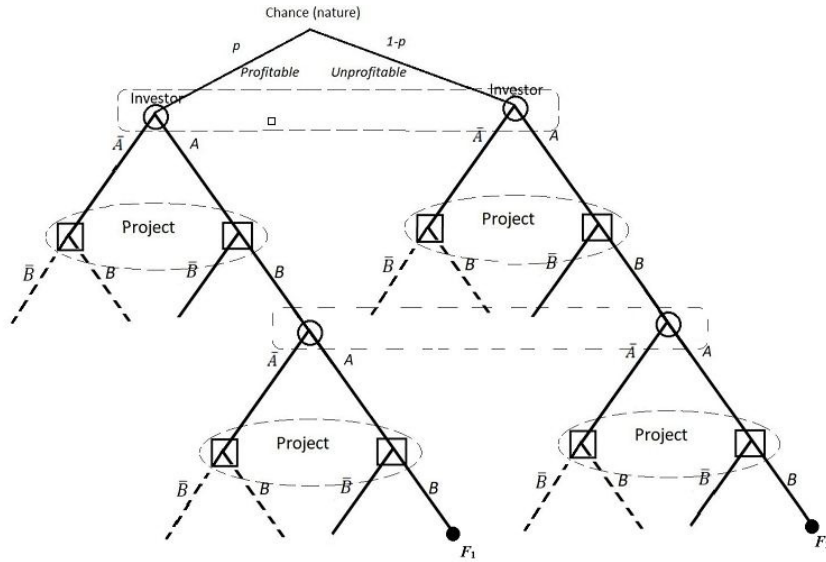


Fig. 3. The multi-stage game with incomplete information in extensive form

It will have similar matrices and the shape of the payout functions, taking into account the following assumptions:

1. It is assumed that investments in the project are of a one-time nature, i.e. I is the only amount invested at the start of the initial stage for all future stages;

2. The revenue and profit from the project may vary significantly at different stages; if R_m is the revenue generated by the Project in the Middle stage, let's assume that $R_m \geq R$, where R now is the revenue of the Initial stage. Therefore, even if at the end of the first stage the project turned out to be unprofitable for the investor, it may turn out to be profitable in the second stage.

It is important to understand that, at this stage of the project and game, choosing the strategy A does not mean “to invest again” for the Investor. Instead, it means continuing to engage with the Project. In this case, the investor would not sell her stake in the project but would reinvest the profits from the Initial stages into subsequent ones.

But what can be used as a criterion for making investment decisions? Typically, the net present value of a project (NPV) or its payback period (T) are considered as characteristics for evaluating the effectiveness of a project. If we follow the methodology discussed in section 3, it seems more logical to consider NPV as a criterion.

According to [3], within the income approach, **Net Present Value (NPV)** is used as a measure of the current value of a project or a business, computed as the sum of discounted future cash flows generated by the asset without the initial investment.

In our model it can be expressed as

$$NPV = -I + \beta R e^{-r\tau} + \beta R_m e^{-rT}, \quad (6)$$

where βR and βR_m are the cash flows, discounted to the start of the game, τ and T are the time period of Initial state and the total period before IPO correspondingly.

It is also worth considering that the revenue received at the Initial stage is then reinvested in the Middle stage, thus (6) can be rewritten in the following form:

$$NPV = -I + \beta R e^{-r\tau} + \beta R e^{-rT} \left(e^{\rho(T-\tau)} - 1 \right),$$

or,

$$NPV = -I + \beta R e^{-rT} \left(e^{r(T-\tau)} + e^{\rho(T-\tau)} - 1 \right), \quad (7)$$

where ρ is rate of return on reinvested funds. ρ and r have different values because they reflect the profitability of the project in different periods (τ and $T - \tau$) with different internal and external conditions, which include both the profitability of the project itself, as well as the levels of inflation, key interest rate and other economic characteristics. For a static game considered in the section 3, NPV can be obtained as the difference between the left and right sides of formula (5). Thus, the positivity of (7) can be considered as a condition for the effectiveness of the project.

4.2 Game Analysis and Results

Now let's get back to studying the game in extensive form 3. From the point of view of the full implementation of the Project, only the nodes F_1 and F_2

are of interest. The path leads to them when the Investor consistently chooses strategies A, A in each decision node, and the Project chooses B, B in her nodes. Thus, for both types of the Project two nodes $ABAB$ correspond to the case of a fully implemented project. But when is it effective?

Node F_1 corresponds to perfect Bayesian equilibrium (PBE) [11] in a two-stage game, as it is achieved by a set of equilibrium strategies in both subgames. But for the case when the Project is unprofitable (has type t_2) it is not an equilibrium, because in this case, the subgame of the second stage begins when, in her information set, the Investor makes a choice from node A, B from a losing position. Thus, F_2 represents an equilibrium only in a subgame related to the second stage of the game.

Additionally, it is worth noting that the necessary condition for the possibility of launching the Middle Stage is

$$I + ((1 - \beta)R^u - C) > C_m, \quad (8)$$

which means that the Project, despite the unprofitability of the Initial stage, has enough funds to launch the second stage of the game.

In node F_1 , when PBE is realized, the Investor's payoff is

$$U_I(ABAB) = -I + \beta R^p e^{\rho_p(T-\tau)}, \quad (9)$$

where ρ_p is reinvestment rate for the Project's type t_1 . The Project's payoff is

$$U_P(ABAB) = I - (C + C_m) + (1 - \beta)R^p e^{\rho_p(T-\tau)}, \quad (10)$$

where C_m is the total cost of the implementation in the Middle stage.

According to the type of the Project, the value R^p satisfies inequalities (1) and (2), thus the value ρ_p does not require additional restrictions. Actually:

$$I < \beta R < \beta R^p \left(e^{\rho_p(T-\tau)} - 1 \right).$$

Remark 2. The parameters r , ρ_p and ρ_u governing the subsequent stage behavior specifically refer to the investment rate and reinvestment rate. The investment rate r represents the capital allocation in the initial stage, determined by the investor's available funds, risk tolerance, and project requirements. The reinvestment rate ρ_p or ρ_u represents the rate at which first-stage returns are reinvested, which can be estimated using market benchmarks (current rates for similar opportunities), the company's cost of capital (WACC), historical reinvestment performance, or MIRR (Modified Internal Rate of Return adjusted for reinvestment). In practice, the reinvestment rate is uncertain and creates reinvestment risk, so investors typically use conservative estimates and conduct sensitivity analysis across plausible values of these parameters.

It is clear that in this scenario, the NPV , as expressed in equation (7), is positive, making the Project an attractive investment for the Investor.

In node F_2 the payoff functions are similar to the profitable case. The Investor's payoff is

$$U_I(ABAB) = -I + \beta R^u e^{\rho_u(T-\tau)}, \quad (11)$$

where ρ_u is reinvestment rate for the Project's type t_2 . The Project's payoff is

$$U_P(ABAB) = I - (C + C_m) + (1 - \beta)R^u e^{\rho_u(T-\tau)}. \quad (12)$$

But in this case, it is necessary to formulate the requirements for the reinvestment rate more carefully. They arise from the fact that the Project was not profitable at the Initial stage. Therefore, the total two-stage revenue $R^u + R_m^u$ should exceed the investment I . It means that

$$\beta R^u e^{\rho_u(T-\tau)} > I.$$

Thus ρ_u should satisfy the following condition:

$$\rho_u > \frac{1}{T-\tau} (\ln I - \ln(\beta R^u)). \quad (13)$$

However, even if it is so, in the case of t_2 , the expression (7) is not necessarily positive. This condition requires additional restrictions on the parameters r and ρ_u , which may not always be satisfied. This corresponds to a higher level of risk associated with the unprofitability of the project in the initial stage.

Mixed strategies leading to the implementation of the project in two stages provide a representation of the players' payoff functions.

The Investor's expected payoff:

$$EU_I = -I + \beta \left(p R^p e^{\rho_p(T-\tau)} + (1-p) R^u e^{\rho_u(T-\tau)} \right). \quad (14)$$

The Project's expected payoff:

$$EU_P = I - (C + C_m) + (1 - \beta) \left(p R^p e^{\rho_p(T-\tau)} + (1-p) R^u e^{\rho_u(T-\tau)} \right). \quad (15)$$

Let us summarize all of the above in the following proposition.

Proposition 2. *In the multi-stage game of Venture Capital Investments with incomplete information*

1. *The expected players' payoffs are defined according to (14) and (15);*
2. *Perfect Bayesian Equilibrium (PBE) is reached in the profile (AA, BB) for Project type t_1 with a probability of p . This equilibrium corresponds to a positive NPV as determined in (7), without any additional restrictions on the discount rate r and reinvestment rate ρ_p ;*
3. *The profile (AA, BB) for project type t_2 represents an equilibrium in the second-stage subgame, which can be reached if condition (8) is met; in this case, the Project pays off only during both stages when the reinvestment rate satisfies (13), which does not guarantee the positivity of (7).*

5 Numerical Examples

Let us consider a few numerical examples to illustrate the results obtained in Sections 3 and 4.

5.1 Example 1

Let's consider the following values of model parameters: initial amount of investments is $I = 100$ conventional monetary units (hereafter referred to as "m.u."), future revenue is $R = 150$ m.u., the Investor's share in the project $\beta = 0.6$, costs for the Initial stage of the project $C = 30$ m.u.. With this set of parameters, if the Investor decides to invest in a project (chooses strategy A) and the Project is implemented (strategy B), we get that the Investor's payoff is $U_I(A, B) = -I + \beta R = -10$ m.u., the Project payoff is $U_P(A, B) = I - C + (1 - \beta)R = 130$ m.u..

The fact that the Investor's income is negative is due to a violation of the condition (1), while condition (2) has been met. It can be concluded that in this case, the Project has type t_1 and is not profitable at the Initial stage.

Let's move on to the Middle stage of the Project in the framework of the game we discussed in section 4. The following parameters are required: periods $\tau = 1$ and $T = 2$, rates $\rho = \rho_u = 0.7$ and $r = 0.6$ and costs for the Middle stage of the project $C_m = 20$ m.u..

First of all, let's check condition (8) to understand, whether the Middle stage is possible or not. For this parameter ratio, the inequality holds ($100 + 30 = 130 > 20$), thus, the next stage is possible and we can check the condition (13) to verify the profitability of both stages. It is also true ($90 \cdot 2.01375 = 181.24 > 100$).

Then we will calculate the winnings of both players, assuming that the investment was made and the project was implemented. According to (11) and (12), the Investor's payoff is $U_I(ABAB) = 81.24$ m.u., the Project's payoff is $U_P(ABAB) = 170.825$ m.u.. At the same time NPV obtained according to (7) is $NPV = -23.13$. It is negative because a positive NPV implies a high discounted income for the Investor, which is not achievable in this case due to the Project's unprofitability at the Initial stage of the game.

Thus, we have examined an example where a project that was unprofitable in its early stages was implemented through venture capital investment and began to generate profits in the middle stages prior to an IPO. Even in this scenario, however, the NPV of the project was negative, indicating a low return on investment.

5.2 Example 2

Now let's consider an example where we keep all of the model parameter values from the previous example, except for the Initial stage revenue R , which we will slightly increase: $R = 200$ m.u..

We immediately get that the Project is already profitable at the Initial Stage (condition (1) is satisfied, since $100 < 120$). In this case, there is no need to check

the conditions (8) and (13), as shown in subsection 4.2. When the players choose the strategies A and B twice, a perfect Bayesian equilibrium is achieved. In PBE the Investor's payoff (9) is $U_I(ABAB) = 141.65$ m.u., the Project's payoff (10) is $U_P(ABAB) = 211.10$ m.u., NPV (7) is $NPV = 2.49$ m.u..

All of them have positive values, albeit small ones, which indicates the effectiveness of the project's investment. Naturally, a greater increase in the value of revenue R and the reinvestment rate ρ would have a more significant impact.

5.3 Example 3

Now, let's solve the problem of finding the expected payoffs (14) and (15) of both players in the entire two-stage game. Consider combining the examples studied in subsections 5.1 and 5.2.

In order to determine the probability p of a project being profitable (has type t_1), we can refer to Russian statistics [18]. According to a study of the current state of the Russian venture investment market, 30% of invested projects are unsuccessful (completely losing their investments), 30% generate moderate returns, and 30% are highly profitable (generating significant profits). This means that approximately 60% of projects provide at least a moderate or high return, while around 40% either fail or generate only a small return that does not cover the overall risk. So, let's take $p = 0.6$ as a value of the Investor's belief in profitability of the Project before the Initial stage of the game.

According to (14) and (15), the Investor's expected payoff is $EU_I = 117.49$ m.u., the Project's expected payoff is $EU_P = 194.99$ m.u..

6 Conclusions and Prospects

The paper examines a one-stage and two-stage game-theoretical models of venture investment. In this model, there are two main players: the Investor, who invests her capital in a future project, and the Project, which is the project team or management, that is launched when the appropriate level of investment has been received. The cases of profitable and unprofitable Project are studied, while the Investor does not have accurate information, but only beliefs in this property of the Project, described in terms of probability. It is also assumed that despite the fact that the Project may not initially be profitable, it can generate the necessary profit during two consequent stages of implementation. In the current work, these two stages are conventionally called Initial (it includes Pre-seed, Seed, Start-up and Early stages of a venture project) and Middle (Expansion and pre-IPO).

For both models, the conditions for the implementation of the project (in one and two stages) were obtained, the corresponding payoffs of the players and restrictions on the parameters of the model were found. These results, including the Bayes-Nash equilibrium for a one-stage game and the perfect Bayesian equilibrium for a two-stage game, were presented in the form of two propositions. All the results were illustrated with numerical examples.

The venture capital investment model, first discussed in [10] and developed here, has great prospects.

First, the probabilistic component of the existing game deserves special attention and, accordingly, the Bayesian approach to the assessment and re-evaluation of a prior and a posterior probabilities. While the static model assumes that parameter p is known to both players, this represents a simplifying assumption for analytical tractability. In practical applications, investors typically estimate p based on historical data, market research, expert opinions, or due diligence processes. Future extensions of this model could incorporate Bayesian learning mechanisms where players update their beliefs about p based on observed outcomes, or consider p as a random variable with a known distribution (Bayesian game framework). Such extensions would better capture the uncertainty inherent in real investment decisions.

Secondly, we can consider co-financing the project by many investors (as is now customary in crowdfunding, crowdinvesting etc.) and also construct a model in terms of game theory. Thirdly, it is possible to analyze existing statistics on venture projects and use them to evaluate the parameters of an existing model.

At the moment, venture capital investment is actively growing, and the models that can be used to explore it should also be developed and improved.

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References

1. Aumann R., Hart S.: Handbook of Game Theory with Economic Applications, vol. 1, pp. 1–733. Elsevier, Amsterdam (1992).
2. Aumann R., Hart S.: Handbook of Game Theory with Economic Applications, vol. 2, pp. 735–1520. Elsevier, Amsterdam (1994).
3. Damodaran A.: Investment Valuation: Tools and Techniques for Determining the Value of Any Asset. 3rd edn. Wiley, Hoboken, NJ (2012).
4. Gladstone D.: Venture Capital Investing: The Complete Handbook for Investing in Private Businesses for Outstanding Profits. FT Press, Upper Saddle River, NJ (2004).
5. Greenfield M. F.: Risk management: “Risk as a resource”. Technical Report, NASA Code Q. Available: <http://www.hq.nasa.gov/office/codeq/risk/risk.pdf> (1998). Accessed 17 May 2026.
6. Gibbons R.: Game Theory for Applied Economists. Princeton University Press, Princeton, NJ, pp. 144–152 (1992).
7. Friedman M., Savage L. J.: Utility Analysis of Choices Involving Risk. *Journal of Political Economy* **56**(4), 279–304 (1948). <https://doi.org/10.1086/256692>
8. Harsanyi J. C.: Games with Incomplete Information Played by Bayesian Players, I–III. *Management Science* **14**(3):159–183; **14**(5):320–334; **14**(7):486–502 (1967/68).

9. Hu Y., Loo C. K.: A Generalized Quantum-Inspired Decision Making Model for Intelligent Agent. *The Scientific World Journal*, Article ID 240983, 8 pp. (2014). <https://doi.org/10.1155/2014/240983>
10. Kumacheva S. Sh.: Game Theoretic Model of Venture Capital Investment. In: *Contributions to Game Theory and Management*, vol. 17, pp. 74–85 (2025). <https://doi.org/10.21638/11701/spbu31.2024.07>
11. Mazalov V. V.: *Mathematical Game Theory and Applications*. Lan, St. Petersburg, 448 pp. (2010). (in Russian)
12. Markowitz H.: Portfolio Selection. *The Journal of Finance* **7**, 77–91 (1952).
13. Petrosyan L. A., Kuzyutin D. V.: *Games in Extensive Form: Optimality and Stability*. St. Petersburg: St. Petersburg State University Press (2000). 222 pp. (in Russian)
14. Petrosyan L., Zenkevich N., Shevkoplyas E.: *Game theory*. BHV-Peterburg, St. Petersburg, 432 pp. (2012). (in Russian)
15. Samuelson P. A., Nordhaus W. D.: *Economics*. 19th edn. McGraw-Hill Education, New York (2009).
16. Sharpe W. F., Alexander G. J., Bailey J. V.: *Investments*. Prentice Hall, Upper Saddle River, NJ, 962 pp. (1999).
17. Zamir S.: Bayesian Games: Games with Incomplete Information. In: *Encyclopedia of Complexity and Systems Science*, pp. 426–454 (2008). https://doi.org/10.1007/978-0-387-30440-3_29
18. Russian Venture Capital Investment Association (RVIA): Official website. <http://www.rvca.ru/rus/> (2024). Accessed 17 May 2026.