

Comparison of D -criteria and metrics in the space of Keplerian orbits

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Abstract. This work is devoted to the analysis of the properties of the classical D -criteria of orbital dissimilarity, the g -metrics of Kholoshevnikov, and the g -metric based on physical integrals of motion. The behavior of these metrics is analyzed as the Keplerian orbital elements change. This analysis shows how well the available tools distinguish between different orbits. It also allows for the selection of a criterion or metric that exhibits better sensitivity under specific conditions.

Introduction

The assessment of the proximity of celestial bodies' orbits is a fundamental task of celestial mechanics, necessary for the search of asteroid families, meteoroid streams, and the identification of bodies of common origin. The paper provides a comparative analysis of existing criteria and metrics, and studies their local extremes and asymptotics, which allows for the selection of the optimal tool for specific astronomical tasks.

Definitions of D -criteria and metrics

Let us introduce the notations: a is the semi-major axis, e is the eccentricity, i is the inclination, Ω is the longitude of the ascending node, ω is the argument of pericenter, $q = a(1 - e)$, $p = a(1 + e)$. The indices 1 and 2 refer to the compared orbits.

The main formulas are summarized in table 1.

Here, I is the mutual inclination, P is the angle between the Laplace–Runge–Lenz vectors, Π and Π' are the differences in the longitudes of the pericenters, $E = -\mu/(2a)$, $L = \sqrt{\mu p}$, and q is the unit orientation quaternion.

TABLE 1. Main criteria and metrics

Name	Formula	Features
D_1 (Southworth-Hawkins)	$D_1^2 = \frac{(q_1 - q_2)^2}{L^2} + 4 \sin^2 \frac{I}{2} + (e_1 - e_2)^2 + (e_1 + e_2)^2 \sin^2 \frac{\Pi}{2}$	Not defined for circular orbits [1]
D_2 (Southworth-Hawkins)	$D_2^2 = \frac{(q_1 - q_2)^2}{L^2} + 4 \sin^2 \frac{I}{2} + (e_1 - e_2)^2 + (e_1 + e_2)^2 \sin^2 \frac{\Pi'}{2}$	Depends on the choice of the main plane [1]
D_3 (Drummond)	$D_3^2 = \left(\frac{q_1 - q_2}{q_1 + q_2} \right)^2 + \left(\frac{e_1 - e_2}{e_1 + e_2} \right)^2 + I^2 + \left(\frac{e_1 + e_2}{2} \right)^2 P^2$	Not defined when $e_1 = 0, e_2 > 0$ [2]
D_J (Jopek)	$D_J^2 = \left(\frac{q_1 - q_2}{q_1 + q_2} \right)^2 + (e_1 - e_2)^2 + 4 \sin^2 \frac{I}{2} + (e_1 + e_2)^2 \sin^2 \frac{\Pi}{2}$	Combination of D_1 and D_3 [3]
ρ_1	$\rho_1^2 = \frac{1}{L}(p_1 + p_2 - 2\sqrt{p_1 p_2} \cos I) + (e_1^2 + e_2^2 - 2e_1 e_2 \cos P)$	Contains the arbitrary parameter L [4]
ρ_2	$\rho_2^2 = (1 + e_1^2)p_1 + (1 + e_2^2)p_2 - 2\sqrt{p_1 p_2}(\cos I + e_1 e_2 \cos P)$	Basic Kholshchevnikov metric [5]
ρ_3	$\rho_3^2 = \min_{\Omega_1, \Omega_2} \rho_2^2$	Factor space over Ω [5]
ρ_4	$\rho_4^2 = \min_{\omega_1, \omega_2} \rho_2^2$	Factor space over ω [5]
ρ_5	$\rho_5^2 = \min_{\Omega_1, \Omega_2, \omega_1, \omega_2} \rho_2^2$	Factor space over Ω and ω [5]
g (new)	$g^2 = \left(\frac{E_1 - E_2}{E_{\text{scale}}} \right)^2 + \left(\frac{L_1 - L_2}{L_{\text{scale}}} \right)^2 + d_q^2(q_1, q_2)$	Based on integrals of motion; quaternion representation [6]

Analysis results

Variations of the major semi-axis a_1

The extremum conditions and asymptotics are presented in table 2.

TABLE 2. Properties when varying a_1

Metric	Extremum condition	Asymptotics as $a_1 \rightarrow \infty$
D_1, D_2	$q_1 = q_2$	$\sim a_1^2$ (quadratic growth)
D_3, D_J	$q_1 = q_2$	$\rightarrow \text{const}$ (loss of sensitivity)
ρ_1	$\sqrt{p_2/p_1} \cos I = 1$	$\sim a_1$ (linear growth)
$\rho_2 - \rho_5$	$1 + e_1^2 = \sqrt{p_2/p_1} \mathcal{F}$	$\sim a_1$
g	polynomial of the 6th degree	$\sim a_1$

Conclusion: D_3 and D_J are unsuitable for distant objects, while the Kholshchevnikov and g metrics provide a balanced sensitivity.

Variations of eccentricity e_1

Complexity of extremum equations: D_1, D_2 — linear; D_J — polynomial of 4th degree; D_3 — 7th; ρ_1 – ρ_5 — irrational, reduced to polynomial of 8th degree; g — $e_1 = 0$ or $L_1 = L_2$ (simplest).

Limits:

- When $e_1 \rightarrow 0^+$: for D_3 , the term $\left(\frac{e_1 - e_2}{e_1 + e_2}\right)^2 \rightarrow 1$ is non-physical; the others have finite limits.
- When $e_1 \rightarrow 1^-$: ρ has a derivative $\sim 1/\sqrt{1 - e_1}$ (strong divergence); g has the same, but with a smaller coefficient, which gives better stability.

Variations of the inclination i_1

Analytical conditions for extremum are obtained for D_2 ($i_1 = \arctan(\tan i_2 \cos(\Omega_1 - \Omega_2))$), ρ_4 (the same without πk) and ρ_5 ($i_1 = i_2$). The rest are transcendental equations.

When $i_1 \rightarrow 0^+$ and $i_1 \rightarrow \pi^-$, all metrics have finite limits. The metric g stands out because its derivative does not depend on the eccentricities, which is important for highly eccentric orbits.

Variations of the node Ω_1

All functions are 2π -periodic. The derivatives at $\Omega_1 = 0$ and 2π have opposite signs. The metrics ρ_3 and ρ_5 do not depend on Ω_1 . For ρ_4 , the extremum condition is $\sin i_1 \sin i_2 \sin(\Omega_1 - \Omega_2) = 0$, which implies $\Omega_1 = \Omega_2$ or $\Omega_1 = \Omega_2 + \pi$.

Variations of the pericenter argument ω_1

For D_1, D_2, D_J , the extremum condition is $\Pi = \pi n$ or $\Pi' = \pi n$, which is the simplest. For D_3 , it is transcendental. The metrics ρ_4 and ρ_5 do not depend on ω_1 . The derivatives ρ_1 – ρ_3 and g are proportional to $e_1 e_2$, so they become zero when $e_1 = 0$ or $e_2 = 0$.

Recommendations for selection

- **For general application** the metric g is preferred: physically justified, free from singularities, and has simple conditions for e and L .
- **For circular orbits** use only ρ or g (classical D is not applicable).
- **For highly eccentric orbits**, g is better than ρ due to lower divergence.
- **For coplanar orbits**, g gives an advantage (the derivative does not depend on e).
- **For fast calculations** D_1, D_2, D_J give simple conditions for ω_1 ; ρ_4, ρ_5 — for i_1 .
- **For quotient spaces** ρ_3, ρ_5 are independent of Ω ; ρ_4, ρ_5 are independent of ω .

Conclusion

The conducted comparative analysis shows that the classical D -criteria, despite their historical value, have serious mathematical drawbacks. The Kholoshevnikov metrics are strict metrics, but they give complex extremum equations and strong divergence in the parabolic limit. The g metric, based on physical integrals, combines strictness, absence of singularities, and simplicity of the mathematical structure, which makes it the most universal tool for orbit identification problems.

References

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