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ASYMPTOTIC EFFICIENCY OF NEW DISTRIBUTION-FREE TESTS OF SYMMETRY FOR GENERALIZED SKEW ALTERNATIVES

G. T. Bookiya* **and Ya. Yu. Nikitin*** UDC 519.2

The Bahadur efficiency of new nonparametric tests of symmetry recently proposed by Nikitin and Ahsanullah is calculated. In contrast to this result, where only location alternatives were discussed, in the present paper generalized skew alternatives are of interest. It is shown that the new tests are highly efficient for a large class of skew alternatives. The problem of most favorable alternatives is also studied. Bibliography: 35 titles.

1. Introduction

A symmetry of the distribution of a random variable with continuous distribution function (d.f.) F means that

$$F(x) = 1 - F(-x) \qquad \text{for all } x \in \mathbb{R}^1. \tag{1}$$

Assume that we are given a sample $X_1, \dots, X_n$ with continuous d.f. F and a problem is to test a hypothesis $\mathcal{H}$ that F is symmetric, i.e., relation (1) holds, against the alternative that this relation is violated at least at one point. Testing the symmetry hypothesis is one of classical problems of nonparametric statistics. There is a number of symmetry tests, e.g., the sign test and the signed-rank Wilcoxon test, and more complicated tests based on very different ideas (see [18, 20], [28, Chap. 4]). Among them only a few tests are based on characterizations of symmetry and hence use internal features defining the properties of symmetry; in this regard, we mention only papers [10, 12, 23].

In contrast to the criteria of symmetry, in recent years constructing and studying *goodness-of-fit tests* based on characterization properties of the distributions is a rapidly evolving field. Examples include papers [19, 26, 27, 32–34] and several others.

In paper [1], Ahsanullah proves that symmetry is characterized by the identity of the distributions of absolute values for extremal order statistics.

Theorem 1. *Let $X_1, \dots, X_n$ be independent observations with common continuous d.f. F. Consider the extreme order statistics*

$$X_{1,n} := \min(X_1, \dots, X_n), \quad X_{n,n} := \max(X_1, \dots, X_n).$$

The distribution of X_i is symmetric, i.e., the d.f. F meets condition (1) *if and only if* $|X_{1,n}| \stackrel{d}{=} |X_{n,n}|$.

This property forms a basis for a distribution-free test of symmetry developed by Nikitin and Ahsanullah in [30].

*St.Petersburg State University, St.Petersburg, Russia, e-mail: gregorybookia@yandex.ru, y.nikitin@spbu.ru.

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Fix an arbitrary integer $k > 1$ and consider V-empirical distribution functions of the variables $|X_{1,n}|$ and $|X_{n,n}|$:

$$G_n^{(k)}(t) = n^{-k} \sum_{1 \le i_1, \dots, i_k \le n} \mathbf{1}\{|\min(X_{i_1}, \dots, X_{i_k})| < t\}, \quad t \in \mathbb{R},$$
$$H_n^{(k)}(t) = n^{-k} \sum_{1 \le i_1, \dots, i_k \le n} \mathbf{1}\{|\max(X_{i_1}, \dots, X_{i_k})| < t\}, \quad t \in \mathbb{R}.$$

By the Glivenko–Cantelli theorem for V-empirical distribution functions [17], the validity of the hypothesis $\mathcal{H}$ implies that the functions $G_n^{(k)}(t)$ and $H_n^{(k)}(t)$ should be infinitely close with increasing n, and this enables us to use their differences in order to construct statistics for tests of symmetry.

We will consider two types of statistics proposed in [1]: the integral one

$$J_n^{(k)} = \int_{\mathbb{R}} [G_n^{(k)}(t) - H_n^{(k)}(t)]\, dF_n(t), \tag{2}$$

where F_n is the empirical distribution function of the sample $|X_i|$, and the Kolmogorov type statistic

$$D_n^{(k)} = \sup_t |G_n^{(k)}(t) - H_n^{(k)}(t)|. \tag{3}$$

Thus for each value of the parameter $k = 2, \dots, n-1$, we consider two criteria corresponding to the integral and Kolmogorov statistics. The number in parenthesis appearing in superscript corresponds to the degree of an appropriate kernel of U- or V-statistic.

In [30], Nikitin and Ahsanullah calculated the local Bahadur asymptotic efficiency of these statistics for location alternatives. The Bahadur efficiency is chosen among other types of efficiency, because it can be applied to statistics whose distribution under the null hypothesis differs from normal. This is especially valuable for the Kolmogorov type statistics that have nonnormal asymptotic distributions.

For most of the considered distributions, the specified effectiveness turns to be unusually high. For example, for the sequence of the integral statistics $\{J_n^{(3)}\}$ its value is equal to 0.977 in the case of normal distribution, and is equal to 0.938 in the case of logistic distribution. At the same time, the value for the Kolmogorov statistic $D_n^{(3)}$, which is usually somewhat less effective, is equal to 0.764 and 0.750 for the above distributions.

It would be interesting to test whether the new tests are equally effective for more complicated and practically important class of alternatives. Therefore in the present paper, we continue to study the local Bahadur efficiency of the statistics in question for a broad class of alternatives, which can be called *generalized skew-symmetric alternatives.* For these alternatives, the density of observations is asymmetric and has the form

$$h(x,\theta) := c(\theta)\, G(w(x,\theta)) f(x), \tag{4}$$

where the even function f is the density of distribution under the validity of the hypothesis $\mathcal{H}$, G is a smooth nonnegative function, $c(\theta)$ is a normalizing factor, and the function $w(\theta, x)$, called the skew function, will be defined later. It is assumed that if the parameter θ equals zero, then the density of the generalized skew distribution coincides with $f(x)$. Therefore the greatest interest is the case, where the values of the parameter are close to zero. Below these alternatives are discussed in more detail.

Note that literature concerning the asymptotic behavior of symmetry tests for skew alternatives is pretty poor; in this regard, we can mention only paper [14] dedicated to ranking tests mainly. Also there are interesting papers [11] and [22], vaguely related to the topic of our research, but they address other criteria, other alternatives, and other type of efficiency.

We calculate the values of local Bahadur efficiencies of statistics (2) and (3) for alternatives (4). Based on these calculations and using a technique developed in previous papers [14–16], we finally construct alternatives of the considered class that are the most effective for these statistics in the Bahadur sense. For brevity, during calculations we omit unimportant conditions of regularity that are easy to recover.

2. The structure of generalized skew-symmetric alternatives

As flexible and realistic models of alternatives, the skew-symmetric distributions are known in Statistics for over a century. The first studies of the mechanisms of skew for the symmetric distributions were carried out by De Helguero in [13], see also [21] and especially [7]. Skew-symmetric distributions became popular after Azzalini's famous paper [3] on skewed normal distribution. In this paper, the density

$$\varphi(x,\theta) = 2\,\varphi(x)\,\Phi(x\theta), \qquad -\infty < x < \infty, \tag{5}$$

have been considered, where Φ and φ are the distribution function and density of the standard normal law, respectively.

This paper has given rise to numerous studies and generalizations of skewed distributions. Among many papers on this subject, we mention [2, 4, 6, 24, 35] and a generalizing monograph of Azzalini and Capitanio [8].

A broad generalization of skew-symmetric distribution to multi-dimensional case was proposed in [35]. For one-dimensional case, the densities considered there have the form

$$2\,G(x-\theta)\,f(x-\theta), \tag{6}$$

where the skew function G satisfies the condition $0 \le G \le 1$ and the symmetry condition $G(x) = 1 - G(-x)$, $x \in \mathbb{R}$, f is the even density, and θ is an arbitrary real parameter. The authors of [35] proved that every continuous density can be represented in form (6). In other words, any distribution, in a certain sense, is skewed with respect to some symmetric distribution. The same class of densities, but in the other notation, was considered in [5], see [35, Proposition 2].

We use a slightly different, but similar, method of construction of generalized skew-symmetric alternatives. Let us come back to density (4) and describe its components in more detail. The normalizing multiplier c depends only on the parameter θ. For convenience, we introduce the inverse value:

$$\widetilde{c}(\theta) = 1/c(\theta) = \int\limits_{-\infty}^{\infty} G(w(x,\theta))\,f(x)\,dx.$$

Assume that the structure function $G : \mathbb{R} \longrightarrow \mathbb{R}^+$ and function of skew $w(\theta,x) : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ satisfy the following conditions:

(1) G is twice differentiable in a neighborhood of zero and $G(0) > 0$;
(2) $\int\limits_{-\infty}^{\infty} G(w(x,\theta))\,f(x)\,dx < \infty$ for all θ;
(3) $w(x,\theta) = -w(-x,\theta)$ for any x;
(4) w is twice differentiable in θ in a neighborhood of zero;
(5) $w(x,0) = w(0,\theta) = 0$ for all x and θ;
(6) $\int\limits_{-\infty}^{\infty} (w'_\theta(x,0))^2 f(x)\,dx < \infty$.

Denote by $H(x,\theta)$ the d.f. corresponding to the density $h(x,\theta)$, and set $g(x) = G'(x)$.

Let us give some concrete examples of generalized skew-symmetric alternatives.

1) The classical skew-normal density (5) is at the same time a generalized skew-symmetric density in our sense. In fact, if we set $w(x,\theta) := x\theta$, $c(\theta) := 2$, and $G(x) = \Phi(x)$ in formula (4), then conditions (1)–(6) are satisfied.

2) In general, the shift alternative is not a generalized skewed alternative. However, the normal density with shift $\varphi(x-\theta)$ satisfies the conditions of a generalized (but not classic) skew. Indeed,

$$\varphi(x-\theta) = \frac{e^{-(x-\theta)^2/2}}{\sqrt{2\pi}} = e^{-\theta^2/2} \cdot e^{\theta x} \cdot \varphi(x),$$

where the functions $c(\theta) = e^{-\theta^2/2}$, $G(x) = e^x$, and $w(x,\theta) = x\theta$ satisfy conditions (1)–(6).

3) A special case of the generalized skew normal density. Consider an alternative density

$$h(x,\theta) = 2\,\varphi(x)\,\Phi\left(\frac{x\theta}{\sqrt{1+x^2\theta^2}}\right), \quad x \in \mathbb{R}.$$

Here,

$$w(x,\theta) = \frac{x\theta}{\sqrt{1+x^2\theta^2}},$$

and all required properties are fulfilled. This model was presented in [2] and then studied many times, see [8, p. 48].

4) A flexible class of densities with cubic function of skew. The density

$$h(x,\theta) = 2\,\varphi(x)\,\Phi((x+x^3)\,\theta), \quad x \in \mathbb{R},$$

was first considered in paper [24], and then in many publications (see [8, p. 49] for more details).

The distribution function of an alternative can be expanded in series for small values of the parameter θ if the skew alternative is close to symmetrical distribution. From the definition of the functions w, it follows that

$$w'_\theta(x,\theta) = -w'_\theta(-x,\theta), \qquad w''_\theta(x,\theta) = -w''_\theta(-x,\theta).$$

The twice differentiable factor $c(\theta)$ has the corresponding derivatives in θ:

$$\begin{aligned} c'(\theta) &= -\widetilde{c}'(\theta)/\widetilde{c}^2(\theta), \\ c''(\theta) &= 2\,(\widetilde{c}'(\theta))^2/(\widetilde{c}(\theta))^3 - (\widetilde{c}(\theta))''/(\widetilde{c}(\theta))^2, \end{aligned}$$

where

$$\widetilde{c}(\theta) = \int_{-\infty}^{\infty} G(w(x,\theta))\,f(x)\,dx,$$

$$(\widetilde{c}(\theta))' = \int_{-\infty}^{\infty} w'_\theta(x,\theta) g(w(x,\theta))\,f(x)\,dx,$$

$$(\widetilde{c}(\theta))'' = \int_{-\infty}^{\infty} w''_\theta(x,\theta) g(w(x,\theta))\,f(x)\,dx + \int_{-\infty}^{\infty} (w'_\theta)^2(x,\theta) g'(w(x,\theta))\,f(x)\,dx,$$

which yields

$$c(0) = 1/G(0); \quad c'(0) = 0; \quad c''(0) = -g'(0)/G^2(0) \int_{-\infty}^{\infty} (w'_\theta(0,x))^2 \, f(x) \, dx.$$

Clearly, if $\theta = 0$, then $H(x,0) = F(x)$. Let us find the value of $H'_\theta(x,0)$. Since

$$H'_\theta(x,\theta) = \int_{-\infty}^{x} \big(c'(\theta) \, G(w(u,\theta) + c(\theta) \, w'_\theta(u,\theta) \, g(w(u,\theta))\big) \, f(u) \, du,$$

we get

$$H'_\theta(x,0) = \frac{g(0)}{G(0)} \int_{-\infty}^{x} w'_\theta(u,0) \, f(u) \, du =: \frac{g(0)}{G(0)} \, F_w(x),$$

where for brevity,

$$F_w(x) = \int_{-\infty}^{x} w'_\theta(u,0) \, f(u) \, du, \quad x \in \mathbb{R}.$$

The oddness of the function $w(x,\theta) f(x)$ implies that for all $x \geq 0$,

$$\int_{-\infty}^{x} w'_\theta(u,0) \, f(u) \, du = -\int_{x}^{\infty} w'_\theta(u,0) \, f(u) \, du.$$

In particular,

$$F_w(-\infty) = F_w(\infty) = 0, \qquad F_w(-x) = F_w(x).$$

Now we can find the main terms of the expansion in Taylor series of the distribution function under the alternative hypothesis. Now, if $\theta \to 0$ and $k > 1$, then

$$\begin{aligned}
H^k(x,\theta) &\sim F^k(x) + k \, \frac{g(0)}{G(0)} \, F_w(x) \, F^{k-1}(x) \cdot \theta, \\
H^k(-x,\theta) &\sim F^k(-x) + k \, \frac{g(0)}{G(0)} \, F_w(x) \, F^{k-1}(-x) \cdot \theta, \\
(1 - H(x,\theta))^k &\sim F^k(-x) - k \, \frac{g(0)}{G(0)} \, F_w(x) \, F^{k-1}(-x) \cdot \theta, \\
(1 - H(-x,\theta))^k &\sim F^k(x) - k \, \frac{g(0)}{G(0)} \, F_w(x) \, F^{k-1}(x) \cdot \theta.
\end{aligned} \tag{7}$$

3. The Kullback–Leibler information

Denote by F and H the d.f. of observations under the null hypothesis and under alternative, respectively; the corresponding densities are f and h. As the "information distance" between the distribution functions, we take the Kullback–Leibler information, which plays an important role in the asymptotic theory of testing hypotheses, see for example [9]:

$$K(h,f) = \int_{-\infty}^{\infty} h(x) \, \ln \frac{h(x)}{f(x)} \, dx$$

(it is assumed that the d.f. H is absolutely continuous with respect to F). In our case, for the density of the symmetric distribution f and the skewed density for the alternative h given in (4), in testing a complex null-hypothesis H one may use the following upper bound for exact slopes of test statistics (discussed below), see [28, Chap. 4]:

$$
\begin{aligned}
K(\theta) &= \int\limits_{-\infty}^{\infty} \ln \frac{2\,h(x,\theta)}{h(x,\theta)+h(-x,\theta)}\, h(x,\theta)\, dx \\
&= \int\limits_{-\infty}^{\infty} c(\theta)\, G(w(\theta,x))\, \ln\{c(\theta)\, G(w(\theta,x))\}\, f(x)\, dx.
\end{aligned}
\tag{8}
$$

Let us find the value of the main term of this expression as $\theta \to 0$ under given regularity conditions for G and w. For this, we decompose the expression $c(\theta)\, G(w(\theta,x))$ in the Taylor series in the neighborhood from $\theta = 0$ up to θ^2. Taking into account the above properties of $c(\theta), w(x,\theta), F_w(x)$, and $G(x)$, we obtain the following asymptotics for the Kullback–Leibler information:

$$
K(\theta) \sim \frac{g^2(0)}{2\,G^2(0)} \int\limits_{-\infty}^{\infty} (w'_\theta(x,0))^2\, f(x)\, dx \cdot \theta^2, \quad \theta \to 0. \tag{9}
$$

In a special case for a classical model of the Azzalini skew (5), for $w(x,\theta) = x\theta$ this relation takes the form

$$
K(\theta) \sim 2\, g^2(0) \cdot \mathbf{E}_F\, X_1^2 \cdot \theta^2
$$

that coincides with calculations in [15].

4. Calculation the Bahadur efficiency

We continue to calculate the local Bahadur efficiency for the statistics $J_n^{(k+1)}$ and $D_n^{(k)}$, see (2) and (3). A more detailed analysis of this technique is presented in [9] and [28]. Note also that the key concept in the Bahadur theory is the notion of exact slope for a sequence of test statistics $\{T_n\}$. This value shows the rate of exponential decrease of the reached level of $\{T_n\}$ under the alternative. As $n \to \infty$, the sequence $\{T_n\}$ should satisfy the following two properties:

1. $T_n \to b_T(\theta)$ as $n \to \infty$, in probability under the alternative;
2. $n^{-1} \ln P(T_n \geq u) \to -\zeta(u)$ as $n \to \infty$, for $\mathcal{H}$,

where $\zeta(u)$ is a continuous function on the range of $b_T(\theta)$. Then the exact Bahadur slope [9] is defined as $c_T(\theta) = 2\,\zeta(b_T(\theta))$, and the absolute local Bahadur efficiency is expressed as follows:

$$
e_T = \lim_{\theta \to 0+} \frac{c_T(\theta)}{2K(\theta)}, \tag{10}
$$

where $K(\theta)$ is as in (8).

When checking conditions 1 and 2, one should take into account that the first of them is, as a rule, a simple consequence of the law of large numbers, whereas the second condition (a form of logarithmic asymptotics of large deviations) is nontrivial. In our case, this asymptotics has already been calculated in [30]. The following sections focus on the local efficiency of the integral and Kolmogorov statistics.

5. Efficiency of the integral statistics

Most of arguments required in the beginning of this section repeat those contained in [30]. Therefore we confine ourselves to a brief description.

Statistics (2) does not depend on the distribution. It can be represented in the form

$$J_n^{(k+1)}=n^{-k-1}\sum_{1\le i_1,\dots,i_{k+1}\le n}\big(\mathbf{1}\{|\min(X_{i_1},\dots,X_{i_k})|<|X_{i_{k+1}}|\}-\mathbf{1}\{|\max(X_{i_1},\dots,X_{i_k})|<|X_{i_{k+1}}|\}\big),$$

which reduces to V-statistics by using symmetrization. According to [31], the asymptotics of large deviations for this statistic is expressed as follows:

$$\zeta_{k+1}(t)\sim\frac{t^2}{2(k+1)^2\sigma_{k+1}^2},\qquad t\to 0,$$

where

$$\sigma_{k+1}^2=\frac{1}{2^{2k-2}(k+1)^2}\int_0^1\left((1+s)^k+(1-s)^k-2\right)^2ds>0.$$

In a similar way, the expression

$$b_J^{(k+1)}(\theta)=\mathbf{P}_\theta\{|\min(Y_1,\dots,Y_k)|<|X|\}-\mathbf{P}_\theta\{|\max(Y_1,\dots,Y_k)|<|X|\}\tag{11}$$

can also be represented as a V-statistic.

Now we need to extract the main part of (11) as $\theta\to 0$ under alternative (4). Note that

$$\mathbf{P}_\theta\{|\min(Y_1,\dots,Y_k)|<|Z|\}=\int_0^\infty\Big((1-H(-x,\theta))^k-(1-H(x,\theta))^k\Big)\,d(H(x,\theta)-H(-x,\theta)).$$

For brevity, set $\overline{H}(x,\theta):=1-H(x,\theta)$ and $H_0(x,\theta):=H(x,\theta)-H(-x,\theta)$. Then using the obtained representations, we get

$$b_J^{(k+1)}(\theta)=\int_0^\infty\Big(\overline{H}^k(-x,\theta)-\overline{H}^k(x,\theta)-H^k(x,\theta)+H^k(-x,\theta)\Big)\,dH_0(x,\theta).$$

From (7), it follows that

$$H_0(x,\theta)\sim F(x)-F(-x),\quad \theta\to 0,$$

and hence,

$$dH_0(x,\theta)\sim 2\,f(x)\,dx.$$

Now with the help of (7), we obtain on asymptotic for $b_J^{(k+1)}(\theta)$:

$$b_J^{(k+1)}(\theta)\sim 4\,k\,\theta\,\frac{g(0)}{G(0)}\int_0^\infty F_w(x)\,(F^{k-1}(-x)-F^{k-1}(x))\,f(x)\,dx.$$

Taking into account that

$$d/dx\ \Big(F^k(x)+F^k(-x)\Big)=k\,f(x)\,\Big(F^{k-1}(x)-F^{k-1}(-x)\Big),$$

we can integrate by parts. Consequently, using (2) one can get

$$b_J^{(k+1)}(\theta) \sim 4\,k\,\theta \frac{g(0)}{G(0)} \int_0^\infty F_w(x)(F^{k-1}(-x) - F^{k-1}(x))\, f(x)\, dx$$

$$\sim -4\,\theta\, \frac{g(0)}{G(0)} \int_0^\infty F_w(x)\, d(F^k(-x) + F^k(x))$$

$$= 4\,\theta\, \frac{g(0)}{G(0)} \Big(\frac{1}{2^{k-1}} F_w(0) + \int_0^\infty (F^k(-x) + F^k(x))\, w'_\theta(x,0)\, f(x)\, dx \Big)$$

$$= 4\,\theta\, \frac{g(0)}{G(0)} \Big(\int_0^\infty \Big(F^k(-x) + F^k(x) - \frac{1}{2^{k-1}}\Big) w'_\theta(x,0)\, f(x)\, dx \Big).$$

Now, taking into account asymptotics (5), we find that as $\theta \to 0$,

$$c_J^{(k+1)}(\theta) \sim \frac{16\, g^2(0) \Big(\int_0^\infty \big(F^k(-x) + F^k(x) - 1/2^{k-1}\big)\, w'_\theta(0,x)\, f(x)\, dx \Big)^2}{G^2(0)(k+1)^2 \sigma_{k+1}^2}\, \theta^2.$$

Finally, taking into account asymptotics (9) for the Kullback–Leibler information, one can write down the local exact absolute Bahadur efficiency (10) of the sequence of the integral statistics $\{J_n^{(k+1)}\}$ as follows:

$$e_{J^{(k+1)}} = \frac{2^{2k+1} \Big(\int_0^\infty \big(F^k(-x) + F^k(x) - 1/2^{k-1}\big)\, w'_\theta(0,x)\, f(x)\, dx \Big)^2}{\int_0^\infty (w'_\theta(0,x))^2\, f(x)\, dx \int_0^1 ((1+s)^k + (1-s)^k - 2)^2\; ds}. \tag{12}$$

6. The efficiency of the Kolmogorov statistics

Now we calculate the efficiency of sequence (3) of the Kolmogorov statistics $D_n^{(k)}$. As in the first case, they are distribution-free under $\mathcal{H}$, and the asymptotics of large deviations was obtained in [30] with the help of a result in [29]. It was proved there that

$$\zeta_{k+1}(t) \sim \frac{t^2}{2k^2 s_k^2}, \qquad t \to 0,$$

where

$$s_k^2 = \sup_{0<s<1} (1-s) \Big((1+s)^{k-1}/2^{k-1} - (1-s)^{k-1}/2^{k-1} \Big)^2.$$

Calculating the limit under the alternative, the authors of [30] proved that

$$b_D^{(k)}(\theta) = \lim_{n\to\infty} D_n^{(k+1)} = \sup_{x\ge 0} \Big| \mathbf{P}_\theta\{|\min(Y_1,\dots,Y_k)| < x\} - \mathbf{P}_\theta\{|\max(Y_1,\dots,Y_k)| < x\} \Big|.$$

Similar to arguments in the previous section and using (7), one can find that for skew alternatives,

$$b_D^{(k)}(\theta) = \sup_{x\ge 0} \Big| H^k(x,\theta) + H_0^k(x,\theta) - H^k(-x,\theta) - H_0^k(-x,\theta) \Big|$$

$$\sim \theta \cdot 2k \frac{g(0)}{G(0)} \sup_{x\ge 0} |(F^{k-1}(x) - F^{k-1}(-x))\, F_w(x))|.$$

Then the exact slope for the Kolmogorov statistics admits the asymptotics

$$c_{D^{(k)}}(\theta) \sim \frac{4\, g^2(0)\,|\sup_{x\ge 0} |(F^{k-1}(x) - F^{k-1}(-x))\, F_w(x))|^2}{G^2(0)\, s_k^2}.$$

Finally, its local absolute Bahadur efficiency is equal to

$$e_{D^{(k)}} = \frac{4 \sup_{x\ge 0} |(F^{k-1}(-x) - F^{k-1}(x)) \int\limits_{-\infty}^{x} w'_\theta(0,u)\ f(u)du|^2}{s_k^2 \int\limits_0^\infty (w'_\theta(0,x))^2\ dF}. \tag{13}$$

7. A discussion of the found efficiencies

Although the generalized skew-symmetric distributions form a quite wide class of alternatives, the asymptotic efficiency of both integral and Kolmogorov criteria depends only on the derivative at zero of the function of skew $w'_\theta(0,x)$ and does not depend on the "structural" function G completely.

In order to obtain another useful result associated with the efficiency criteria, we introduce an auxiliary function $R(x) = 2F(x) - 1$. Then

$$F(x) = \frac{1+R(x)}{2}, \quad F(-x) = \frac{1-R(x)}{2}, \quad dF(x) = \frac{1}{2}dR(x), \quad R(0)=0, \quad R(\infty)=1.$$

Now the efficiencies (12) and (13) are represented as follows:

$$e_{J^{(k+1)}} = \frac{\left(\int\limits_0^\infty \big((1+R(x))^k + (1-R(x))^k - 2\big)\, w'_\theta(0,x)\ dR(x)\right)^2}{\int\limits_0^\infty (w'_\theta(0,x))^2\ dR(x) \int\limits_0^\infty ((1+s)^k + (1-s)^k - 2)^2\ ds},$$

$$e_{D^{(k)}} = \frac{\sup\limits_{x>0} |((1+R(x))^{k-1} - (1-R(x))^{k-1}) \int\limits_{-\infty}^{x} w'_\theta(0,u)\ dR(u)|^2}{s_k^2 \int\limits_0^\infty (w'_\theta(0,x))^2\ dR(x)}.$$

Finally, after some simplifications we get

$$e_{J^{(k+1)}} = \frac{\Big(\int\limits_0^\infty \Big(2\sum\limits_{l=1}^{\lfloor k/2\rfloor} \binom{k}{2l} R(x)^{2l}\Big) w'_\theta(x,0)\ dR(x)\Big)^2}{\int\limits_0^\infty (w'_\theta(x,0))^2\ dR(x) \int\limits_0^\infty \Big(2\sum\limits_{l=1}^{\lfloor k/2\rfloor} \binom{k}{2l} R^{2l}(x)\Big)^2\ dR(x)}, \tag{14}$$

$$e_{D^{(k)}} = \frac{\sup\limits_{x>0} \Big|\Big(2\sum\limits_{l=1}^{\lfloor k/2\rfloor} \binom{k}{2l-1} R(x)^{2l-1}\Big) \int\limits_{-\infty}^{x} w'_\theta(u,0)\ dR(u)\Big|^2}{\int\limits_0^\infty (w'_\theta(x,0))^2\ dR(x) \sup\limits_{x>0}(1-x)\Big(2\sum\limits_{l=1}^{\lfloor k/2\rfloor} \binom{k}{2l-1} x^{2l-1}\Big)^2}. \tag{15}$$

From (14)–(15), it follows that if $k = 2$ and $k = 3$, then the values of the efficiencies are the same for the integral and Kolmogorov criteria. This is consistent with the results for these criteria obtained in [23] and [30] for location alternatives.

Now we calculate the local Bahadur efficiency for several standard distributions. Consider the following symmetric densities of the original sample:

(1) Normal: $f_1(x) = (2\pi)^{-1/2} e^{-x^2/2}$;

(2) Logistic: $f_2(x) = e^x/(1+e^x)^2$;
(3) Arcsine on $[-1,1]$: $f_3(x) = (\pi\sqrt{1-x^2})^{-1}\mathbf{1}\{-1 < x < 1\}$;
(4) Uniform on $[-1,1]$: $f_4(x) = \frac{1}{2}\mathbf{1}\{-1 < x < 1\}$;
(5) Student-3: $f_5(x) = 2/(\pi(1+x^2)^2)$.

These distributions were considered in [14, 15]. For each of the selected densities of f, we consider the corresponding skew-symmetric distribution (4) with arbitrary function G and a deviation function w such that $w'(x,0) = x$; note that the latter condition corresponds to the most common case $w(x,\theta) = x\theta$. The values of the efficiencies are obtained according to the corresponding formulas (12) and (13) with the help of the Matlab package.

Let us present a table for $k = 2$, 3, 4. In parentheses, we write down (if known) the efficiency of relevant tests obtained by Nikitin and Ahsanullah in [30] for location alternatives.

Table 1. Local efficiency of the integral and Kolmogorov tests.

Test	integral		Kolmogorov	
Distribution	$k = 2,3$	$k = 4$	$k = 2,3$	$k = 4$
Normal	0,977 (0,977)	0,975 (0,975)	0,764 (0,764)	0,733 (0,733)
Logistic	0,962 (0,938)	0,964 (0,925)	0,747 (0,750)	0,725 (0,696)
Arcsine	0,868	0,848	0,698	0,635
Uniform	0,938	0,923	0,750	0,697
Student-3	0,766	0,777	0,585	0,306

First, we note that the tests showed a rather high efficiency even for relatively rare distributions. Therefore we can confidently recommend them for checking the hypothesis of symmetry for generalized skew-symmetric alternatives. More effective integral criterion requires more computational costs. It has already been used in applied problems. It should be noted that sometimes the integral test for $k = 4$ shows better efficiency than for $k = 2$ and 3.

An analysis of this table and Table 2 in paper [30] shows that the values of efficiencies for different criteria for alternative and skewed alternative almost coincide. This is convenient for applications: in practice, the structure of the distribution of alternative is rarely known, and this property allows to use the best criteria regardless of the alternatives. Another interesting observation is that for the normal distribution, the values of the Bahadur efficiencies for shift and skewed alternatives coincide. This property of the normal distribution was observed in paper [15].

For several distributions, we also calculate the local efficiencies for skew-symmetric alternatives with a cubic function of the skew (see example 4 in Sec. 2). As one can see from Table 2, the results are quite comparable.

8. The most favorable alternative

The following question was put by Bahadur and developed in [28, Chap. 6]. What is the distribution at the alternative that leads to local asymptotic optimality (LAO) of new Bahadur statistics? In other words, which alternative is the most favorable and provides a local efficiency equal to one, i.e., $e_{J(k+1)} = 1$ or $e_{D(k)} = 1$ for some $k > 1$.

Table 2. Efficiencies for skewed alternatives with a cubic function of skew.

Test	Integral		Kolmogorov	
Distribution	$k=2,3$	$k=4$	$k=2,3$	$k=4$
Normal	0.670	0.689	0.548	0.575
Arcsine	0.943	0.928	0.778	0.721
Uniform	0.991	0.985	0.803	0.767

Coming back to representing the efficiencies in the form (14)–(15), one can see that the numerator and denominator are connected by Cauchy's inequality. Indeed,

$$\Bigl(\int_0^\infty \Bigl(2\sum_{l=1}^{\lfloor k/2\rfloor}\binom{k}{2l}R^{2l}(x)\Bigr)w'_\theta(x,0)\;dR(x)\Bigr)^2 \le \int_0^\infty (w'_\theta(x,0))^2\;dR(x)\int_0^\infty\Bigl(2\sum_{l=1}^{\lfloor k/2\rfloor}\binom{k}{2l}R^{2l}(x)\Bigr)^2\;dR(x);$$

$$\sup_{x>0}\Bigl|\Bigl(2\sum_{l=1}^{\lfloor k/2\rfloor}\binom{k}{2l-1}R^{2l-1}(x)\Bigr)\int_{-\infty}^{-x} w'_\theta(u,0)\;dR(u)\Bigr|^2$$
$$=\sup_{x>0}\Bigl(2\sum_{l=1}^{\lfloor k/2\rfloor}\binom{k}{2l-1}R^{2l-1}(x)\Bigr)^2\Bigl(\int_0^\infty \mathbf{1}\{u>x\}w'_\theta(u,0)\;dR(u)\Bigr)^2$$
$$\le\sup_{x>0}\Bigl(2\sum_{l=1}^{\lfloor k/2\rfloor}\binom{k}{2l-1}R^{2l-1}(x)\Bigr)^2\int_x^\infty dR(x)\int_0^\infty (w'_\theta(u,0))^2\;dR(u)$$
$$=\int_0^\infty (w'_\theta(x,0))^2\;dR(x)\sup_{x>0}(1-x)\Bigl(2\sum_{l=1}^{\lfloor k/2\rfloor}\binom{k}{2l-1}x^{2l-1}\Bigr)^2.$$

Thus, the distribution is asymptotically optimal in the sense of Bahadur if in the corresponding Cauchy inequality the equality is attained, and this is possible if the integrands are proportional. But for the Kolmogorov statistics, the optimality means that

$$\mathbf{1}\{u>x\}=C(x)\cdot w'_\theta(u,0),$$

which is impossible, because of the properties of the function w. A similar result was obtained in [15] for the classical Kolmogorov test for skew alternatives. Thus, the domain of LAO is empty.

For integral tests, the optimality is achieved if and only if

$$\sum_{l=1}^{\lfloor k/2\rfloor}\binom{k}{2l}R^{2l}(u)=C\cdot w'_\theta(u,0), \tag{16}$$

for a certain positive constant C. Here is an example of the optimum (most favorable) alternatives for $k=2$ and $k=3$. The density of such distributions has the form

$$f_{2,3}(x)=C\cdot|w'_\theta(x,0)|^{-1/2},\quad -b\le x\le b,\quad b>0.$$

If $w'_\theta(x,0) = x$, which corresponds to the classical case of $w(x,\theta) = x\theta$, then the most favorable alternative is a simple symmetric distribution with density

$$f_{2,3}(x) = 1/4\sqrt{b|x|}, \quad -b \le x \le b,$$

which had never before appeared in similar problems. From equality (16), one can also obtain more complicated examples.

Conclusion

We found theoretical values of the local Bahadur efficiency for sequence of statistics (2) and (3) proposed in [30], to check the symmetry of a distribution against the generalized skew-symmetric alternatives. Also we calculated the values for some of the most common cases. The efficiency values were very high, and the same criteria showed the best results for both shift alternatives and skew alternatives. All of this shows that the new criteria are quite successful and can be applied in practice. In addition, we also discuss a question of the most favorable alternatives from a given class.

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