ISSN 1064-5624, Doklady Mathematics, 2025, Vol. 112, No. 4, pp. 698–704. © Pleiades Publishing, Ltd., 2025.
Russian Text © The Author(s), 2025, published in Matematicheskaya Teoriya Igr i Ee Prilozheniya, 2025, Vol. 17, No. 2, pp. 75–89.

MATHEMATICS

About One Differential Game on Hypergraph

A. V. Tur[a,*] **and I. R. Tantlevskij**[a,**]

Received November 21, 2025; revised December 26, 2025; accepted December 26, 2025

Abstract—A cooperative differential game on a hypergraph is considered. A characteristic function of special type is introduced taking into account the network structure of the game, and its properties are studied. The Shapley value is used as a cooperative optimality principle. For games on hypergraphs with a cycle-free line graph, an explicit formula for the Shapley value is obtained. An illustrative example is considered.

Keywords: differential games, cooperative games, games on networks, games on hypergraphs, characteristic function, Shapley value, network models

DOI: 10.1134/S1064562425601052

1. INTRODUCTION

Control of dynamic systems on networks is a rapidly developing field with a wide range of applications in various areas of economics and science. Game-theoretic methods for finding optimal solutions are well suited for analyzing systems controlled by multiple agents competing to achieve their particular goals. In such models, agents (players) are typically identified with nodes of a network whose structure determines the agents' ability to interact with one another.

Cooperative differential games on networks were studied in [2–4, 7]. An important assumption made in these studies was the ability of players to break network connections, thereby excluding interactions with some or all of the other participants. In this context, problems were considered in which the players' equations of motion depend only on their own actions, while other players influence the value of the functional. A special characteristic function that takes into account the network structure was proposed to construct a cooperative solution in games of this class.

The aim of this work is to generalize previously obtained methods to the class of differential games on hypergraphs. Hypergraphs provide a powerful tool for modeling complex structures and relationships in various fields. Their main advantage is the ability to reflect multi-channel connections between elements. Using many-to-many relationships, hypergraphs are well suited for representing complex social interactions. Hypergraph vertices can be simultaneously connected by several types of relations, which are represented by different edges. This property can be used to study social and economic organizational systems. In addition to standard applications of such models for the analysis of relations in social groups, they can also be successfully used to study religious, ethnic, or political movements. In [1, 2] network analysis tools were applied to analyze the interactions between religious movements and government in Judea. In a similar manner, the use of hypergraph structures can underlie an effective approach for analyzing national and religious identity. Various ethnic and religious groups within a nation can be represented as collections of nodes in a hypergraph. Multi-channel connections between these groups can be reflected by hyper edges with varying weights. For example, the multi-aspect nature of complex relationships between the main religious movements and power in Judea in the 2nd–1st centuries BC was noted in [6] (including religious, ideological, economic, political, and other aspects), which can be more fully described using hypergraph.

Furthermore, hypergraphs can help describe the dynamic nature of identity transformations across different historical periods. Nodes can represent cultural and religious elements (such as language, traditions, customs, and religious rituals). The connections of these elements to specific communities, such as Jewish ones, or to historical periods can be depicted as edges.

This paper is organized as follows. In Section 2, we describe the original model of a cooperative differential game on a hypergraph. Cooperative optimality principles for this class of games are studied in Section 3. We introduce a characteristic function of special form and examine its properties. An illustrative example is given in Section 4. The conclusions are drawn in Section 5.

---

[a] *St. Petersburg State University, St. Petersburg, 199034 Russia*

**e-mail: a.tur@spbu.ru*

***e-mail: i.tantlevsky@spbu.ru*

2. COOPERATIVE DIFFERENTIAL GAME ON A HYPERGRAPH

Consider an n-person differential game on a hypergraph. Let $H = (N, E)$ be an undirected hypergraph, where $N = \{1, 2, \ldots, n\}$ is the set of its vertices identified with the players and $E = \{e_1, \ldots, e_m\}$ is the hyperedge set, where $e_i \in 2^N$ for $i = 1, \ldots, m$.

A path of length k between vertices $i_1 \in N$ and $i_2 \in N$ is a sequence of distinct hyperedges

$$p(i_1, i_2) = (e_{j_1}, e_{j_2}, \ldots, e_{j_k}),$$

such that $i_1 \in e_{j_1}$, $i_2 \in e_{j_k}$, and $e_{j_l} \cap e_{j_{l+1}} \neq \varnothing$ for $l = 1, \ldots, k-1$. The length of a path between vertices i_1 and i_2 is equal to the number of hyperedges in this path and is denoted by $|p(i_1, i_2)|$.

The shortest path between two vertices is the path of shortest length between them. We say that player j is at a distance of k from player i if the length of the shortest path between them is k. The length of the shortest path is denoted by $d(i, j)$.

For $l = 1, \ldots, m$, let $K^l(i)$ denote the set of players who are at a distance of l from player i.

Suppose that, at any moment of time, each player $i \in N$ can leave a hyperedge he/she belongs to, i.e., can break the connection to any other agent from the set $K^1(i)$.

Let $x_i(t) \subset R^{k_1}$ be the state variable of player $i \in N$ at time t, and let $u_i(t) \in U_i \subset R^{k_2}$ be the control variable of player $i \in N$.

The equations of motion of the players have the form

$$\dot{x}_i(\tau) = f_i(x_i(\tau), u_i(\tau)), \quad x_i(t_0) = x_i^0, \\ \text{for} \quad \tau \in [t_0, T], \ \ i \in N, \tag{2.1}$$

where $f_i(x_i, u_i)$ are continuously differentiable functions of x_i and u_i.

The payoff of player i is defined as

$$J_i(x_i^0, u_1, \ldots, u_n) = \int_{t_0}^{T} h_i(x_i(\tau), u_i(\tau))d\tau$$

$$+\sum_{l=1}^{m} \sum_{j \in K^l(i)} \delta^{l-1} \int_{t_0}^{T} h_i^j(x_i(\tau), x_j(\tau), u_i(\tau), u_j(\tau))d\tau. \tag{2.2}$$

Here, $\int_{t_0}^{T} h_i(x_i(\tau), u_i(\tau))d\tau$ is the payoff that player i receives independently and $\delta^{l-1}\int_{t_0}^{T} h_i^j(x_i(\tau), x_j(\tau),\ u_i(\tau), u_j(\tau))d\tau$ is the payoff that player i can receive by interacting with player j who is at distance l from i. Assume that $h_i^j(x_i(\tau), x_j(\tau), u_i(\tau), u_j(\tau)) \geq 0$ for any i and j. Let $\delta \in (0, 1)$ be a coefficient indicating that the greater the distance between players on the hypergraph, the less influence they have on each other. Denote the vector $(x_1(t), x_2(t), \ldots, x_n(t))$ by $x(t)$.

Assume that players can cooperate to achieve the maximum total payoff:

$$\sum_{i \in N} J_i(x_i^0, u_1, \ldots, u_n). \tag{2.3}$$

The players' optimal cooperative strategies ($\bar{u}(t) = \bar{u}_1(t)$, ..., $\bar{u}_n(t)$ for $t \in [t_0, T]$ are defined as

$$\bar{u}(t) = \arg \max_{u_1(t), \ldots, u_n(t)} \sum_{i \in N} J_i(x_i^0, u_1, \ldots, u_n). \tag{2.4}$$

The trajectory $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_n(t))$ corresponding to the optimal strategies $\bar{u}_1(t), \ldots, \bar{u}_n(t)$ is the optimal cooperative trajectory. Then the maximum joint payoff of the players is given by

$$\sum_{i \in N} \int_{t_0}^{T} h_i(\bar{x}_i(\tau), \bar{u}_i(\tau))d\tau$$

$$+\sum_{i \in N} \sum_{l=1}^{m} \sum_{j \in K^l(i)} \delta^{l-1} \int_{t_0}^{T} h_i^j(\bar{x}_i(\tau), \bar{x}_j(\tau), \bar{u}_i(\tau), \bar{u}_j(\tau))d\tau. \tag{2.5}$$

The original game is denoted by $\Gamma(x_0, T - t_0)$, and the subgame starting at time t from the point $\bar{x}(t)$ is denoted by $\Gamma(\bar{x}(t), T - t)$.

Let us introduce additional notation and definitions.

Let $S \subset N$. Then (S, E_S) is a subhypergraph of the hypergraph (N, E) that contains vertices only from S and its hyperedge set has the form $E_S = \{e' : e' = e \cap S, e \in E\}$.

3. CHARACTERISTIC FUNCTION

To construct cooperative optimality principles, we use the characteristic function proposed in [3] for differential games on networks, namely,

$$V(\{i\}; x_0, T - t_0) = \int_{t_0}^{T} h_i(\bar{x}_i(\tau), \bar{u}_i(\tau))d\tau, \quad i = 1, \ldots, n,$$

$$V(S; x_0, T - t_0) = \sum_{i \in S} V(\{i\}; x_0, T - t_0) \tag{3.1}$$

$$+\sum_{i \in S} \left(\sum_{l=1}^{m} \sum_{j \in K_S^l(i)} \delta^{l-1} \int_{t_0}^{T} h_i^j(\bar{x}_i(\tau), \bar{x}_j(\tau), \bar{u}_i(\tau), \bar{u}_j(\tau))d\tau \right), \\ S \subset N.$$

Here, $K_S^l(i)$ is the set of players who are at distance l from i in the subhypergraph (S, E_S). Since players can break connections to other players at any time and the instantaneous payoffs h_i^j are nonnegative, formula (3.1) does not involve minimization over the remaining players (minimization over the controls of the players from $N \backslash S$, as in the case of the classical characteristic function [9]), since minimization consists in breaking connections to players from S. Moreover, $V(S; x_0, T - t_0)$ for agents from the coalition S is calculated using optimal cooperative strategies, which significantly simplify the construction of the characteristic function. It was shown in [7] that, in games on cycle-free graphs, this characteristic function is convex. In a game on a hypergraph, the characteristic function constructed according to (6) is not convex in the general case, but it is superadditive for games on hypergraphs of a certain type. To formulate this result, we introduce the following notation. The line graph $L(H)$ of a hypergraph H is defined by the following conditions: the vertices of $L(H)$ correspond bijectively to the hyperedges of H and two vertices are adjacent in $L(H)$ if and only if the corresponding edges intersect. In what follows, we consider hypergraphs whose line graphs are cycle-free.

Lemma 3.1. *In the cooperative differential game* (2.1), (2.2) *on a hypergraph with a cycle-free line graph, the characteristic function constructed according to rule* (3.1) *is superadditive.*

Proof. Let $S_1 \subset N$, $S_2 \subset N$, and $S_1 \cap S_2 = \varnothing$. Then

$$
\begin{gathered}
V(S_1 \cup S_2; x_0, T - t_0) \\
= \sum_{i \in S_1} V(\{i\}; x_0, T - t_0) + \sum_{i \in S_2} V(\{i\}; x_0, T - t_0)
\end{gathered}
$$

$$
+\sum_{i \in S_1} \left(\sum_{l=1}^{m} \delta^{l-1} \sum_{j \in K_{S_1}^l(i)} \int_{t_0}^{T} h_i^j(\overline{x}_i(\tau), \overline{x}_j(\tau), \overline{u}_i(\tau), \overline{u}_j(\tau)) d\tau \right)
$$

$$
+\sum_{i \in S_2} \left(\sum_{l=1}^{m} \delta^{l-1} \sum_{j \in K_{S_2}^l(i)} \int_{t_0}^{T} h_i^j(\overline{x}_i(\tau), \overline{x}_j(\tau), \overline{u}_i(\tau), \overline{u}_j(\tau)) d\tau \right)
$$

$$
+ \sum_{i \in S_1 \cup S_2} \left(\sum_{l=1}^{m} \delta^{l-1} \sum_{\substack{j \in K_{S_1 \cup S_2}^l(i) \\ j \notin K_{S_1}^l(i) \cup K_{S_2}^l(i)}} \int_{t_0}^{T} h_i^j(\overline{x}_i(\tau), \overline{x}_j(\tau), \overline{u}_i(\tau), \overline{u}_j(\tau)) d\tau \right)
$$

$$
= V(S_1; x_0, T - t_0) + V(S_2; x_0, T - t_0)
$$

$$
+ \sum_{i \in S_1 \cup S_2} \left(\sum_{l=1}^{m} \delta^{l-1} \sum_{\substack{j \in K_{S_1 \cup S_2}^l(i) \\ j \notin K_{S_1}^l(i) \cup K_{S_2}^l(i)}} \int_{t_0}^{T} h_i^j(\overline{x}_i(\tau), \overline{x}_j(\tau), \overline{u}_i(\tau), \overline{u}_j(\tau)) d\tau \right).
$$

Since $h_i^j(x_i(\tau), x_j(\tau), u_i(\tau), u_j(\tau)) \geq 0$ for any i and j, we obtain

$$
\begin{gathered}
V(S_1 \cup S_2; x_0, T - t_0) \\
\geq V(S_1; x_0, T - t_0) + V(S_2; x_0, T - t_0).
\end{gathered}
$$

The superadditivity of the characteristic function is proved.

Similarly, for any subgame $\Gamma(\overline{x}(t), T - t)$, we can prove the superadditivity of the characteristic function constructed according to the rule

$$
V(\{i\}; \overline{x}(t), T - t) = \int_{t}^{T} h_i(\overline{x}_i(\tau), \overline{u}_i(\tau)) d\tau, \quad i = 1, \cdots, n,
$$

$$
V(S; \overline{x}(t), T - t) = \sum_{i \in S} V(\{i\}; \overline{x}(t), T - t) \tag{3.2}
$$

$$
\begin{gathered}
+\sum_{i \in S} \left(\sum_{l=1}^{m} \sum_{j \in K_S^l(i)} \delta^{l-1} \int_{t}^{T} h_i^j(\overline{x}_i(\tau), \overline{x}_j(\tau), \overline{u}_i(\tau), \overline{u}_j(\tau)) d\tau \right), \\
S \subset N.
\end{gathered}
$$

3.1. Shapley Value

After constructing the characteristic function, optimal imputations can be obtained using any known cooperative optimality principle. In this paper, a fair distribution of the total payoff for a maximal coalition of players is constructed using the Shapley value [5]. The component of the Shapley value for each player is defined as the average marginal profit added by this player to each coalition of the remaining players.

Let π be a permutation of players $\{1, \ldots, n\}$, $\pi(i)$ be the index of player i in π, and Π be the set of all possible permutations of elements of the set N. We define

$$
S_{\pi,k} = \{i \in N : \pi(i) \leq k\}, \quad k = 0, 1, \ldots, n.
$$

The vector of marginal contributions is defined as

$$
\begin{gathered}
\alpha_i^{\pi}(x_0, T - t_0) \\
= V(S_{\pi,\pi(i)}; x_0, T - t_0) - V(S_{\pi,\pi(i)-1}; x_0, T - t_0).
\end{gathered}
$$

Then the Shapley value is given by

$$
Sh_i(x_0, T - t_0) = \frac{1}{n!} \sum_{\pi \in \Pi} \alpha_i^{\pi}(x_0, T - t_0)
$$

(see [5]).

In the considered class of games for hypergraphs of a certain type, the formula for the Shapley value can be rewritten in a form more convenient for calculation.

Before formulating the theorem, we introduce the necessary notation. Let $i \in N, j \in N$, and $d(i, j) = l$, i.e., the shortest path $p(i, j)$ is constructed using l edges: $e_{i_1}, \ldots, e_{i_l}$. Consider the intersections of these edges: $e_1(i, j), \ldots, e_{l-1}(i, j)$. Here, $e_k(i, j) = e_{i_k} \cap e_{i_{k+1}}$, $k = 1, \ldots, l - 1$. Let $|e_k(i, j)| = n_k(i, j)$ and $P(i,\ j) = i \cup j \cup e_1(i, j) \cup \ldots \cup e_{l-1}(i, j)$.

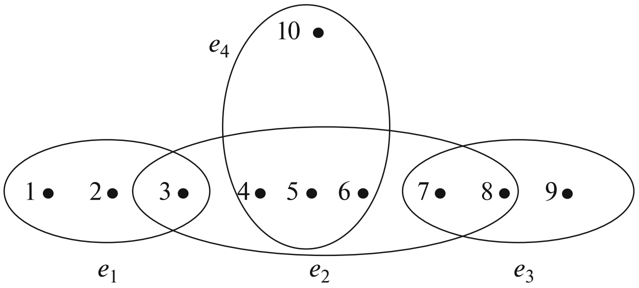

Fig. 1. Game on a hypergraph.

Consider these concepts as applied to a game on the hypergraph shown in Fig. 1. Here, $N = \{1,2,3,4,5,6,7,8,9,10\}$, $E = \{e_1,e_2,e_3,e_4\}$, $e_1 = \{1, 2, 3\}$, $e_2 = \{3,4,5,6,7,8\}$, $e_3 = \{7,8,9\}$, and $e_4 = \{4,5,6,10\}$. For the pair of players 1 and 9, we have $d(1,9) = 3$, $p(1,9) = (e_1,e_2,e_3)$, $e_1(1,9)=e_1 \cap e_2 = \{3\}$, $n_1(1,9) = 1$, $e_2(1,9)=e_2 \cap e_3 = \{7, 8\}$, $n_2(1,9) = 2$, and $P(1,9) = \{1,3,7,8,9\}$.

For convenience, we introduce the notation

$$\begin{aligned}\bar{h}_i^j(t) &= h_i^j(\bar{x}_i(t),\bar{x}_j(t),\bar{u}_i(t),\bar{u}_j(t)),\\ \bar{h}_i(t) &= h_i(\bar{x}_i(t),\bar{u}_i(t)),\\ \bar{h}_{i,j}(t) &= \bar{h}_i^j(t)+\bar{h}_j^i(t).\end{aligned}$$

The following result is valid.

Theorem 3.1. *Given a cooperative differential game* (2.1), (2.2) *on a hypergraph with a cycle-free line graph, if the characteristic function of the game is defined by* (3.1), *then the Shapley value has the form*

$$\begin{aligned}&Sh_i(x_0,T-t_0)\\ &= \int_{t_0}^{T}\bar{h}_i(\tau)d\tau + \sum_{\{r,q\}:i\in P(r,q)}\delta^{d(r,q)-1}\phi_i(r,q)\int_{t_0}^{T}\bar{h}_{r,q}(\tau)d\tau,\end{aligned}$$

where

$$\phi_i(r,q)=\begin{cases} 2\left(\dfrac{(n_j(r,q)-1)!}{(n_j(r,q)+2)!}-\displaystyle\sum_{g\neq j}\dfrac{(n_j(r,q)+n_g(r,q)-1)!}{(n_j(r,q)+n_g(r,q)+2)!}\right. & \\ +\displaystyle\sum_{g,d:g\neq j,d\neq j,g\neq d}\dfrac{(n_j(r,q)+n_g(r,q)+n_d(r,q)-1)!}{(n_j(r,q)+n_g(r,q)+n_d(r,q)+2)!} & \\ \left.+\ldots+(-1)^l\dfrac{\left(\displaystyle\sum_{g=1}^{l-1}n_g(r,q)-1\right)!}{\left(\displaystyle\sum_{g=1}^{l-1}n_g(r,q)+2\right)!}\right) & \text{for}\quad i\neq r,\quad i\neq q,\quad i\in e_j(r,q),\\ \dfrac{1}{2}-\displaystyle\sum_{g=1}^{l-1}\dfrac{(n_g(r,q))!}{(n_g(r,q)+2)!}+\displaystyle\sum_{g,d:g\neq d}\dfrac{(n_g(r,q)+n_d(r,q))!}{(n_g(r,q)+n_d(r,q)+2)!}+\ldots & \\ +(-1)^{l+1}\dfrac{\left(\displaystyle\sum_{g=1}^{l-1}n_g(r,q)\right)!}{\left(\displaystyle\sum_{g=1}^{l-1}n_g(r,q)+2\right)!} & \text{for}\quad i=r\quad\text{or}\quad i=q.\end{cases}\tag{3.3}$$

Proof. Consider arbitrary players $\{r,q\}\in N$. Let $\delta^{d(r,q)-1}\int_{t_0}^{T}\bar{h}_{r,q}(\tau)d\tau$ be the payoff received by the maximal coalition when these players interact. Note that the term $\xi\delta^{d(r,q)-1}\int_{t_0}^{T}\bar{h}_{r,q}(\tau)d\tau$, where ξ is a coefficient taking values from the interval (0, 1), can appear in the expression for the ith component of the Shapley value if there exists a permutation π such that the coalition $S_{\pi,\pi(i)}$ loses a value of $\xi\delta^{d(r,q)-1}\int_{t_0}^{T}\bar{h}_{r,q}(\tau)d\tau$ in its payoff if player i leaves $S_{\pi,\pi(i)}$. In other words, a path between players r and q exists in the subhypergraph ($S_{\pi,\pi(i)}$, $E_{S_{\pi,\pi(i)}}$), but does not in the subhypergraph ($S_{\pi,\pi(i)-1}$,

$E_{S_{\pi,\pi(i)-1}}$). Thus, according to the Shapley value, $\delta^{d(r,q)-1}\int_{t_0}^{T}\overline{h}_{r,q}(\tau)d\tau$ must be divided only among players of the set $P(r,q)$.

Let $\phi_i(r,q)$ denote the value of the coefficient determining the share of $\delta^{d(r,q)-1}\int_{t_0}^{T}\overline{h}_{r,q}(\tau)d\tau$ received by player $i \in P(r,q)$.

Let us find $\phi_i(r,q)$ for an arbitrary player $i \in P(r,q)$. First, we assume that $i \neq r$ and $i \neq q$. Let $i \in e_j(r,q)$ for some $j = 1,\ldots,l-1$. For a path between players r and q to exist in the subhypergraph $(S_{\pi,\pi(i)}, E_{S_{\pi,\pi(i)}})$, but not in the subhypergraph $(S_{\pi,\pi(i)-1}, E_{S_{\pi,\pi(i)-1}})$, the permutation π must satisfy the following conditions:

1. $\pi(r) < \pi(i)$, $\pi(q) < \pi(i)$;
2. for all $k \in e_j(p,q), k \neq i$: $\pi(k) > \pi(i)$;
3. for all $e_s(p,q), s = \overline{1, l-1}, s \neq j$, there is a player $i_s \in e_s(p,q)$ such that $\pi(i_s) < \pi(i)$.

Note that $\phi_i(r,q)$ can be interpreted as the probability of a permutation satisfying the above conditions. Let us find this probability.

We define the following events:

$A_j = \{$Permutation satisfies conditions 1, 2$\}$;

$B = \{$Permutation satisfies conditions 3$\}$;

$\overline{B}$ is the opposite of an event B.

For all $g = \overline{1, l-1}, g \neq j$: $A_g = \{$Permutation satisfies the condition: $\forall k \in e_g(p,q) : \pi(k) > \pi(i)\}$. Note that $\overline{B} = \bigcup_{\substack{g=1\\ g\neq j}}^{l-1} A_g$.

Then

$$\phi_i(p,q) = P(A_j \cap B)$$

$$= P(A_j) - P(A_j \cap \overline{B}) = P(A_j) - P\left(\bigcup_{\substack{g=1\\ g\neq j}}^{l-1} A_j \cap A_g\right)$$

$$= P(A_j) - \sum_{\substack{g=1\\ g\neq j}}^{l-1} P(A_j \cap A_g)$$

$$+ \ldots + (-1)^l P(A_1 \cap A_2 \cap \ldots \cap A_{l-1})$$

$$= 2\left(\frac{(n_j(r,q)-1)!}{(n_j(r,q)+2)!} - \sum_{g:g\neq j}\frac{(n_j(r,q)+n_g(r,q)-1)!}{(n_j(r,q)+n_g(r,q)+2)!}\right.$$

$$+ \sum_{g,d:g\neq j,d\neq j,g\neq d}\frac{(n_j(r,q)+n_g(r,q)+n_d(r,q)-1)!}{(n_j(r,q)+n_g(r,q)+n_d(r,q)+2)!}$$

$$\left. + \ldots + (-1)^l \frac{\left(\sum_{g=1}^{l-1} n_g(r,q) - 1\right)!}{\left(\sum_{g=1}^{l-1} n_g(r,q) + 2\right)!}\right).$$

For players r and q, we obtain the following coefficients:

$$\phi_r(r,q) = \phi_q(r,q) = \frac{1}{2}\left(1 - \sum_{\substack{i\in P(r,q)\\ i\neq r, i\neq q}} \phi_i(r,q)\right)$$

$$= \frac{1}{2} - \sum_{g=1}^{l-1}\frac{(n_g(r,q))!}{(n_g(r,q)+2)!}$$

$$+ \sum_{g,d:g\neq d}\frac{(n_g(r,q)+n_d(r,q))!}{(n_g(r,q)+n_d(r,q)+2)!} + \ldots$$

$$+ (-1)^{l+1}\frac{\left(\sum_{g=1}^{l-1} n_g(r,q)\right)!}{\left(\sum_{g=1}^{l-1} n_g(r,q) + 2\right)!}.$$

Since all pairs of players $\{r,q\}$ are such that $i \in P(r,q)$, taking into account that $\int_{t_0}^{T}\overline{h}_i(\tau)d\tau$ is the payoff received by player i irrespective of the other players, we derive formula (3.3). The theorem is proved.

Corollary 3.1. *For a pair of players* i_1, i_2 *such that* $i_1 \in e_k$ *and* $i_2 \in e_k$ *for some* $k = \overline{1,m}$ (*i.e., they belong to the same hyperedge*), *it is true* $\phi_{i_1}(i_1,i_2) = \phi_{i_2}(i_1,i_2) = \frac{1}{2}$.

Corollary 3.2. *Let* $d(r,q) = l$ *and* $i \in e_j(r,q)$ *for some* $j = \overline{1, l-1}$. *If* $n_j(r,q) = 1$, *then* $\phi_i(r,q) = \phi_r(r,q) = \phi_q(r,q)$.

If $\{r,q\}$ *a pair of players such that* $n_j(r,q) = 1$ *for all* $j = \overline{1, l-1}$, *then* $\phi_i(r,q) = \frac{1}{d(r,q)+1}$ *for all* $i \in P(r,q)$.

This particular case of the problem under consideration corresponds to the formula for Shapley value calculation in a differential game on an acyclic graph (see [8], where it was proven that the payoff received by a pair of players must be divided equally among all players lying on the path between them).

An important property of cooperative solutions for dynamic games is dynamic stability. It was proved in [4] that the Shapley value in differential games on networks with a characteristic function constructed in a similar manner is dynamically stable. Note that this property also holds for the considered class of games on hypergraphs. This means that the following formula is valid for any $t \in [t_0, T]$ and any $i \in N$:

$$Sh_i(x_0, T - t_0)$$

$$= \int_{t_0}^{t} \bar{h}_i(\tau)d\tau + \sum_{\{r,q\}: i \in P(r,q)} \delta^{d(r,q)-1}\phi_i(r,q)\int_{t_0}^{t} \bar{h}_{r,q}(\tau)d\tau$$

$$+ Sh_i(\bar{x}(t), T - t),$$

where $Sh_i(\bar{x}(t), T - t)$ is the ith component of the Shapley value in the subgame $\Gamma(\bar{x}(t), T - t)$ with the characteristic function constructed according to (3.2).

4. EXAMPLE

To illustrate the resulting solution, we consider an example.

Let us find $Sh_5(x_0, T - t_0)$ in the game on the hypergraph shown in Fig. 1. According to Corollary 3.1, we have

$$\phi_5(3,5) = \phi_5(4,5) = \phi_5(6,5)$$

$$= \phi_5(7,5) = \phi_5(8,5) = \phi_5(10,5) = \frac{1}{2}.$$

According to Corollary 3.2,

$$\phi_5(1,5) = \phi_5(2,5) = \frac{1}{3}.$$

For the other pairs of players, using formula (3.3), we obtain

$$\phi_5(9,5) = \frac{1}{2} - \frac{2!}{4!} = \frac{5}{12},$$

$$\phi_5(3,10) = \phi_5(7,10) = \phi_5(8,10) = 2\frac{(3-1)!}{(3+2)!} = \frac{1}{30},$$

$$\phi_5(9,10) = 2\left(\frac{(3-1)!}{(3+2)!} - \frac{(5-1)!}{(5+2)!}\right) = \frac{1}{42},$$

$$\phi_5(1,10) = \phi_5(2,10)$$

$$= 2\left(\frac{(3-1)!}{(3+2)!} - \frac{(4-1)!}{(4+2)!}\right) = \frac{1}{60}.$$

Then

$$Sh_5 = \int_{t_0}^{T} \bar{h}_5(\tau)d\tau + \frac{\delta^0}{2}\left(\int_{t_0}^{T} \bar{h}_{3,5}(\tau)d\tau + \int_{t_0}^{T} \bar{h}_{4,5}(\tau)d\tau + \int_{t_0}^{T} \bar{h}_{6,7}(\tau)d\tau\right.$$

$$\left. + \int_{t_0}^{T} \bar{h}_{7,5}(\tau)d\tau + \int_{t_0}^{T} \bar{h}_{8,5}(\tau)d\tau + \int_{t_0}^{T} \bar{h}_{10,5}(\tau)d\tau\right) + \delta^1\left(\frac{1}{3}\int_{t_0}^{T} \bar{h}_{1,5}(\tau)d\tau\right.$$

$$+ \frac{1}{3}\int_{t_0}^{T} \bar{h}_{2,5}(\tau)d\tau + \frac{5}{12}\int_{t_0}^{T} \bar{h}_{9,5}(\tau)d\tau$$

$$+ \frac{1}{30}\int_{t_0}^{T} \bar{h}_{8,10}(\tau)d\tau + \frac{1}{30}\int_{t_0}^{T} \bar{h}_{3,10}(\tau)d\tau$$

$$\left. + \frac{1}{30}\int_{t_0}^{T} \bar{h}_{7,10}(\tau)d\tau\right)$$

$$+ \delta^2\left(\frac{1}{60}\int_{t_0}^{T} \bar{h}_{1,10}(\tau)d\tau + \frac{1}{60}\int_{t_0}^{T} \bar{h}_{2,10}(\tau)d\tau + \frac{1}{42}\int_{t_0}^{T} \bar{h}_{9,10}(\tau)d\tau\right).$$

5. CONCLUSIONS

Cooperative differential games on a hypergraph have been considered. A special characteristic function that takes into account the game's network structure was used to construct a cooperative solution The Shapley value was used as a cooperative optimality principle. In contrast to previously considered graph games, in this class of games, the exit of a single player from a coalition can be insufficient for breaking a path between a pair of players; instead, the exit of a group of players is required. Of course, this complicates the calculation of the Shapley value. For the case where the original hypergraph has a cycle-free edge graph, we derived a formula for explicitly calculating this imputation. This formula significantly simplifies Shapley value calculation, since there is no need to calculate the characteristic function for each coalition. The resulting solutions were illustrated by an example.

Note that the mathematical methods for analyzing games on hypergraphs developed in this paper enrich the researcher's toolkit based on network models for analyzing relationships in social groups and for studying religious, ethnic, or political movements.

FUNDING

This work was supported by the Russian Science Foundation, project no. 24-18-00479, https://rscf.ru/project/24-18-00479/, St. Petersburg State University.

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. I. Tantlevskij, D. Kuzyutin, N. Smirnova, and E. I. Tantlevskaya, "Models of mutual influence of Jewish sects and ideological struggle in Judea in the 2nd–1st centuries BC," Vestn. Russ. Khrist. Gumanit. Akad. **24** (4), 136–148 (2023).

2. L. Petrosyan and D. Yeung, “Shapley value for differential network games: Theory and application,” J. Dyn. Games **8** (2), 151–166 (2020).
3. L. A. Petrosyan and D. Yeung, “Construction of dynamically stable solutions in differential network games,” in *Stability, Control, and Differential Games*, Ed. by A. Tarasyev, V. Maksimov, and T. Filippova (Springer, Cham, 2020).
4. L. Petrosyan, D. W. K. Yeung, and Y. Pankratova, “Cooperative differential games with partner sets on networks,” Tr. Inst. Mat. Mekh. Ural. Otd. Ross. Akad. Nauk **27** (3), 286–295 (2021).
5. L. S. Shapley, “A value for *n*-person games,” in *Contributions to the Theory of Games*, Vol. 2: *Annals of Mathematics Studies* (Princeton Univ. Press, Princeton, NJ, 1953), pp. 307–317.
6. I. Tantlevskij, D. Kuzyutin, and N. Smirnova, “A signed network model of the interaction between religious movements and authority in Judea,” in *Modeling and Simulation of Social-Behavioral Phenomena in Creative Societies, MSBC 2024*, Ed. by N. Agarwal, L. Sakalauskas, and U. Tukeyev, *Communications in Computer and Information Science* (2024). Vol. 2211.
7. A. Tur and L. Petrosyan, “The core of cooperative differential games on networks,” in *Mathematical Optimization Theory and Operations Research, MOTOR 2022*, Ed. by P. Pardalos, M. Khachay, and V. Mazalov, *Lecture Notes in Computer Science* (Springer, Cham, 2022). Vol. 13367.
8. A. Tur and L. Petrosyan, “Strong time-consistent solution for cooperative differential games with network structure,” Mathematics **9**, 755 (2021).
9. J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behavior* (Princeton Univ. Press, Princeton, 1953).

Translated by I. Ruzanova

Publisher’s Note. Pleiades Publishing remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
AI tools may have been used in the translation or editing of this article.