Dimensional transmutation and nonconventional scaling behaviour in a model of self-organized criticality

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Stochastic models of non-equilibrium critical behaviour

Stochastic PDE's for a scalar field h:

$$h = h(x), \quad x = \{t, \mathbf{x}\}, \quad \mathbf{x} = \{x_i\}, \quad i = 1, \dots d.$$

Kardar–Parisi–Zhang model (1986, in fact 1976):

$$\partial_t h = \varkappa_0 \, \partial^2 h + (\lambda_0/2)(\partial h)^2 + \eta, \tag{1}$$

Diaz-Guilera–Pavlik model (1992):

$$\partial_t h = \varkappa_0 \, \partial^2 h + (\lambda_0/2) \, \partial^2 h^2 + \eta, \tag{2}$$

where

$$\partial_t = \partial/\partial t, \quad \partial_i = \partial/\partial x_i, \quad \partial^2 = \partial_i \partial_i.$$

Hwa–Kardar model (1989):

$$\partial_t h = \varkappa_{\perp 0}\, \partial_\perp^2 h + \varkappa_{\| 0}\, \partial_\|^2 h - \left(\lambda_0/2\right) \partial_\| h^2 + \eta. \tag{3}$$

Pastor-Satorras–Rothman model (1998):

$$\partial_t h = \varkappa_{\perp 0}\, \partial_\perp^2 h + \varkappa_{\| 0}\, \partial_\|^2 h + \left(\lambda_0/3!\right) \partial_\|^2 h^3 + \eta, \tag{4}$$

where $\mathbf{n}$ defines a certain distinguished direction and

$$\mathbf{x} = \mathbf{x}_\perp + \mathbf{n} x_\|, \quad x_\| = \mathbf{n}\mathbf{x}, \quad \mathbf{n}\mathbf{x}_\perp = 0$$

for coordinates and

$$\partial = \partial_\perp + \mathbf{n}\partial_\|, \quad \partial_\| = \mathbf{n}\partial, \quad \mathbf{n}\partial_\perp = 0$$

for gradients.

Random Gaussian noise with pair correlator

Ordinary white noise:

$$\langle \eta(x)\eta(x')\rangle = C_0\,\delta(t-t')\,\delta^{(d)}(\mathbf{x}-\mathbf{x}'), \quad C_0 > 0, \tag{5}$$

Conservation case:

$$\langle \eta(x)\eta(x')\rangle = -C_0\,\delta(t-t')\,\partial^2\delta^{(d)}(\mathbf{x}-\mathbf{x}'), \tag{6}$$

Static noise:

$$\langle \eta(x)\eta(x')\rangle = C_0\,\delta^{(d)}(\mathbf{x}-\mathbf{x}'), \tag{7}$$

Quenched noise:

$$\langle \eta(x)\eta(x')\rangle = \Delta(h-h')\,\delta^{(d)}(\mathbf{x}-\mathbf{x}'), \quad \text{e.g.} \quad \Delta(h-h') = \delta(h-h'). \tag{8}$$

Quantities of interest and expected IR behaviour

Structure functions, isotropic cases:

$$\mathcal{S}_n(t,r) = \langle [h(t,\mathbf{x}) - h(0,0)]^n \rangle \simeq r^{-n\Delta_h} F\left(tr^{-\Delta_\omega}\right), \quad r = |\mathbf{x}| \tag{9}$$

Strongly anisotropic cases:

$$\mathcal{S}_n(t,r_\perp,r_\parallel) \simeq r_\perp^{-n\Delta_h} F\left(tr_\perp^{-\Delta_\omega}, r_\parallel r_\perp^{-\Delta_\parallel}\right), \quad r_\perp = |\mathbf{x}_\perp|, \quad r_\parallel = |x_\parallel| \tag{10}$$

with $\Delta_\parallel \neq 1$.

Standard notation:

$$\chi = -\Delta_h, \quad z = \Delta_\omega, \quad \zeta = 1/\Delta_\parallel.$$

Moving medium or interaction with environment

Lagrangean (Galilean covariant) derivative:

$$\partial_t h \to \nabla_t h = \partial_t + v_i \partial_i h \quad \text{or} \quad \nabla_t h = \partial_t + \partial_i (v_i h)$$

with a certain velocity field $v_i(x) = v_i(t, \mathbf{x})$.

Kazantsev–Kraichnan Gaussian ensemble:

$$\langle v_i(x) v_j(x') \rangle = D_0 \, \delta(t - t') \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^{d+\xi}} \left\{ P_{ij}^{\perp}(\mathbf{k}) + a P_{ij}^{\|}(\mathbf{k}) \right\} \exp\{ i\mathbf{k}(\mathbf{x} - \mathbf{x}') \}. \tag{11}$$

Kolmogorov scaling: $\xi = 4/3$, Batchelor's limit: $\xi \to 2$.

Stirred Navier–Stokes equation for incompressible fluid:

$$\nabla_t v_i = \upsilon_0 \partial^2 v_i - \partial_i \wp + f_i, \qquad (12)$$

pressure $\wp(v)$ makes the dynamics consistent with incompressibility condition $\partial_i v_i = 0$.

The correlator of the Gaussian stirring force:

$$\langle f_i(x) f_j(x') \rangle = \delta(t - t') \int \frac{d\mathbf{k}}{(2\pi)^d} D(k)\, P^{\perp}_{ij}(\mathbf{k}) \exp\{\mathrm{i}\mathbf{k}(\mathbf{x} - \mathbf{x}')\} \qquad (13)$$

where, e.g.

$$D(k) = D_0\, k^2 \quad \text{or} \quad D(k) = D_0 \quad \text{or} \quad D(k) = D_0\, k^{4-d-y}\,.$$

Note that

$$\lim_{y \to 4} D_0\, k^{4-d-y} \sim W\, \delta^{(d)}(\mathbf{k}) \quad \text{with} \quad D_0 \sim (4 - y)\, W.$$

HK model: Field theoretic treatment

$$\partial_t h = \varkappa_{\perp 0}\, \partial_\perp^2 h + \varkappa_{\|0}\, \partial_\|^2 h - (\lambda_0/2)\, \partial_\| h^2 + \eta, \tag{14}$$

$$\langle \eta(x)\eta(x') \rangle = C_0\, \delta(t-t')\, \delta^{(d)}(\mathbf{x}-\mathbf{x}'), \quad C_0 > 0, \tag{15}$$

Actual expansion parameter is $\lambda_0^2 C_0$ so we set $\lambda_0 = 1$.

The De Dominicis–Janssen action functional:

$$S(\Phi) = \frac{1}{2} h' C_0\, h' + h' \left\{ -\partial_t h + \varkappa_{\|0}\, \partial_\|^2 h + \varkappa_{\perp 0}\, \partial_\perp^2 h - \frac{1}{2}\, \partial_\| h^2 \right\}. \tag{16}$$

Integration over $x = \{t, \mathbf{x}\}$ implied, e.g.,

$$\frac{1}{2}\, h' C_0\, h' = \frac{1}{2}\, C_0 \int dt \int d\mathbf{x}\, (h')^2(t, \mathbf{x}). \tag{17}$$

Three independent canonical dimensions:

$$[F] \sim [T]^{-d_F^{\omega}}[L_{||}]^{-d_F^{||}}[L_{\perp}]^{-d_F^{\perp}}\,. \tag{18}$$

Then

$$\begin{aligned} d_{x_{||}}^{||} &= -1, \quad d_{x_{\perp}}^{\perp} = -1, \quad d_t^{\omega} = -1, \quad d_{x_{\perp}}^{||} = d_{x_{||}}^{\perp} = 0, \\ d_{x_{||}}^{\omega} &= d_{x_{\perp}}^{\omega} = 0, \quad d_{\omega}^{||} = d_{\omega}^{\perp} = 0, \end{aligned} \tag{19}$$

the others are given in the table.

Convenient coupling constant

$$g_0 = C_0 \varkappa_{\perp 0}^{-3/2} \varkappa_{||0}^{-3/2} \quad \text{so that} \quad g_0 \sim \Lambda^{\varepsilon}.$$

Model is logarithmic in $d = 4$, UV divergences = poles in $\varepsilon = 4 - d$.

Canonical dimensions of the fields and the parameters

F	h'	h	C_0	$\varkappa_{\|\|0}$	$\varkappa_{\perp 0}$	u_0	g_0	g	μ, Λ
d_F^{ω}	-1	1	3	1	1	0	0	0	0
$d_F^{\|\|}$	2	-1	-3	-2	0	-2	0	0	0
$d_F^{\perp}$	$d-1$	0	$1-d$	0	-2	2	ε	0	1
$d_F^k = d_F^{\perp} + d_F^{\|\|}$	$d+1$	-1	$-d-2$	-2	-2	0	ε	0	1
$d_F = d_F^k + 2d_F^{\omega}$	$d-1$	1	$4-d$	0	0	0	ε	0	1

Here

- $\varepsilon = 4 - d$;
- $u_0 = \varkappa_{\|\|0}/\varkappa_{\perp 0}$;
- the renormalization mass μ is an additional arbitrary parameter in the MS scheme.

Renormalization

$$S_R(\Phi) = \frac{1}{2} h' D\, h' + h' \left(-\partial_t h + Z_{\varkappa_\|} \varkappa_\| \partial_\|^2 h + \varkappa_\perp \partial_\perp^2 h - \partial_\| h^2/2 \right). \quad (20)$$

$$\varkappa_{\|0} = \varkappa_\| Z_{\varkappa_\|}, \quad C_0 = C = g\mu^\varepsilon \varkappa_\|^{3/2} \varkappa_\perp^{3/2}, \quad g_0 = g\mu^\varepsilon Z_g, \quad Z_g = Z_{\varkappa_\|}^{-3/2}, \quad (21)$$

Symmetries of the model: combined reflection $x_\| \to -x_\|$,

$$h'(t, \{x_\|, \mathbf{x}_\perp\}) \to -h'(t, \{-x_\|, \mathbf{x}_\perp\}), \quad h(t, \{x_\|, \mathbf{x}_\perp\}) \to -h(t, \{-x_\|, \mathbf{x}_\perp\}) \quad (22)$$

and a tilt symmetry = the Galilean symmetry in the preferred direction $\mathbf{n}$: $\mathbf{x} \to \mathbf{x} + Ut\,\mathbf{n}$,

$$h'(t, \mathbf{x}) \to h'(t, \mathbf{x} + Ut\,\mathbf{n}), \quad h(t, \mathbf{x}) \to h(t, \mathbf{x} + Ut\,\mathbf{n}) - U, \quad U = \text{const.} \quad (23)$$

Canonical scale invariance of a certain renormalized Green function $G = \langle \Phi \dots \Phi \rangle$ is expressed by three differential equations:

$$\left(\sum_i d_i^{\omega} \mathcal{D}_i - d_G^{\omega} \right) G = 0,$$
$$\left(\sum_i d_i^{\perp} \mathcal{D}_i - d_G^{\perp} \right) G = 0, \qquad (24)$$
$$\left(\sum_i d_i^{\parallel} \mathcal{D}_i - d_G^{\parallel} \right) G = 0.$$

Here i is the full set of the arguments of G, namely, t, $r_\perp$, $r_\parallel$, $\varkappa_\perp$, $\varkappa_\parallel$ and μ.

The operator $\mathcal{D}_a$ is defined as $\mathcal{D}_a = a\partial_a$ for any a.

The RG equation:

$$\left(\mathcal{D}_\mu + \beta_g \partial_g - \gamma_{\varkappa_\parallel} \mathcal{D}_{\varkappa_\parallel} - \gamma_{\varkappa_\perp} \mathcal{D}_{\varkappa_\perp} - \gamma_G\right) G = 0. \tag{25}$$

The anomalous dimensions γ and β functions (referred to as RG functions) are:

$$\gamma_F = \widetilde{\mathcal{D}}_\mu \ln Z_F, \quad \beta_g = \widetilde{\mathcal{D}}_\mu g \tag{26}$$

for any quantity F and any coupling constant g, where $\widetilde{\mathcal{D}}_\mu$ is $\mathcal{D}_\mu$ at fixed bare parameters.

The one-loop β function:

$$\beta_g = g\left(-\varepsilon + 9g/32\right), \quad g^* = 32\varepsilon/9 > 0, \quad \beta'(g^*) = \varepsilon > 0. \tag{27}$$

At the fixed point $g = g^*$ the RG equation becomes:

$$\left(\mathcal{D}_\mu - \gamma^*_{\varkappa_\parallel}\mathcal{D}_{\varkappa_\parallel} - \gamma^*_{\varkappa_\perp}\mathcal{D}_{\varkappa_\perp} - \gamma^*_G\right) G = 0. \tag{28}$$

Combining with (24) excluding "IR irrelevant"parameters μ and $\varkappa$ gives the equation of IR scaling:

$$\left(-\mathcal{D}_{r_\perp} - \Delta_\parallel \mathcal{D}_{r_\parallel} - \Delta_\omega \mathcal{D}_t - \Delta_G\right) G = 0. \tag{29}$$

The solution reads:

$$G(t, r_\perp, r_\parallel) \simeq r_\perp^{-\Delta_G} F\left(t r_\perp^{-\Delta_\omega}, r_\parallel r_\perp^{-\Delta_\parallel}\right). \tag{30}$$

For a general quantity G:

$$\Delta_G = d_G^{\perp} + \Delta_{\parallel} d_G^{\parallel} + \Delta_{\omega} d_G^{\omega} + \gamma_G^*,$$

where

$$\Delta_{\omega} = 2 - \gamma^*_{\varkappa_\perp}, \quad \Delta_{\parallel} = 1 - \gamma^*_{\varkappa_\parallel}\,.$$

The critical dimensions are known exactly:

$$\Delta_h = 1 - \varepsilon/3, \quad \Delta_{h'} = 3 - \varepsilon/3, \quad \Delta_{\omega} = 2, \quad \Delta_{\parallel} = 1 + \varepsilon/3. \quad (31)$$

Full model with moving medium

Consider the system:

$$\nabla_t h = \varkappa_{\perp 0}\, \partial_\perp^2 h + \varkappa_{\| 0}\, \partial_\|^2 h - \partial_\| h^2/2 + \eta, \tag{32}$$

$$\langle \eta(x)\eta(x') \rangle = C_0\, \delta(t-t')\, \delta^{(d)}(\mathbf{x}-\mathbf{x}'), \tag{33}$$

$$\langle v_i(x) v_j(x') \rangle = \delta(t-t')\, D_{ij}(\mathbf{x}-\mathbf{x}'), \tag{34}$$

$$D_{ij}(\mathbf{x}-\mathbf{x}') = D_0 \int \frac{d\mathbf{k}}{(2\pi)^d} \frac{1}{k^{d+\xi}}\, P_{ij}^\perp(\mathbf{k})\, \exp\{i\mathbf{k}(\mathbf{x}-\mathbf{x}')\}. \tag{35}$$

The field theoretic action:

$$S(\Phi) = \frac{1}{2} h' C_0\, h' + h' \Big(-\nabla_t h + \varkappa_{\| 0} \partial_\|^2 h + \varkappa_{\perp 0} \partial_\perp^2 h - \partial_\| h^2/2 \Big) + S_{\mathbf{v}}, \tag{36}$$

with an averaging over Gaussian distribution of $v_i(x)$:

$$S_{\mathbf{v}} = -\frac{1}{2}\int dt \int d\mathbf{x} \int d\mathbf{x}' \quad v_i(t,\mathbf{x}) D_{ij}^{-1}(\mathbf{x}-\mathbf{x}') v_j(t,\mathbf{x}'). \tag{37}$$

The symmetries are augmented by

$$\mathbf{v}(t,\{x_{\|},\mathbf{x}_{\perp}\}) \to -\mathbf{v}(t,\{-x_{\|},\mathbf{x}_{\perp}\}), \qquad \mathbf{v}(t,\mathbf{x}) \to \mathbf{v}(t,\mathbf{x}+Ut\mathbf{n}).$$

Another realization of the Galilean symmetry becomes possible:

$$\mathbf{x} \to \mathbf{x}+Ut\mathbf{n}, \quad h'(t,\mathbf{x}) \to h'(t,\mathbf{x}+Ut\mathbf{n}), \quad h(t,\mathbf{x}) \to h(t,\mathbf{x}+Ut\mathbf{n}),$$
$$\mathbf{v}(t,\mathbf{x}) \to \mathbf{v}(t,\mathbf{x}+Ut\mathbf{n}) - U\mathbf{n}, \quad U = \text{const.} \tag{38}$$

Full model: Canonical dimensions

$$[F] \sim [T]^{-d_F^\omega}[L]^{-d_F^k} \qquad (39)$$

F	h'	h	C_0	$\varkappa_{\|\|0}$	$\varkappa_{\perp 0}$	g_0	v	D_0	x_0	u_0,u,g,x
d_F^ω	-1	1	3	1	1	0	1	0	0	0
d_F^k	$d+1$	-1	$-2-d$	-2	-2	ε	-1	$\xi-2$	ξ	0
d_F	$d-1$	1	$4-d$	0	ε	1	ξ	ξ	0	0

Canonical dimensions of the fields and the parameters in the full model; $\varepsilon = 4-d$; $d_F = d_F^k + 2d_F^\omega$.

Coupling constants:

$$g_0 = C_0 \varkappa_{\perp 0}^{-3/2} \varkappa_{\| 0}^{-3/2}, \quad x_0 = D_0 \varkappa_{\perp 0}^{-1}, \quad u_0 = \varkappa_{\| 0} / \varkappa_{\perp 0}$$

$$\text{so that} \quad g_0 \sim \Lambda^{\varepsilon}, \quad x_0 \sim \Lambda^{\xi}.$$

Logarithmic at $\varepsilon = \xi = 0$, double expansion in ε and ξ.

Full model: Renormalization

$$S_R(\Phi) = \frac{1}{2}h' D\, h' + h'\left(-\nabla_t h + Z_{\varkappa_\parallel}\varkappa_\parallel \partial_\parallel^2 h + Z_{\varkappa_\perp}\varkappa_\perp \partial_\perp^2 h - \partial_\parallel h^2/2\right) + S_{\mathbf{v}}. \tag{40}$$

$$\begin{gathered}\varkappa_{\parallel 0} = \varkappa_\parallel Z_{\varkappa_\parallel}, \quad \varkappa_{\perp 0} = \varkappa_\perp Z_{\varkappa_\perp}, \\ g_0 = Z_g g \mu^{\varepsilon}, \quad x_0 = Z_x x \mu^{\xi}, \quad u_0 = Z_u u. \end{gathered} \tag{41}$$

$$D = \varkappa_\parallel^{3/2} \varkappa_\perp^{3/2} g \mu^{\varepsilon}, \quad B = x \varkappa_\perp \mu^{\xi}. \tag{42}$$

The renormalization constants are related to each other as follows:

$$Z_g = Z_{\varkappa_\parallel}^{-3/2} Z_{\varkappa_\perp}^{-3/2}, \quad Z_x = Z_{\varkappa_\perp}^{-1}, \quad Z_u = Z_{\varkappa_\parallel} Z_{\varkappa_\perp}^{-1}. \tag{43}$$

One-loop β functions:

$$\begin{aligned}
\beta_g &= g\left(-\varepsilon + \frac{9}{32}g + \frac{9}{16}\frac{x}{u} + \frac{9}{16}x\right), \\
\beta_x &= x\left(-\xi + \frac{3}{8}x\right), \\
\beta_u &= u\left(-\frac{3}{16}g - \frac{3}{8}\frac{x}{u} + \frac{3}{8}x\right).
\end{aligned} \tag{44}$$

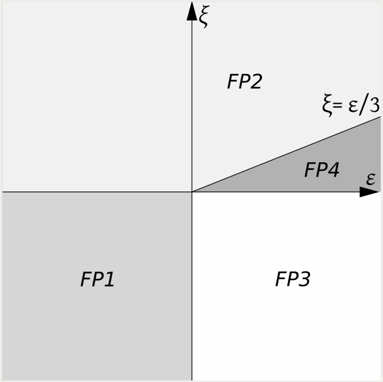

Рис.: Regions of stability of the fixed points in the full model. Different areas correspond to the values of the parameters for which points FP1–FP4 are IR attractive. The region FP1 corresponds to the mean field theory while the regions FP2, FP3, and FP4 correspond to the regimes where only the turbulent advection is relevant, only the nonlinearity of the HK equation is relevant, both the advection and the nonlinearity are relevant, respectively.

IR scaling behaviour

Canonical scale invariance:

$$\left(\sum_i d_i^\omega \mathcal{D}_i - d_G^\omega\right) G = 0,$$
$$\left(\sum_i d_i^k \mathcal{D}_i - d_G^k\right) G = 0. \tag{45}$$

The differential RG equation:

$$\left(\mathcal{D}_\mu + \beta_g \partial_g + \beta_x \partial_x + \beta_u \partial_u - \gamma_{\upsilon_\perp} \mathcal{D}_{\upsilon_\perp} - \gamma_G\right) G = 0. \tag{46}$$

At the fixed points $\beta_i = 0$ for all $i = g$, x, u.

Excluding IR irrelevant parameters μ and $\varkappa$ gives:

$$\left(-\mathcal{D}_r - \Delta_\omega \mathcal{D}_t - \Delta_G\right) G = 0, \tag{47}$$

where $\Delta_\omega = 2 - \gamma^*_{\nu_\perp}$.

For a general quantity G:

$$\Delta_G = d^k_G + \Delta_\omega d^\omega_G + \gamma^*_G.$$

For example, for FPII:

$$\Delta_h = 1 - \xi, \quad \Delta_v = 1 - \xi, \quad \Delta_{h'} = 3 - \varepsilon + \xi, \quad \Delta_\omega = 2 - \xi. \tag{48}$$

Restoration/Emergence of anisotropic scaling: point FP3

FP3 (and FP4) correspond to $u \to u^* = \infty$, so we pass to new charge $a = 1/u$ with

$$\beta_a = \widetilde{\mathcal{D}}_\mu a = a\left(-\frac{3}{8}x + \frac{3}{8}xa + \frac{3}{16}g\right).$$

The differential RG equation:

$$\left(\mathcal{D}_\mu + \beta_g \partial_g + \beta_x \partial_x + \beta_a \partial_a - \gamma_{\upsilon_\perp} \mathcal{D}_{\upsilon_\perp} - \gamma_G\right) G = 0. \qquad (49)$$

At the fixed point we set $\beta_g = \beta_x = 0$ but retain

$$\beta_a \partial_a \simeq \ \Omega_a \mathcal{D}_a, \quad \Omega_a = \partial\beta_a/\partial a \quad \text{at FP3}$$

The IR scaling equation becomes:

$$\left(-\mathcal{D}_{r_\parallel} - \mathcal{D}_{r_\perp} - \Delta_\omega \mathcal{D}_t - \beta_a \mathcal{D}_a - d_G^k - \Delta_\omega d_G^\omega - \gamma_G^*\right) G = 0, \qquad (50)$$

For FP3 this gives:

$$\left(-\mathcal{D}_{r_\parallel} - \mathcal{D}_{r_\perp} - 2\mathcal{D}_t - \frac{2\varepsilon}{3}\mathcal{D}_a - d_G^k - 2d_G^\omega - \gamma_G^*\right) G = 0. \qquad (51)$$

We recall two canonical scale equations:

$$\begin{aligned} \left(\sum_i d_i^\omega \mathcal{D}_i - d_G^\omega\right) G = 0, \\ \left(\sum_i d_i^k \mathcal{D}_i - d_G^k\right) G = 0. \end{aligned} \qquad (52)$$

Solutions to (homogeneous analog of) set of equations (51) and (52):

$$z_1 = \frac{r_\perp^2}{t\upsilon_\perp}, \quad z_2 = \frac{r_\perp}{r_\parallel} \quad \text{and} \quad z_3 = a\,(\mu r_\perp)^{-2\varepsilon/3}\,. \tag{53}$$

To obey canonical symmetries of pure HK, choose two of them: z_1 and $z_0 = z_2 z_3^{-1/2}$:

$$z_0 = \frac{r_\perp^{\Delta_\parallel}}{r_\parallel}\,\mu^{\varepsilon/3} a^{-1/2} \quad \text{with} \quad \Delta_\parallel = 1 + \varepsilon/3. \tag{54}$$

For fixed IR irrelevant parameters μ and a we arrive at strongly anisotropic scaling:

$$G(t, r_\perp, r_\parallel) \simeq r_\perp^{-\Delta_G} F\left(t r_\perp^{-\Delta_\omega}, r_\parallel r_\perp^{-\Delta_\parallel}\right), \quad r_\perp = |\mathbf{x}_\perp|, \quad r_\parallel = |x_\parallel| \tag{55}$$

(we recall that $\Delta_\omega = 2$ for FP3.)

Thank you for attention, interest and patience!

Another example:

$$\partial_t h = \varkappa_0 \, \partial^2 h + (\lambda_0/2)(\partial h)^2 + \eta, \tag{56}$$

$$\langle \eta(x)\eta(x') \rangle = C_0 \, \delta^{(d)}(\mathbf{x} - \mathbf{x}'), \tag{57}$$

$$\nabla_t v_i = \nu_0 \partial^2 v_i - \partial_i \wp + f_i, \tag{58}$$

$$D(k) = D_0 + D_0' \, k^{4-d-y} \, .$$

The β functions in the MS scheme ($\varepsilon = 4 - d$ and y):

$$
\begin{aligned}
\beta_g &= -g\left(y - \frac{3}{8}(g + g')\right), \\
\beta_{g'} &= -g'\left(\varepsilon - \frac{3}{8}(g + g')\right), \\
\beta_w &= -w\left(\frac{\varepsilon}{2} + \frac{w^2}{2} - \frac{3a^2(g + g')}{8u(1 + u)}\right), \\
\beta_a &= -a\left(\frac{3(g + g')zw}{8u(1 + u)} - \frac{w^2}{4}\right), \\
\beta_u &= -u\left(\frac{3a^2(g + g')}{8u(1 + u)} - \frac{w^2}{4} - \frac{1}{8}(g + g')\right), \\
\beta_z &= -z\left(-\frac{\varepsilon}{2} + \frac{3(g + g')}{8u(1 + u)}(wz + 2a^2) - \frac{3}{4}w^2 - \frac{wa^2}{4z}\right).
\end{aligned}
\tag{59}
$$