ISSN 1990-4789, Journal of Applied and Industrial Mathematics, 2024, Vol. 18, No. 4, pp. 812–823. © Pleiades Publishing, Ltd., 2024.
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# Analysis of the Dynamics of Solutions for Hybrid Difference Lotka–Volterra Systems

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Received June 28, 2023; revised September 4, 2024; accepted November 6, 2024

**Abstract**—A difference system of the Lotka–Volterra type is considered. It is assumed that this system can operate both in some program and perturbed modes. The restrictions on the time of the system's stay in these modes, providing the desired dynamical behavior, are investigated. In particular, the conditions of the ultimate boundedness of solutions and the permanence of the system are obtained. The direct Lyapunov method is used, and different Lyapunov functions are constructed in different parts of the state space. The sizes of the domain of permissible initial values of solutions and the domain of the ultimate bound of solutions corresponding to the required dynamics of the system are estimated. Constraints are set on the size of the digitization step of the system.

Keywords: *difference Lotka–Volterra systems, switching, ultimate boundedness of solutions, permanence*

**DOI**: 10.1134/S199047892404015X

## INTRODUCTION

Lotka–Volterra type systems are widely used to model various interactions between entities in biological, chemical, or economic environments [1, 2]. These systems are most commonly applied to construct population models describing the dynamics of population sizes in biological communities. Both continuous and discrete models are employed in applied problems [3, 4]. Difference equations can be obtained by discretizing continuous equations using computational schemes or can be constructed directly without reference to continuous models. Since population dynamics is essentially a discrete process, the mathematical framework of difference equations often proves more suitable for describing such processes [4].

One important problem in population dynamics is the ultimate boundedness of solutions [1]. This involves studying conditions that guarantee the existence of a bounded region in the system state space such that any solution will enter this region in finite time and remain there thereafter. From a biological perspective, this means population sizes will not exceed certain specified values. In discrete models, such ultimate boundedness can often only be guaranteed for solutions starting in some finite neighborhood of the origin. However, since the size of this neighborhood can be varied, this does not pose fundamental problems for modeling real processes.

Another key problem is system persistence [1]. Persistence means that during biological interactions, populations do not go extinct, and their sizes do not fall below certain positive thresholds. A system exhibiting both ultimate boundedness and persistence is said to be permanent [1, 5, 6].

These problems have been extensively studied in recent decades for Lotka–Volterra systems with constant and variable coefficients [4–6], under random or nonrandom disturbances [7–9], with time delays or diffusion [10, 11], etc. Of particular practical interest is the study of systems with hybrid structure [12, 13], especially when system coefficients can switch between different sets of values due to external influences, changes in control schemes, etc. Many papers have been devoted to developing dynamic analysis methods for switched systems (see [14, 15]), Lyapunov's direct method being a standard tool. For example, switched Lotka–Volterra systems were examined in [16–18].

The present paper studies a hybrid discrete Lotka–Volterra type system that can operate either in a planned mode with an asymptotically stable equilibrium or in a perturbed mode where this equilibrium and stability properties are lost. The system dynamics is analyzed, sufficient conditions for ultimate boundedness and permanence are established, and estimates are obtained for admissible initial and limiting solution values corresponding to the studied dynamics. A distinctive feature of this paper is the partitioning of the system state space and the construction of different Lyapunov functions in different regions of this space. This approach yields improved results both qualitatively and quantitatively [19, 20].

## 1. STATEMENT OF THE PROBLEM

Consider the discrete system

$$x_i(k+1) = x_i(k)\exp\left(h\Big(c_i + \sum_{j=1}^{n} p_{ij} f_j(x_j(k))\Big)\right), \quad i = 1,\dots,n. \tag{1}$$

Equations (1) represent a difference analog of the well-known generalized Lotka–Volterra differential system [3, 4]. They are typically used to model interactions between multiple populations in a biological community. Here the variable $x_i(k)$ describes the size of the $i$th population at the $k$th iteration ($k = 0,1,\dots$; $i = 1,\dots,n$). The parameter $h > 0$ determines the discretization step. The constant coefficients $c_i$ and $p_{ij}$ characterize the natural growth/decline rates of populations, intraspecific competition, and interspecific interactions ($i,j = 1,\dots,n$). The functions $f_i(z_i)$ defined for $z_i \in [0,+\infty)$ ($i = 1,\dots,n$) are chosen to ensure that the modeling results are consistent with experimental observations.

Note that Eqs. (1) are also applied to model certain chemical and economic processes [1, 2].

In accordance with the physical meaning of the variables, we consider system (1) in the nonnegative orthant $K^+ = \{\mathbf{z} = (z_1,\dots,z_n)^T : z_i \geq 0, i = 1,\dots,n\}$. The interior of $K^+$ is denoted by $K_0^+ = \{\mathbf{z} = (z_1,\dots,z_n)^T : z_i > 0, i = 1,\dots,n\}$. Note that both $K^+$ and $K_0^+$ are invariant sets of system (1).

Following standard assumptions (see [1–3]), we consider that the functions $f_i(z_i)$ are continuous and strictly increasing for $z_i \in [0,+\infty)$, with $f_i(0) = 0$ and $f_i(z_i) \to +\infty$ as $z_i \to +\infty$ ($i = 1,\dots,n$).

Additionally, we assume that for any constant $H$ the functions $\tilde{f}_i(z_i) = f_i(e^{z_i})$ satisfy the Lipschitz condition on $(-\infty, H]$ with some Lipschitz constant $L(H) > 0$, and moreover, $\int_0^1 \frac{f_i(\tau)}{\tau}\,d\tau < +\infty$ ($i = 1,\dots,n$).

**Remark 1.** All the assumptions made are satisfied, for example, for power-law functions $f_i(z_i) = z_i^{\mu_i}$, where $\mu_i > 0$, $i = 1,\dots,n$. In particular, the case of $\mu_1 = \ldots = \mu_n = 1$ corresponds to the classical Lotka–Volterra difference system.

Let us denote $f(\mathbf{z}) = (f_1(z_1),\dots,f_n(z_n))^T$, $\mathbf{c} = (c_1,\dots,c_n)^T$, and $\mathbf{P} = (p_{ij})_{i,j=1}^n$.

We assume that $\det\mathbf{P} \neq 0$, $\mathbf{P}^{-1}\mathbf{c} < 0$ (component-wise), and there exist positive numbers $\lambda_1,\dots,\lambda_n$ such that the matrix $\Lambda\mathbf{P} + \mathbf{P}^T\Lambda$, where $\Lambda = \operatorname{diag}\{\lambda_1,\dots,\lambda_n\}$, is negative definite.

Under these assumptions, system (1) has a unique equilibrium position $\bar{\mathbf{x}} = (\bar{x}_1,\dots,\bar{x}_n)^T \in K_0^+$, and this equilibrium is asymptotically stable [1–4].

The dynamics of system (1) will be called the planned (programmed) regime. Furthermore, we assume that at certain moments, due to external influences, the system switches to the perturbed regime

$$x_i(k+1) = x_i(k)\exp\left(h\Big(\hat{c}_i(k) + \sum_{j=1}^{n} \hat{p}_{ij}(k) f_j\big(x_j(k)\big)\Big)\right), \quad i = 1,\dots,n. \tag{2}$$

Here the coefficients $\hat{c}_i(k)$ and $\hat{p}_{ij}(k)$ are bounded quantities,

$$\big|\hat{c}_i(k)\big| \leq \hat{c}_i, \quad \big|\hat{p}_{ij}(k)\big| \leq \hat{p}_{ij}, \quad k = 0,1,\dots \quad (\hat{c}_i = \text{const} \geq 0,\ \hat{p}_{ij} = \text{const} \geq 0).$$

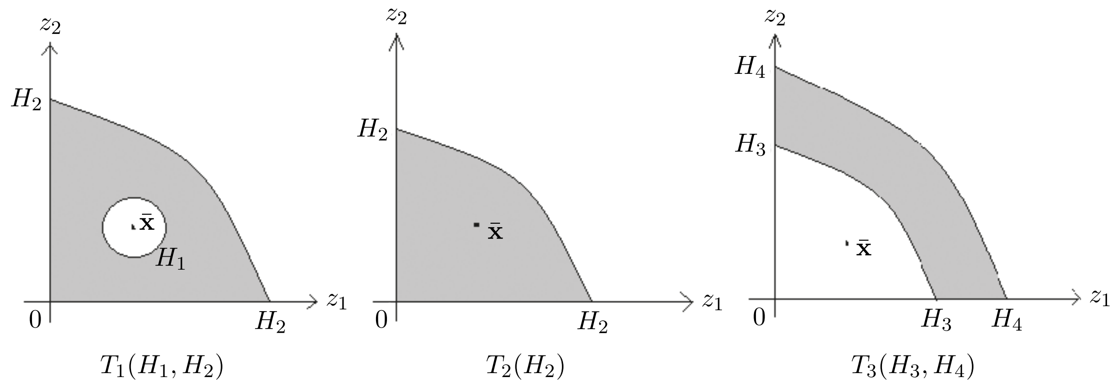


**Fig. 1.** Domains of the state space.

After the perturbations cease, the system returns to the planned regime (1). Alternatively, one can interpret the problem as the system operating in the perturbed regime (2), but at certain times we apply special stabilizing control to bring the system to the planned regime (1). After the control is turned off, the system returns to the regime (2).

The aim of the present paper is to determine constraints on the times (numbers of consecutive iterations) for which the system stays in regimes (1) and (2) which guarantee the desired dynamic characteristics of the system solutions. In particular, we will study the problem on system solutions entering some compact neighborhood $G \in K_0^+$ of the point $\bar{\mathbf{x}}$ and remaining there. Thus, the paper will establish conditions for the ultimate boundedness of solutions and the permanence of the hybrid system consisting of subsystems (1) and (2).

## 2. CONSTRUCTION AND ESTIMATES OF LYAPUNOV FUNCTIONS

To solve the problem, we use two Lyapunov functions [3, 17]

$$V_1(\mathbf{z}) = \sum_{i=1}^{n} \lambda_i \int_{\bar{x}_i}^{z_i} \frac{f_i(\tau) - f_i(\bar{x}_i)}{\tau}\, d\tau,$$
$$V_2(\mathbf{z}) = \sum_{i=1}^{n} \lambda_i \int_{\bar{x}_i}^{z_i} \frac{f_i(\tau)}{\tau}\, d\tau,$$

where for the coefficients $\lambda_1, \dots, \lambda_n$ we take the numbers indicated in the previous section of the paper.

We have $V_1(\bar{\mathbf{x}}) = 0$ and $V_1(\mathbf{z}) > 0$ for $\mathbf{z} \neq \bar{\mathbf{x}}$, and $V_1(\mathbf{z}) \to +\infty$ as $z_i \to +0$ for at least one $i \in \{1, \dots, n\}$ and also as $\|\mathbf{z}\| \to +\infty$. At the same time, the function $V_2(\mathbf{z})$ maintains a finite value as $z_i \to +0$ for $i \in \{1, \dots, n\}$, but $V_2(\mathbf{z}) \to +\infty$ as $\|\mathbf{z}\| \to +\infty$. Thus, there exists an $\bar{H} > 0$ such that

$$V_2(\mathbf{z}) > 0 \quad \text{for} \quad \|\mathbf{z}\| \geq \bar{H}. \tag{3}$$

We will compute the finite differences of the functions $V_1(\mathbf{z})$ and $V_2(\mathbf{z})$ on the solutions of subsystems (1) and (2).

Let us take arbitrary values $0 < H_1 < \min\limits_{i=1,\dots,n} \bar{x}_i$ and $H_2 > \|\bar{\mathbf{x}}\|$ and specify the domain (see Fig. 1)

$$T_1(H_1, H_2) = \{\mathbf{z} \in K_0^+ :\ \|\mathbf{z} - \bar{\mathbf{x}}\| \geq H_1,\ \|\mathbf{z}\| \leq H_2\}.$$

We obtain

$$\Delta V_1\big|_{(1)} = V_1\big(\mathbf{x}(k+1)\big) - V_1\big(\mathbf{x}(k)\big)$$

$$= \sum_{i=1}^{n} \lambda_i \int_{x_i(k)}^{x_i(k+1)} \frac{f_i(\tau) - f_i(\bar{x}_i)}{\tau}\, d\tau = \sum_{i=1}^{n} \lambda_i \int\limits_{y_i(k)}^{y_i(k+1)} \Big(\tilde{f}_i(\xi) - \tilde{f}_i(\bar{y}_i)\Big)\, d\xi$$

$$= h \sum_{i=1}^{n} \lambda_i \left(\tilde{f}_i\big(y_i(k) + \theta_{ik}\Delta y_i(k)\big) - \tilde{f}_i(\bar{y}_i)\right) \sum_{j=1}^{n} p_{ij} \left(\tilde{f}_j\big(y_j(k)\big) - \tilde{f}_j(\bar{y}_j)\right)$$

$$= h \sum_{i=1}^{n} \lambda_i \left(\tilde{f}_i\big(y_i(k)\big) - \tilde{f}_i(\bar{y}_i)\right) \sum_{j=1}^{n} p_{ij} \left(\tilde{f}_j\big(y_j(k)\big) - \tilde{f}_j(\bar{y}_j)\right)$$

$$+ h \sum_{i=1}^{n} \lambda_i \left(\tilde{f}_i\big(y_i(k) + \theta_{ik}\Delta y_i(k)\big) - \tilde{f}_i\big(y_i(k)\big)\right) \sum_{j=1}^{n} p_{ij} \left(\tilde{f}_j\big(y_j(k)\big) - \tilde{f}_j(\bar{y}_j)\right).$$

Here $y_i(k) = \ln x_i(k)$, $\bar{y}_i = \ln \bar{x}_i$, $\Delta y_i(k) = y_i(k+1) - y_i(k)$, and $\theta_{ik} \in (0,1)$, $i = 1, \dots, n$.

Hence if $\|\mathbf{x}(k)\| \le H_2$ and $\|\mathbf{x}(k+1)\| \le H_2$, then

$$\Delta V_1\big|_{(1)} \le \Big( - b_1 h + b_2 L(\ln H_2) h^2 \Big) \big\|\mathbf{f}(\mathbf{x}(k)) - \mathbf{f}(\bar{\mathbf{x}})\big\|^2,$$

where $b_1$ and $b_2$ are some positive constants independent of $h$ and $H_2$.

Take $0 < h_1(H_2) < b_1/\big(b_2\, L(\ln H_2)\big)$. Now, if $h \in (0, h_1(H_2))$, then, as long as the solution of system (1) remains in the domain $T_1(H_1, H_2)$, one has the estimates

$$\Delta V_1\big|_{(1)} \le -h\, b_3(H_2) \Big\|\mathbf{f}\big(\mathbf{x}(k)\big) - \mathbf{f}(\bar{\mathbf{x}})\Big\|^2 \le -h\, \alpha_1(H_1, H_2), \tag{4}$$

where $b_3(H_2)$ and $\alpha_1(H_1, H_2)$ are positive constants depending on $H_2$ and $(H_1, H_2)$, respectively.

Using a similar argument, one can specify a nonnegative constant $\beta_1(H_2)$ depending on $H_2$ such that if $h \in (0, h_1(H_2))$, then, as long as the solution of system (2) remains in the domain (see Fig. 1)

$$T_2(H_2) = \big\{\mathbf{z} \in K_0^+ :\ \|\mathbf{z}\| \le H_2\big\},$$

one has the estimate

$$\Delta V_1\big|_{(2)} \le h\, \beta_1(H_2). \tag{5}$$

Now consider a domain of the form (see Fig. 1)

$$T_3(H_3, H_4) = \big\{\mathbf{z} \in K_0^+ :\ H_3 \le \|\mathbf{z}\| \le H_4\big\},$$

where $H_3$ and $H_4$ are some positive constants and $H_3 < H_4$.

We have

$$\Delta V_2\big|_{(1)} = V_2\big(\mathbf{x}(k+1)\big) - V_2\big(\mathbf{x}(k)\big)$$

$$= \sum_{i=1}^{n} \lambda_i \int_{x_i(k)}^{x_i(k+1)} \frac{f_i(\tau)}{\tau}\, d\tau = \sum_{i=1}^{n} \lambda_i \int\limits_{y_i(k)}^{y_i(k+1)} \tilde{f}_i(\xi)\, d\xi$$

$$= h \sum_{i=1}^{n} \lambda_i \tilde{f}_i\big(y_i(k) + \hat{\theta}_{ik}\Delta y_i(k)\big) \left(c_i + \sum_{j=1}^{n} p_{ij} \tilde{f}_j\big(y_j(k)\big)\right)$$

$$= h \sum_{i=1}^{n} \lambda_i \tilde{f}_i\big(y_i(k)\big) \left(c_i + \sum_{j=1}^{n} p_{ij} \tilde{f}_j\big(y_j(k)\big)\right)$$

$$+ h \sum_{i=1}^{n} \lambda_i \left(\tilde{f}_i\big(y_i(k) + \hat{\theta}_{ik}\Delta y_i(k)\big) - \tilde{f}_i\big(y_i(k)\big)\right) \left(c_i + \sum_{j=1}^{n} p_{ij} \tilde{f}_j\big(y_j(k)\big)\right).$$

Here, as before, $y_i(k) = \ln x_i(k)$, $\Delta y_i(k) = y_i(k+1) - y_i(k)$, and $\hat{\theta}_{ik} \in (0,1)$, $i = 1, \dots, n$.

Hence if $\|\mathbf{x}(k)\| \le H_4$ and $\|\mathbf{x}(k+1)\| \le H_4$, then one has the inequality

$$\Delta V_2\big|_{(1)} \le -b_4 h \Big\|\mathbf{f}\big(\mathbf{x}(k)\big)\Big\|^2 + b_5 h \Big\|\mathbf{f}\big(\mathbf{x}(k)\big)\Big\| + b_6 L(\ln H_4) h^2 \bigg(1 + \Big\|\mathbf{f}\big(\mathbf{x}(k)\big)\Big\|^2\bigg),$$

where $b_4$, $b_5$, and $b_6$ are some positive constants independent of $h$ and $H_4$.

Let us find a number $\widetilde{H} > 0$ such that

$$\big\|\mathbf{f}(\mathbf{z})\big\| > b_5/b_4 \quad \text{for} \quad \|\mathbf{z}\| \ge \widetilde{H}. \tag{6}$$

Based on the quantities defined by formulas (3) and (6), we find the constant $\hat{H} = \max\{\bar{H};\, \widetilde{H}\}$. We will assume that $H_4 > H_3 > \hat{H}$. Then it is not difficult to find positive constants $h_2(H_3, H_4)$ and $b_7(H_3, H_4)$ such that if $h \in (0, h_2(H_3, H_4))$, then, as long as the solution of system (1) remains in the domain $T_3(H_3, H_4)$, the inequality

$$\Delta V_2\big|_{(1)} \le -h\, b_7(H_3, H_4) \Big\|\mathbf{f}\big(\mathbf{x}(k)\big)\Big\|^2$$

holds. Consequently, for any $\eta \ge 0$ in the domain $T_3(H_3, H_4)$ we can construct the estimate

$$\Delta V_2\big|_{(1)} \le -h\, \alpha_2(\eta, H_3, H_4) V_2^{\eta}\big(\mathbf{x}(k)\big), \tag{7}$$

where $\alpha_2(\eta, H_3, H_4)$ is a positive constant depending on the choice of $\eta$, $H_3$, and $H_4$.

In a similar way, we can find a nonnegative constant $\beta_2(\eta, H_3, H_4)$ depending on the choice of $\eta$, $H_3$, and $H_4$ such that for $h \in (0, h_2(H_3, H_4))$ the estimate

$$\Delta V_2\big|_{(2)} \le h\, \beta_2(\eta, H_3, H_4) V_2^{\eta}\big(\mathbf{x}(k)\big) \tag{8}$$

holds in the domain $T_3(H_3, H_4)$.

**Remark 2.** The coefficients $\alpha_1(H_1, H_2)$, $\beta_1(H_2)$, $\alpha_2(\eta, H_3, H_4)$, and $\beta_2(\eta, H_3, H_4)$ occurring in inequalities (4)–(8), as well as the quantities $h_1(H_2)$ and $h_2(H_3, H_4)$, can be estimated in a straightforward manner for selected values of $H_1$, $H_2$, $H_3$, $H_4$, and $\eta$ and for a specifically given family of subsystems (1) and (2) via numerical analysis.

## 3. ESTIMATES OF SOLUTIONS OF THE HYBRID SYSTEM

Now consider the hybrid system formed by subsystems (1) and (2). Let a discrete function $\sigma(k)\colon \{0, 1, \dots\} \to \{1, 2\}$ determine the switching order between these subsystems. Thus, if $\sigma(k) = 1$, we consider that during the $k$th iteration the hybrid system operates in the planned mode (1), while if $\sigma(k) = 2$, then the hybrid system operates in the perturbed mode (2), $k = 0, 1, \dots$.

Relations (4)–(8) established earlier for the selected Lyapunov functions permit one to obtain estimates for solutions of the hybrid system in the corresponding regions of the positive orthant $K_0^+$.

**Lemma.** *For any $\eta$, $y$, and $x$ such that $\eta > 0$, $y > 0$, and $x < y^{-\eta+1}$, one has the inequalities*

$$\big(y - x y^{\eta}\big)^{-\eta+1} \ge y^{-\eta+1} + (\eta - 1)x \quad \text{if} \quad \eta > 1,$$

*and*

$$\big(y - x y^{\eta}\big)^{-\eta+1} \le y^{-\eta+1} + (\eta - 1)x \quad \text{if} \quad 0 < \eta < 1.$$

**Proof.** Take $\eta > 0$ and $y > 0$ and consider the function

$$\varphi(x) = \big(y^{-\eta+1} + (\eta - 1)x\big)\big(y - x y^{\eta}\big)^{\eta-1}.$$

We have

$$\varphi'(x) = -\eta(\eta-1)xy^{\eta}\big(y - xy^{\eta}\big)^{\eta-2}.$$

If $\eta > 1$, then $\varphi'(x) > 0$ for $x < 0$ and $\varphi'(x) < 0$ with $0 < x < y^{-\eta+1}$, and hence

$$\max_{(-\infty,\, y^{-\eta+1})} \varphi(x) = \varphi(0) = 1.$$

Likewise, if $0 < \eta < 1$, then $\varphi'(x) < 0$ for $x < 0$ and $\varphi'(x) > 0$ for $0 < x < y^{-\eta+1}$, and hence

$$\min_{(-\infty,\, y^{-\eta+1})} \varphi(x) = \varphi(0) = 1.$$

The proof of the lemma is complete. □

Let some values $0 < H_1 < \min\limits_{i=1,\dots,n} \bar{x}_i$ and $H_2 > \|\bar{\mathbf{x}}\|$ be chosen, the quantity $h_1(H_2)$ be defined, and the estimates (4) and (5) in the domain $T_1(H_1, H_2)$ be constructed.

Define $\bar{\omega}_k(H_1, H_2) = \alpha_1(H_1, H_2)$ if $\sigma(k) = 1$ and $\bar{\omega}_k(H_1, H_2) = -\beta_1(H_2)$ if $\sigma(k) = 2$, $k = 0, 1, \dots$.

Take $k_0 \ge 0$ and $\mathbf{x}_0 \in T_1(H_1, H_2)$ and consider a solution $\mathbf{x}(k)$ of the hybrid system (1), (2) such that $\mathbf{x}(k_0) = \mathbf{x}_0$. Assume that for $k = k_0, \dots, \tilde{k}$ the solution $\mathbf{x}(k)$ remains in the domain $T_1(H_1, H_2)$. Now if $h \in (0, h_1(H_2))$, then the following estimates hold for $k = k_0 + 1, \dots, \tilde{k}$:

$$V_1(\mathbf{x}(k)) \le V_1(\mathbf{x}(k-1)) - h\bar{\omega}_{k-1}(H_1, H_2) \le \dots \le V_1(\mathbf{x}_0) - h\sum_{i=k_0}^{k-1} \bar{\omega}_i(H_1, H_2). \tag{9}$$

In a similar way, let some values $\eta \ge 0$ and $H_4 > H_3 > \hat{H}$ be selected, the quantity $h_2(H_3, H_4)$ be defined, and the estimates (7) and (8) in the domain $T_3(H_3, H_4)$ be constructed. Define $\widetilde{\omega}_k(\eta, H_3, H_4) = \alpha_2(\eta, H_3, H_4)$ if $\sigma(k) = 1$ and $\widetilde{\omega}_k(\eta, H_3, H_4) = -\beta_2(\eta, H_3, H_4)$ if $\sigma(k) = 2$, $k = 0, 1, \dots$.

Take $k_0 \ge 0$ and $\mathbf{x}_0 \in T_3(H_3, H_4)$ and consider a solution $\mathbf{x}(k)$ of the hybrid system (1), (2) such that $\mathbf{x}(k_0) = \mathbf{x}_0$. Assume that for $k = k_0, \dots, \tilde{k}$ the solution $\mathbf{x}(k)$ remains in the domain $T_3(H_3, H_4)$. Then, applying the lemma, it is straightforward to find a positive $\bar{h}_2(\eta, H_3, H_4) \le h_2(H_3, H_4)$ such that if $h \in (0, \bar{h}_2(\eta, H_3, H_4))$, then the following estimates hold for $k = k_0 + 1, \dots, \tilde{k}$:

$$\begin{aligned} V_2^{-\eta+1}\big(\mathbf{x}(k)\big) &\ge \Big(V_2\big(\mathbf{x}(k-1)\big) - h\widetilde{\omega}_{k-1}(\eta, H_3, H_4)V_2^{\eta}\big(\mathbf{x}(k-1)\big)\Big)^{-\eta+1} \\ &\ge V_2^{-\eta+1}\big(\mathbf{x}(k-1)\big) + (\eta-1)h\widetilde{\omega}_{k-1}(\eta, H_3, H_4) \\ &\ge \dots \ge V_2^{-\eta+1}(\mathbf{x}_0) + (\eta-1)h\sum_{i=k_0}^{k-1} \widetilde{\omega}_i(\eta, H_3, H_4) \end{aligned} \tag{10}$$

if $\eta > 1$;

$$\begin{aligned} V_2^{-\eta+1}\big(\mathbf{x}(k)\big) &\le \Big(V_2\big(\mathbf{x}(k-1)\big) - h\widetilde{\omega}_{k-1}(\eta, H_3, H_4)V_2^{\eta}\big(\mathbf{x}(k-1)\big)\Big)^{-\eta+1} \\ &\le V_2^{-\eta+1}\big(\mathbf{x}(k-1)\big) + (\eta-1)h\widetilde{\omega}_{k-1}(\eta, H_3, H_4) \\ &\le \dots \le V_2^{-\eta+1}(\mathbf{x}_0) + (\eta-1)h\sum_{i=k_0}^{k-1} \widetilde{\omega}_i(\eta, H_3, H_4) \end{aligned} \tag{11}$$

if $0 \le \eta < 1$;

$$V_2\big(\mathbf{x}(k)\big) \le \Big(1 - h\widetilde{\omega}_{k-1}(1, H_3, H_4)\Big)V_2\big(\mathbf{x}(k-1)\big) \le \dots \le \prod_{i=k_0}^{k-1}\Big(1 - h\widetilde{\omega}_i(1, H_3, H_4)\Big)V_2(\mathbf{x}_0) \tag{12}$$

if $\eta = 1$.

Taking into account the specific form of the Lyapunov functions $V_1(\mathbf{z})$ and $V_2(\mathbf{z})$, inequalities (9)–(12) permit one to estimate $\|\mathbf{x}(k)\|$ as long as the solution $\mathbf{x}(k)$ of the hybrid system remains in the corresponding domains of the positive orthant.

## 4. ANALYSIS OF THE DYNAMICS OF THE HYBRID SYSTEM

Now let us study the dynamics of solutions of the hybrid system (1), (2). First, we establish sufficient conditions for the ultimate boundedness of solutions using the Lyapunov function $V_2(\mathbf{z})$.

**Theorem 1.** *Suppose that for some selected values $H_4 > H_3 > \hat{H}$, $\eta \geq 0$, the estimates* (7) *and* (8) *have been constructed and the following conditions hold*:

1. *There exist numbers $\Delta_1$ and $\Delta_2$ such that*

$$H_3 < B(H_3) < \Delta_2 < \Delta_1 < B(\Delta_1) < H_4, \tag{13}$$

*where*

$$B(s) = \max_{\mathbf{z}\in K_0^+:\ V_2(\mathbf{z})=A(s)} \|\mathbf{z}\|,$$
$$A(s) = \max_{\mathbf{z}\in K_0^+:\ \|\mathbf{z}\|=s} V_2(\mathbf{z}).$$

2. *The time* (*number of consecutive iterations*) *for which the hybrid system stays in mode* (1) *is bounded below by a positive integer $L_1$, the time* (*number of consecutive iterations*) *for which the system stays in mode* (2) *is bounded above by a positive integer $L_2$, and one has the relation*

$$-\alpha_2(\eta, H_3, H_4)L_1 + \beta_2(\eta, H_3, H_4)L_2 < 0. \tag{14}$$

*Then there exists an $h_{01} > 0$ such that if $h \in (0, h_{01})$ and $\|\mathbf{x}_0\| \leq \Delta_1$, then, for some $K \geq 0$, the inequality $\|\mathbf{x}(k)\| \leq \Delta_2$ holds for all $k \geq k_0 + K$. Here $\mathbf{x}(k)$ is the solution of the hybrid system* (1), (2) *with the initial condition $\mathbf{x}(k_0) = \mathbf{x}_0$; $k_0 \geq 0$, $\mathbf{x}_0 \in K_0^+$.*

**Proof.** Let $\mathbf{x}_0 \in T_3(H_3, H_4)$. Now if $h \in (0, \bar{h}_2(\eta, H_3, H_4))$, then, as long as the solution of the hybrid system remains in the domain $T_3(H_3, H_4)$, the estimates (10)–(12) will hold (depending on the chosen value of $\eta$).

The choice of $\Delta_1$ according to inequalities (13), along with the assumption about the bounded time of the system staying in the mode (2), guarantees the existence of a value

$$0 < h_{21}(\eta, H_3, H_4, \Delta_1, L_2) \leq \bar{h}_2(\eta, H_3, H_4)$$

such that for $h \in (0, h_{21}(\eta, H_3, H_4, \Delta_1, L_2))$ and $\|\mathbf{x}_0\| \leq \Delta_1$ the condition $\|\mathbf{x}(k)\| \leq H_4$ will be maintained for all $k = k_0, k_0 + 1, \ldots$ (i.e., the solution of the hybrid system will not leave the domain $T_3(H_3, H_4)$ through the upper boundary $\|\mathbf{z}\| = H_4$).

On the other hand, according to condition (14), there exists a $\bar{k} \geq k_0$ such that for $k = \bar{k}$ the estimates (10)–(12) become inconsistent in the domain $T_3(H_3, H_4)$ (which means that the solution of the hybrid system will leave the region $T_3(H_3, H_4)$ through the lower boundary $\|\mathbf{z}\| = H_3$ at time $k = \bar{k}$).

Finally, the choice of $\Delta_2$ according to inequalities (13), taking into account the assumption about the bounded time of the system staying in the mode (2), guarantees the existence of a value $0 < h_{22}(\eta, H_3, H_4, \Delta_2, L_2) \leq \bar{h}_2(\eta, H_3, H_4)$ such that for $h \in (0, h_{22}(\eta, H_3, H_4, \Delta_2, L_2))$ the condition $\|\mathbf{x}(k)\| \leq \Delta_2$ will be maintained for all $k \geq \bar{k}$ (i.e., the solution of the hybrid system, once entering the region $T_2(\Delta_2)$ at some time, will not leave it thereafter).

Setting $h_{01} = \min\{h_{21}(\eta, H_3, H_4, \Delta_1, L_2);\ h_{22}(\eta, H_3, H_4, \Delta_2, L_2)\}$, we arrive at the desired result. The proof of the theorem is complete. $\square$

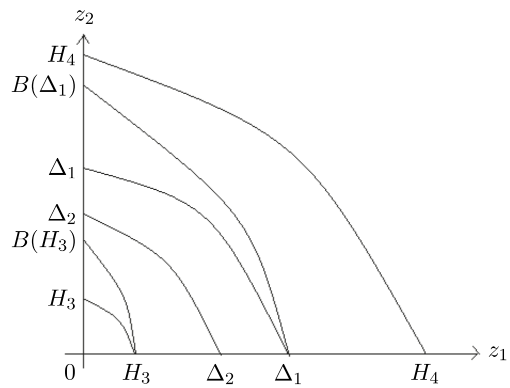


**Fig. 2.** Constraints on the selection of $H_3$ and $H_4$ values.

**Remark 3.** One can readily verify that the quantity $K$ in Theorem 1 can be chosen independently of $k_0$, which means that the conditions of Theorem 1 ensure the uniform ultimate boundedness of solutions.

**Remark 4.** Inequalities (13) impose constraints on the selection of $H_3$ and $H_4$ (see Fig. 2). Note that for the existence of $\Delta_1$ and $\Delta_2$ satisfying (13) it is necessary and sufficient that $H_3$ and $H_4$ are chosen so that the inequality $B(B(H_3)) < H_4$ holds.

**Remark 5.** By choosing $\eta$, one can optimize relation (14) by means of finding the value of $\max\limits_{\eta \geq 0} \alpha_2(\eta, H_3, H_4)/\beta_2(\eta, H_3, H_4)$. Note that condition 1 of Theorem 1 is independent of $\eta$. However, the choice of $\eta$ affects the estimate for the limiting admissible discretization step $h_{01}$.

**Remark 6.** If the functions $\tilde{f}_i(z_i) = f_i(e^{z_i})$, $i = 1, \ldots, n$, satisfy the Lipschitz condition on any interval $(-\infty, H]$ with a Lipschitz constant $L > 0$ independent of $H$, then by setting $H_4 = +\infty$, one can consider the problem of uniform dissipativity of the hybrid system (1), (2). This only requires the additional assumption that there exists an $\eta \geq 0$ for which the estimates (7) and (8) hold in the domain $T_3(H_3, +\infty)$. In this case, $\Delta_1 = +\infty$ can be chosen in Theorem 1.

We now use the Lyapunov function $V_1(\mathbf{z})$ for the dynamic analysis to guarantee permanence of the hybrid system (1), (2).

**Theorem 2.** *Let the time (number of consecutive iterations) of the hybrid system staying in the mode (1) be bounded below by a positive integer $L_1$, let the time (number of consecutive iterations) of the system staying in mode (2) be bounded above by a positive integer $L_2$, and let the following conditions be satisfied:*

1. *For some selected values $H_4 > H_3 > \hat{H}$, $\eta \geq 0$, the estimates (7) and (8) have been constructed, there exist numbers $\Delta_1$ and $\Delta_2$ satisfying inequalities (13), and relation (14) holds true as well.*
2. *For some $0 < H_1 < \min\limits_{i=1,\ldots,n} \bar{x}_i$ and $H_2 = \Delta_2$, the estimates (4), (5) have been constructed, and*

$$-\alpha_1(H_1, \Delta_2)L_1 + \beta_1(\Delta_2)L_2 < 0. \tag{15}$$

*Then for any constant $C > 0$ there exists an $h_{02} > 0$ such that if $h \in (0, h_{02})$, then there exists an $N \geq 0$ for which any solution of the hybrid system (1), (2) starting in the domain $T_2(\Delta_1)$ enters the domain*

$$G = \left\{\mathbf{z} \in K_0^+ : \ V_1(\mathbf{z}) \leq D(H_1) + C\right\} \cap T_2(\Delta_2),$$

*where $D(H_1) = \max\limits_{\mathbf{z} \in K_0^+ :\ \|\mathbf{z} - \bar{\mathbf{x}}\| = H_1} V_1(\mathbf{z})$, no later than the $N$th iteration and remains there.*

**Proof.** By Theorem 1, for $h \in (0, h_{01})$ any solution of system (1), (2) starting in the domain $T_2(\Delta_1)$ will eventually enter the domain $T_2(\Delta_2)$ and remain there. Condition (15), together with estimate (9), guarantees that if $h \in (0, h_1(\Delta_2))$, then the solution will eventually enter the $H_1$-neighborhood of the point $\bar{\mathbf{x}}$. Finally, the discretization step can be chosen sufficiently small so that the solutions starting in the $H_1$-neighborhood of the point $\bar{\mathbf{x}}$ do not jump beyond the domain $G$ within $L_2$ iterations. The proof of the theorem is complete. □

**Remark 7.** The use of the Lyapunov function $V_2(\mathbf{z})$ enables driving solutions of the hybrid system (1), (2) starting in the domain $T_2(\Delta_1)$ into the domain $T_2(\Delta_2)$ (ensuring their ultimate boundedness). The subsequent application of the Lyapunov function $V_1(\mathbf{z})$ then drives these solutions into a neighborhood $G$ of point $\bar{\mathbf{x}}$ (ensuring system permanence). The size of neighborhood $G$, constraints on $L_1$ and $L_2$, and the estimate of admissible discretization step can be adjusted through selection of the constants $C$, $H_1$, $H_3$, $H_4$, $\eta$, $\Delta_1$, and $\Delta_2$. Figure 3 illustrates the solution behavior guaranteed by Theorem 2. Note that using only one of the Lyapunov functions $V_1(\mathbf{z})$ or $V_2(\mathbf{z})$ would not suffice to guarantee system permanence under the given assumptions.

## 5. NUMERICAL EXAMPLE

Consider a vector of the form (1) describing the interaction of two ($n = 2$) populations in a biological community:

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\Big(h\big(-1 - x_1(k) + 2x_2(k)\big)\Big), \\ x_2(k+1) &= x_2(k)\exp\Big(h\big(3 - 2x_1(k) - x_2(k)\big)\Big). \end{aligned} \tag{16}$$

Here $f_i(z_i) = z_i$, $i = 1, 2$. System (16) has the equilibrium position $\bar{\mathbf{x}} = (1, 1)^T \in K_0^+$.

Assume that the perturbed system (2) is represented in the form

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\Big(h\big(-\cos(k) + \cos(k)\,x_1(k) + 2\sin(k)\,x_2(k)\big)\Big), \\ x_2(k+1) &= x_2(k)\exp\Big(h\big(3\cos(k) - 2\sin(k)\,x_1(k) + \cos(k)\,x_2(k)\big)\Big). \end{aligned} \tag{17}$$

Take $\Lambda = \mathrm{diag}\{1,\ 1\}$ and construct the Lyapunov functions

$$\begin{aligned} V_1(\mathbf{z}) &= z_1 - \ln z_1 + z_2 - \ln z_2 - 2, \\ V_2(\mathbf{z}) &= z_1 + z_2 - 2. \end{aligned}$$

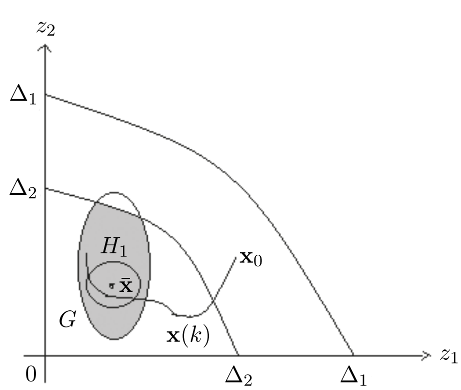


**Fig. 3.** Dynamics of solutions for the hybrid system.

Set $H_3 = 4$. Then $B(H_3) = 4\sqrt{2}$, and hence one can take, say, $\Delta_2 = 6$, $\Delta_1 = 10$. We obtain $B(\Delta_1) = 10\sqrt{2}$; consequently, it suffices to set $H_4 = 15$ for inequalities (13) to be satisfied.

For $\eta = 2$ and for a sufficiently small discretization step $h$, in the domain $T_3(4, 15)$ we arrive at the estimates

$$\begin{aligned} V_2(\mathbf{z}) &> 0, \\ \Delta V_2\big|_{(16)} &\le -0.15\, h\, V_2^2(\mathbf{x}(k)), \\ \Delta V_2\big|_{(17)} &\le 7.1\, h\, V_2^2(\mathbf{x}(k)). \end{aligned}$$

By Theorem 1, if the inequality

$$-0.15L_1 + 7.1L_2 < 0, \tag{18}$$

holds, then, for the hybrid system consisting of subsystems (16) and (17), the solutions starting in the domain $T_2(10)$ will eventually enter the domain $T_2(6)$ and remain there. Thus, these solutions will be ultimately bounded.

Now take $H_1 = 0.75$. For a sufficiently small discretization step $h$, in the domain $T_1(0.75, 6)$ we obtain the estimates

$$\begin{aligned} \Delta V_1\big|_{(16)} &\le -0.56\, h, \\ \Delta V_1\big|_{(17)} &\le 53\, h. \end{aligned}$$

Therefore, if in addition to inequality (18), the relation

$$-0.56L_1 + 53L_2 < 0, \tag{19}$$

holds; then, according to Theorem 2, the hybrid system studied will be permanent for solutions starting in the domain $T_2(10)$. Note that inequality (18) follows from inequality (19). Thus, condition (19) will guarantee nonextinction of the considered populations and their bounded numbers. Moreover, using Theorem 2, we can estimate the boundaries for the limiting numbers of these populations. The best accuracy of estimates can be achieved by numerically enumerating possible values of the parameters $H_1$, $H_3$, $H_4$, $\Delta_1$, $\Delta_2$, and $\eta$.

## CONCLUSIONS

The paper obtained constraints on the switching law between the planned and perturbed modes of operation of a given discrete dynamical system, guaranteeing the ultimate boundedness of the solutions and the permanence of the system. The Lyapunov direct method was used to solve the problem. Since for nonhomogeneous systems, the estimates for the chosen Lyapunov function generally depend significantly on the considered region of the state space, it is often advisable to partition the state space into parts and use separate estimates in each part. It is even possible to construct a separate Lyapunov function in each part. In the present paper, two Lyapunov functions were employed for the analysis. It was noted that only the joint application of these functions leads to the desired result. The obtained relations connect the constraints on the switching law, the sizes of the domain of initial solution values, the sizes of the domain of ultimate solution behavior, and the discretization step size. These relations are determined by the choice of certain auxiliary parameters, which allows formulating optimization problems for this choice.

## FUNDING

This work was supported by ongoing institutional funding. No additional grants to carry out or direct this particular research were obtained.

## CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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