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Oriented Shadowing by Elementary Pseudotrajectories Near Attractors in a Banach Space

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Abstract—We study shadowing properties of differentiable mappings in Banach spaces in neighborhoods of attractors. The properties of oriented shadowing of a pseudotrajectory are defined. A pseudotrajectory $\xi = \{x_k : k \in \mathbb{Z}\}$ belonging to an attractor $\mathcal{A}$ of a mapping f is called elementary if the set of indices $\mathbb{Z}$ can be decomposed into a finite family of intervals $I_1, \dots, I_n$ so that for any set I_m there exist two fixed hyperbolic points p and q such that the set of points $\{x_k : k \in I_m\}$ of the pseudotrajectory belongs to a trajectory lying in the intersection of the unstable manifold of the point p and the stable manifold of the point q. The main results of the paper state that if f is gradient-like with hyperbolic nonwandering set in $\mathcal{A}$, then f has the properties of oriented shadowing by elementary pseudotrajectories belonging to $\mathcal{A}$ and to a neighborhood of $\mathcal{A}$. As an application, we consider semigroups generated by parabolic PDEs.

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1. INTRODUCTION

Attractors are one of the basic objects in the global theory of dynamical systems, and a lot of research is devoted to their study.

We work with the following standard definition of an attractor for a continuous mapping f of a metric space (X, dist). A subset $\mathcal{A} \subset X$ is called an attractor of f if

(A1) $\mathcal{A}$ is compact and positively f-invariant (i.e., $f^k(x) \in \mathcal{A}$ for $x \in \mathcal{A}$ and $k \geq 0$);

(A2) $\mathcal{A}$ is Lyapunov stable, i.e., for any neighborhood U of $\mathcal{A}$ there exists a neighborhood V of $\mathcal{A}$ such that $f^k(V) \subset U$, $k \geq 0$;

(A3) there exists a neighborhood W of $\mathcal{A}$ such that

$$\text{dist}\left(f^k(x), \mathcal{A}\right) \to 0, \quad x \in W, \quad k \to +\infty.$$

A general theory of attractors for semigroups in various functional spaces is developed in the book [2].

A property important both for the "internal" theory of dynamical systems and for applications is the shadowing property (see, for example, [9]).

Let us recall that for $d > 0$, a sequence $\xi = \{x_k \in X : k \in \mathbb{Z}\}$ is called a d-pseudotrajectory of a continuous mapping $f : (X, \text{dist}) \to (X, \text{dist})$, where (X, dist) is a metric space, if

$$\text{dist}(f(x_k), x_{k+1}) < d, \quad k \in \mathbb{Z}.$$

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The mapping f has the (standard) shadowing property if for any $\varepsilon > 0$ there exists a $d > 0$ such that for any d-pseudotrajectory ξ there is a point x such that

$$\mathrm{dist}\left(f^k(x), x_k\right) < \varepsilon, \quad k \in \mathbb{Z}.$$

In this case, one says that the pseudotrajectory ξ is ε-shadowed by the trajectory of the point x.

The shadowing property near attractors for mappings of Banach spaces has been studied in the book [9]. In this book, the author considered a global attractor $\mathcal{A}$ of a smooth mapping f generated by a parabolic PDE and found conditions under which f has the Lipschitz shadowing in a neighborhood of $\mathcal{A}$, i.e., there exists a neighborhood U' of $\mathcal{A}$ and constants L' and d' such that any d-pseudotrajectory ξ with $d \le d'$ in U' is $L'd$-shadowed by the trajectory of some point.

It was assumed in [9] that the system has a so-called inertial manifold, i.e., an exponentially attracting, smooth, finite-dimensional, invariant manifold containing the global attractor. Now, a lot of conditions for the existence of inertial manifolds are known (see, for example, [5]). A crucial step in the proof in [9] was projecting to the inertial manifold, which reduced the shadowing problem to a finite-dimensional one. At the same time, it is known [2] that there exist systems generated by PDEs for which compact finite-dimensional global attractors do not belong to any finite-dimensional submanifolds of the phase space.

Thus, it is important to develop methods for establishing shadowing properties in the infinite-dimensional setting without the assumption that an inertial manifold exists. This is the main goal of the present paper.

In [9], the author studied mappings f of class C^1 of a Banach space $\mathcal{X}$ that have gradient-like Morse–Smale structure on an attractor $\mathcal{A}$, i.e., mappings having the following properties (MS1)–(MS3).

(MS1) $\mathcal{A}$ contains hyperbolic fixed points $\pi_1, \dots, \pi_N$.

Recall that a fixed point p of f is called hyperbolic if the spectrum of the derivative $Df(p)$ does not intersect the unit circle. In other words, all the operators

$$(Df(p) - \lambda I)^{-1}, \quad \lambda \in \mathbb{C}, \quad |\lambda| = 1$$

exist and are bounded. Here I is the identical operator.

If a fixed point is hyperbolic, then it has the stable and unstable manifolds $W^s(p)$ and $W^u(p)$ defined as follows:

$$W^s(p) = \left\{ x \in \mathcal{X} : f^k(x) \to p, k \to +\infty \right\}.$$

Since the mapping f is not assumed to be invertible, the definition of the unstable manifold of p is more complicated; $W^u(p)$ is defined as the set of points x for which there exists a sequence $\{x_k : \ k \le 0\}$ such that $x_0 = x$, $f(x_k) = x_{k+1}$ for $k \le -1$, and $x_k \to p$, $k \to -\infty$.

It is known [2] that $W^s(p)$ and $W^u(p)$ are immersed submanifolds of $\mathcal{X}$ of class C^1.

It follows from the Lyapunov stability and compactness of $\mathcal{A}$ that for any fixed point $\pi_i \in \mathcal{A}$, $W^u(\pi_i) \subset \mathcal{A}$.

(MS2) The mapping f is invertible on $\mathcal{A}$ and the inverse map $f^{-1} : \mathcal{A} \mapsto \mathcal{A}$ is smooth. The trajectory $f^n(x)$ of any point $x \in \mathcal{A}$ tends to one of the points π_i as $n \to \pm\infty$ (thus, $\mathcal{A}$ is the union of a finite family of unstable manifolds of the points $\pi_1, \dots, \pi_N$).

(MS3) The stable and unstable manifolds of the points $\pi_1, \dots, \pi_N$ are pairwise transverse.

In this paper, we study a mapping f of class $C^{1+\alpha}$, $\alpha > 0$, of a Banach space having an attractor $\mathcal{A}$ on which f belongs to a different class of mappings. It is assumed that f has properties (MS1), (MS2), and the following property.

(LF) There exists a continuous function (Lyapunov function) V on $\mathcal{A}$ such that $V(f(x)) < V(x)$ if x is not a fixed point of f.

It is easily seen that if f has properties (MS1), (MS2), and (LF), then the chain recurrent set of the restriction of f to $\mathcal{A}$ consists of fixed points of f. At the same time, we work with a class of systems essentially different from gradient-like Morse–Smale systems since it is not assumed that f has property (MS3).

It is natural to call such mappings f with properties (MS1), (MS2), and (LF) gradient-like with hyperbolic nonwandering set in $\mathcal{A}$; let us denote the class of such mappings by GLHNS($\mathcal{A}$).

In this paper, we introduce a weakened variant of the shadowing property (note that the approach based on a close notion for flows has been developed in the paper [12]).

We say that a pseudotrajectory $\xi = \{x_k : k \in \mathbb{Z}\}$ is oriented ε-shadowed by the trajectory of a point x if there exist two mappings $h_-, h_+ : \mathbb{Z} \to \mathbb{Z}$ such that

$$h_-(k) \le h_+(k) = h_-(k+1) - 1, \quad k \in \mathbb{Z},$$

and

$$\operatorname{dist}\left(f^i(x), x_k\right) < \varepsilon, \quad i \in [h_-(k), h_+(k)], \quad k \in \mathbb{Z}.$$

A similar definition is applied in the case of a segment $\xi(l,m) = \{x_k : k \in [l,m]\}$ of a pseudotrajectory, in which case we consider mappings h_- and h_+ defined on $[l,m]$ with obvious modifications for infinite l or m. Obviously, this part of the pseudotrajectory will correspond to the interval $[h_-(l), h_+(m)]$ of the exact trajectory.

Also, we say that a pseudotrajectory $\xi = \{x_k : k \in \mathbb{Z}\}$ is oriented ε-shadowed by a pseudotrajectory $\eta = \{y_k : k \in \mathbb{Z}\}$ if there exist two mappings $h_-, h_+ : \mathbb{Z} \to \mathbb{Z}$ such that

$$h_-(k) \le h_+(k) = h_-(k+1) - 1, \quad k \in \mathbb{Z},$$

and

$$\operatorname{dist}(y_i, x_k) < \varepsilon, \quad i \in [h_-(k), h_+(k)], \quad k \in \mathbb{Z}.$$

Clearly, if a pseudotrajectory ξ is oriented ε-shadowed by a pseudotrajectory $\eta = \{y_k : k \in \mathbb{Z}\}$ and the pseudotrajectory η is oriented ε-shadowed by the trajectory of a point x, then the pseudotrajectory ξ is oriented 2ε-shadowed by the trajectory of the point x.

Let us introduce an important class of pseudotrajectories for a mapping of the class GLHNS($\mathcal{A}$) (here we note that a similar class of pseudotrajectories for flows was studied in [8]).

A pseudotrajectory $\xi = \{x_k : k \in \mathbb{Z}\}$ belonging to the set $\mathcal{A}$ is called elementary if the set of indices $\mathbb{Z}$ can be decomposed into a finite family of intervals $I_1, \dots, I_n$ so that for any set I_m there exist two fixed points p and q such that the set of points $\{x_k : k \in I_m\}$ of the pseudotrajectory belongs to a trajectory lying in the intersection of the unstable manifold of the point p and the stable manifold of the point q. In this definition, we do not exclude the case where $p = q$. In this case, the set $\{x_k : k \in I_m\}$ coincides with the fixed point p.

Note that the existence of a Lyapunov function on the attractor implies that the intersection of the stable and unstable manifolds of a fixed point $p \in \mathcal{A}$ coincides with the point p.

Remark 1. The oriented shadowing by elementary pseudotrajectories is quite similar to the property of piecewise shadowing for flows (see Section 3 of [8]).

Our main result (Theorem 2.1) states that if $f \in C^{1+\alpha}(\mathcal{X})$ is a mapping of the class GLHNS($\mathcal{A}$), then for any $\varepsilon > 0$ there exists a $d > 0$ such that any d-pseudotrajectory $\xi \subset \mathcal{A}$ is oriented ε-shadowed by an elementary pseudotrajectory.

Remark 2. The assumption that $f \in C^{1+\alpha}(\mathcal{X})$ is applied in the description of stable and unstable manifolds of hyperbolic fixed points.

The structure of the paper is as follows. We prove Theorem 2.1 in Section 2. Section 3 contains proof of an analog of our main result for pseudotrajectories belonging to a neighborhood of the attractor. In Section 4, we give an example of the application of our results to a semigroup generated by a parabolic PDE.

2. ORIENTED SHADOWING BY ELEMENTARY PSEUDOTRAJECTORIES ON AN ATTRACTOR

For a pseudotrajectory $\xi = \{x_k : k \in \mathbb{Z}\}$ and a number $k \in \mathbb{Z}$ we denote $\xi(k) = x_k$; for numbers $k_1 < k_2$ we denote

$$\xi(k_1, k_2) = \{x_k : \ k_1 \le k \le k_2\},$$

and $\xi(-\infty, n)$, $\xi(n, \infty)$ denote the sets $\{x_k : \ k \le n\}$, $\{x_k : \ n \le k\}$, respectively.

For $a > 0$ and $x \in \mathcal{X}$ we denote by $B(a, x)$ the open a-ball centered at x.

We take a number $\rho > 0$, define, for a fixed point $p \in \mathcal{A}$ of f, $U(p) := B(\rho, p)$, and assume that the following statements are valid:

• If p and q are different fixed points in $\mathcal{A}$, then $U(q) \cap B(2\rho, p) = \emptyset$;

• f has the Lipschitz shadowing property in $U(p)$ (the possibility of such a choice of ρ is established in [9]).

We define local stable and unstable manifolds of a hyperbolic fixed point p as follows. Let, for definiteness, $p = 0$.

It is known (see [2, Theorem 1 of Section V.3]) that one can introduce coordinates (ζ, η) (where coordinate ζ has the dimension of the unstable manifold of p) near the origin and find a number $r_0 > 0$ and mappings H and Z of class C^1 defined for $|\zeta| < r_0$ and $|\eta| < r_0$, respectively, such that

$$H(0) = 0, \quad \frac{\partial H}{\partial \zeta}(0) = 0, \quad Z(0) = 0, \quad \frac{\partial Z}{\partial \eta}(0) = 0,$$

and the sets M_+ and M_- given by the equalities

$$M_+ : \eta = H(\zeta), \quad |\zeta| < r_0, \quad \text{and} \quad M_- : \zeta = Z(\eta), \quad |\eta| < r_0,$$

have the following properties:

• if $x \in M_+$, then $f(x) \in M_+$;

• if $x \in M_-$, then there exists a point $y \in M_-$ such that $x = f(y)$ (thus, negative semitrajectories of points in M_- belong to M_-);

• there exist numbers $K \ge 1$ and $\mu \in (0, 1)$ such that

$$\left| f^k(x) - f^k(y) \right| \le K\mu^k |x - y|, \quad x, y \in M_+, \quad k \ge 0, \tag{1}$$

and

$$\left| f^{-k}(x) - f^{-k}(y) \right| \le K\mu^k |x - y|, \quad x, y \in M_-, \quad k \ge 0;$$

• if $f^k(x) \in B(r_0, p)$ for $k \ge 0$, then $x \in M_+$;

• if $f^k(x) \in B(r_0, p)$ for $k \le 0$, then $x \in M_-$.

Assume that the number r_0 is chosen so that

$$\left\| \frac{\partial H}{\partial \zeta}(\zeta) \right\| < 1, \quad |\zeta| < r_0, \tag{2}$$

and

$$\left\| \frac{\partial Z}{\partial \eta}(\eta) \right\| < 1, \quad |\eta| < r_0.$$

We assume that $2\rho < r_0$ and define the local stable manifold of size 2ρ of the hyperbolic fixed point p by the formula

$$W^s_{loc}(p) = M_+ \cap B(2\rho, p)$$

and the local unstable manifold of size 2ρ of the hyperbolic fixed point p by the formula

$$W^u_{loc}(p) = M_- \cap B(2\rho, p).$$

In what follows, we fix a ρ having the above properties and the corresponding sets $W^s_{loc}(p)$ and $W^u_{loc}(p)$.

Now we formulate several known or obvious statements. We emphasize that in these statements 2.1–2.4 we consider only pseudotrajectories and fixed points belonging to the attractor $\mathcal{A}$.

Statement 2.1. *If U_1 is any open set containing all the fixed points of f, then there exist numbers d_1^* and T_1^* such that if ξ is a d_1^*-pseudotrajectory and $\xi(u,v) \cap U_1 = \emptyset$ for some $u < v$, then $v - u \le T_1^*$.*

This statement is proved in [10].

Statement 2.2. *For any neighborhoods $U_2(p) \subset U(p)$ there exist d_2^* and δ_2^* such that if ξ is a d_2^*-pseudotrajectory, $u < v$, $\xi(u) \in B(\delta_2^*, p)$, and $\xi(v) \notin U_2(p)$, then $\xi(k) \notin B(\delta_2^*, p)$ for any $k \ge v$.*

This statement is proved in [10].

Statement 2.3. *For any $\varepsilon > 0$ and $T > 0$ there exists a d_3^* such that if ξ is a d_3^*-pseudotrajectory, $n \in \mathbb{Z}$, $y \in \mathcal{A}$, and $|y - \xi_n| < d_3^*$, then*

$$|f^k(y) - \xi(n+k))| < \varepsilon, \quad -T \le k \le T.$$

This statement follows from the continuity of f and compactness of $\mathcal{A}$.

Statement 2.4. *For any indices $-\infty \le u < v < w \le +\infty$, any pseudotrajectory ξ, and any $\varepsilon > 0$, the following holds: if $\xi(u,v)$ and $\xi(v,w)$ are oriented ε-shadowed by elementary pseudotrajectories, then $\xi(u,w)$ is oriented ε-shadowed by an elementary pseudotrajectory.*

This statement obviously follows from the definition.

Lemma 2.1. *For any $T > 0$ and $\Delta > 0$ there exists a $\Delta_1 > 0$ such that if p and q are fixed points, $f^{-T}(y) \in W^u_{loc}(p)$ (hence, $y \in \mathcal{A}$), $z \in W^s_{loc}(q)$, and $|y - z| < \Delta_1$, then there exists a point $x \in W^u(p) \cap W^s(q)$ such that $|x - y| < \Delta$.*

Proof. Since $\mathcal{A}$ contains a finite number of fixed points, it is enough to prove our statement for a fixed pair p, q.

Assume that the statement does not hold. Then, there exist $T_0 > 0$ and $\Delta_0 > 0$ such that for any $n > 0$ there are points y_n and z_n such that $f^{-T_0}(y_n) \in W^u_{loc}(p)$, $z_n \in W^s_{loc}(q)$, and $|y_n - z_n| < 1/n$, while there is no point $x \in W^u(p) \cap W^s(q)$ satisfying the inequality $|x - y_n| < \Delta$.

Since $y_n \in \mathcal{A}$ and $\mathcal{A}$ is compact, passing to a subsequence, if necessary, we may assume that $y_n \to y \in \mathcal{A}$. We claim that $y \in W^u(p)$. Otherwise, for any N there exists an index $s > N$ such that $|f^{-s}(y) - p| > \rho$.

On the other hand, for any $s > 0$ and $y_n' = f^{(-T_0+s)}(y_n) \in W^u_{loc}(p)$ with an arbitrary n we have the inequality $|f^s(y_n') - p| < \rho$. This implies that $|f^{-t}(y) - p| \le \rho$ for any $t > T_0$. The obtained contradiction proves the claim.

Since $z_n \to y$, a similar reasoning shows that $y \in W^s(q)$. Thus, $y \in W^u(p) \cap W^s(q)$, and the contradiction with the relation $|y_n - y| \to 0$ completes the proof of our lemma.

Lemma 2.2. *Let $U'(r)$ be some neighborhoods of fixed points r and let $p \ne q$ be two selected fixed points. For any $\varepsilon > 0$ there exist numbers $d^* = d^*(p,q)$ and $\delta^* = \delta^*(p,q)$ such that if $\xi \subset \mathcal{A}$ is a d^*-pseudotrajectory, $u < s$,*

$$\xi(u) \in B(\delta^*, p), \quad \xi(s) \in B(\delta^*, q), \tag{3}$$

and

$$\xi(u,s) \cap \left(\bigcup_{r \ne p,q} U'(r) \right) = \emptyset,$$

then $\xi(u,s)$ is oriented ε-shadowed by the trajectory of some point $x \in W^u(p) \cap W^s(q)$.

Proof. In the following proof, we select small numbers $d_0, d_1, \dots$ and $\delta_1, \delta_2, \dots$; at every step, it is assumed that δ_{k+1} is less than the previously chosen δ_k and d_k is not greater than δ_k.

Finally, we take as d^* and δ^* the smallest d_k and δ_k chosen in the proof. By the choice of the neighborhoods $U(p)$ and $U(q)$, f is Lipschitz shadowing in $U(p)$ and $U(q)$; fix constants d_0 and L such that any d-pseudotrajectory in $U(p)$ and $U(q)$ with $d \le d_0$ is Ld-shadowed.

Find indices $u < m < n < s$ such that

$$\xi(u,m) \subset U(p), \quad \xi(m+1) \notin U(p),$$

and

$$\xi(n,s) \subset U(q), \quad \xi(n-1) \notin U(q),$$

then by Statements 2.1 and 2.2 there exists a d_1 and a constant $T = T(p,q)$ such that if ξ is a d_1-pseudotrajectory, then $n - m \leq T$.

Apply condition (3) with a small $\delta = \delta_1 < \min(d_0/2, \rho/2)$ to find a point $\xi(u)$ of the pseudotrajectory ξ such that $|\xi(u) - p| < \delta$.

Let $\xi(u) = (\zeta_u, \eta_u)$ in coordinates applied to define the local stable and unstable manifolds of p. Then, $|\zeta_u| < \delta$, and it follows from estimate (2) that $|H(\zeta_u)| < \delta$. Hence, for the point $y_u = (\zeta_u, H(\zeta_u)) \in M_-$, the inequality $|y_u - p| < 2\delta < \rho$ holds, and $y_u \in W^u_{loc}(p)$.

Due to inequalities (1), $|f^k(y_u) - p| \leq 2K\delta$ for $k \geq 0$, and if $2K\delta_1 < \rho$, then $f^k(y_u) \in U(p)$ for $k \leq 0$.

Construct a sequence $\xi_1(k)$ for $-\infty < k \leq m$ by setting

$$\xi_1(u-k) = f^{-k}(y_u), \quad k > 0, \quad \text{and} \quad \xi_1(k) = \xi(k), \quad u \leq k \leq m.$$

Since

$$|\xi(u) - y_u| = |\eta_u - H(\eta_u)| \leq 2\delta,$$

we get the estimates

$$|f(\xi_1(u-1)) - \xi_1(u)| = |y_u - \xi(u)| \leq 2\delta$$

(we take into account that $d \leq \delta$). Thus, the sequence ξ_1 is a 2δ-pseudotrajectory in $U(p)$ with $2\delta < d_0$, and there exists a point Y such that

$$|f^{k-m}(Y) - \xi_1(k)| \leq D_p(\delta) := 2L\delta, \quad -\infty < k \leq m.$$

Note that

$$|\xi_1(k) - p| \leq K|\xi_1(u) - p| = K|\xi(u) - p| \leq K\delta, \quad k \leq 0,$$

and

$$|\xi_1(k) - p| = |\xi(k) - p| \leq \rho, \quad u < k \leq m;$$

hence, we can estimate

$$|f^{k-m}(Y) - p| \leq |f^{k-m}(Y) - \xi_1(k)| + |\xi_1(k) - p| \leq D_p(\delta) + \max(K\delta, \rho), \quad k \leq m,$$

and if $D_p(\delta_1) + \max(K\delta_1, \rho) < 2\rho$, then the negative semitrajectory of Y lies in $B(2\rho, p)$; thus, $Y \in W^u_{loc}(p)$.

Finally,

$$|Y - \xi(m)| = |Y - \xi_1(m)| \leq D_p(\delta). \tag{4}$$

In a similar way, we construct a pseudotrajectory $\xi_1(k)$ for $k \geq n$ by finding a point $z_s \in W^s(q)$ such that $|z_s - \xi(s)| < \delta \leq \delta_2$ with a proper δ_2 and setting

$$\xi_1(s+k) = f^k(z_s), \quad k \geq 0, \quad \text{and} \quad \xi_1(k) = \xi(k), \quad n \leq k < s.$$

Since

$$|f(\xi_1(s-1)) - \xi_1(s)| = |f(\xi(s-1)) - z_s| \leq |f(\xi(s-1))) - \xi(s)| + |\xi(s) - z_s)| \leq d + \delta,$$

and since $d \leq \delta$, the sequence ξ_1 is a 2δ-pseudotrajectory in $U(q)$ with $2\delta < d_0$, and the same reasoning as above shows that there exists a point $Z \in W^s_{loc}(q)$ such that

$$|f^{k-n}(Z) - \xi_1(k)| \leq D_q(\delta), \quad n \leq k < \infty, \tag{5}$$

where $D_q(\delta) \to 0$ as $\delta \to 0$.

To complete the proof, fix an $\varepsilon > 0$. Since $\xi \subset \mathcal{A}$, it follows from Statement 2.3 that there exists a $\Delta > 0$ such that if $t \in \mathcal{A}$ and $|t - \xi(n)| < \Delta$, then

$$|f^k(t) - \xi(n+k)| < \frac{\varepsilon}{2K}, \quad m - n \le k \le 0. \tag{6}$$

Apply Lemma 2.1 to find a $\Delta_1 > 0$ (depending only on Δ and T and not depending on y and z) such that if $f^{-T}(y) \in W^u_{loc}(p)$, $z \in W^s_{loc}(q)$, and $|y - z| < \Delta_1$, then there is a point $x \in W^u(p) \cap W^s(q)$ with

$$|x - y|, |x - z| < \frac{\Delta}{2}.$$

Since $\xi \subset \mathcal{A}$ and $Y \in \mathcal{A}$, it follows from Statement 2.3 and (4) that there is a $\delta_3 > 0$ such that if $\delta < \delta_3$, then

$$|f^k(Y) - \xi(m+k)| < \frac{\Delta_1}{2}, \quad 0 \le k \le n - m,$$

in particular,

$$|f^{n-m}(Y) - \xi(n)| < \frac{\Delta_1}{2}. \tag{7}$$

Take $y = f^{n-m}(Y) \in W^u(p)$ and $z = Z \in W^s(q)$ (note that $f^{-T}(y) \in W^u_{loc}(p)$ and $z \in W^s_{loc}(q)$); by (5) with $k = n$ and (7),

$$|y - z| < \frac{\Delta_1}{2} + D_q(\delta),$$

which gives us a point $x \in W^u(p) \cap W^s(q)$ with $|x - \xi(n)| < \Delta$ if $\Delta_1/2 > D_q(\delta_3)$. Then, relations (6) with $t = x$ and $k - m - n$ are satisfied, and

$$|f^{m-n}(x) - Y| < \frac{\varepsilon}{2K} + D_p(\delta).$$

Denote $\chi = f^{m-n}(x)$. Note that

$$|\chi - p| \le |\chi - Y| + |Y - p| < \frac{\varepsilon}{2K} + 2D_p(\delta) + \rho < 2\rho$$

if

$$\frac{\varepsilon}{2K} + 2D_p(\delta) < \rho$$

(which we assume to hold for $\delta < \delta_4$).

Since $\chi, Y \in W^u_{loc}(p)$,

$$\left| f^{m-n-k}(x) - f^{-k}(Y) \right| \le \frac{\varepsilon}{2} + KD_p(\delta), \quad k \le 0,$$

and

$$\left| \xi(m-k) - f^{-k}(Y) \right| \le \left| \xi_1(m-k) - f^{-k}(Y) \right| + \delta \le D_p(\delta) + \delta, \quad 0 \le k \le m - u;$$

we see that there exists a δ_5 such that if

$$KD_p(\delta) + D_p(\delta) + \delta < \varepsilon/2$$

for $\delta < \delta_5$, then

$$\left| f^{m-n-k}(x) - \xi(m-k) \right| < \varepsilon, \quad 0 \le k \le m - u.$$

A similar estimate holds for $\xi(n+k)$ for $0 \le k \le s - n$, which, combined with estimate (6), completes the proof.

Let p and q be two different fixed points of f. We write $p \succ q$ if $W^u(p) \cap W^s(q) \ne \emptyset$. Note that the relation $p \succ q$ implies the inequality $V(p) > V(q)$. Hence, the "no cycle" condition is satisfied: relations

$$p \succ r_1 \succ \cdots \succ r_m \succ p$$

are impossible, and we can define the number $\mathcal{L}(p,q)$ as the maximal number m for which there exist fixed points $r_1,\dots,r_{m-1}$ such that

$$[p \succ r_1 \succ \cdots \succ r_{m-1} \succ q]$$

If such points $r_1,\dots,r_{m-1}$ do not exist, we set $\mathcal{L}(p,q)=+\infty$.

It is easily seen that if $\mathcal{L}(p,r)<+\infty$ and $\mathcal{L}(r,q)<+\infty$, then $\mathcal{L}(p,q)<+\infty$ and $\mathcal{L}(p,q)\ge\mathcal{L}(p,r)+\mathcal{L}(r,q)$.

Lemma 2.3. *There exist numbers d_4^* and δ_4^* such that if $\xi\subset\mathcal{A}$ is a d_4^*-pseudotrajectory, p and q are different fixed points, and*

$$\xi(u)\in B(\delta_4^*,p),\quad \xi(v)\in B(\delta_4^*,q)$$

for some $u<v$, then $\mathcal{L}(p,q)<+\infty$.

Proof. To get a contradiction, assume that for any natural k there exist $1/k$-pseudotrajectories $\xi_k\subset\mathcal{A}$, different fixed points p_k and q_k, and numbers $u_k<v_k$ such that $\mathcal{L}(p_k,q_k)=+\infty$,

$$\xi_k(u_k)\in B(1/k,p_k),\quad \text{and}\quad \xi_k(v_k)\in B(1/k,q_k).$$

Without loss of generality, we may assume that $p_k=p$ and $q_k=q$ for all k.

Recall that $U(p)$ is a neighborhood of p containing no other fixed points. Denote

$$m_k=\min\{t\in[u_k,v_k]:\ \xi_k(t)\notin U(p)\}.$$

Since $\mathcal{A}$ is compact, passing to a subsequence, we may assume that $\xi_k(m_k)\to z_p\in\mathcal{A}$. Then, $z_p\in W^u(p)$. Besides, there exists a fixed point r_1 such that $z_p\in W^s(r_1)$. If $r_1=q$, then $\mathcal{L}(p,q)<+\infty$, and we get a contradiction with the relation $\mathcal{L}(p,q)=+\infty$. Otherwise, we find indices $w_k\in[u_k,v_k]$ such that $|\xi_k(w_k)-r_1|\to 0$ and repeat the above process getting a fixed point r_2, and so on.

Due to the "no cycle" condition, this process cannot be infinite, and finally we get a fixed point $r_m=q$, so that $\mathcal{L}(p,q)<+\infty$. This contradiction completes the proof.

Lemma 2.4. *For any $\varepsilon>0$ and fixed points $p\neq q$ there exist $d_5^*(p,q)>0$ and $\delta_5^*(p,q)>0$ such that if $\xi\subset\mathcal{A}$ is a $d_5^*(p,q)$-pseudotrajectory,*

$$\xi(u)\in B(\delta_5^*(p,q),p),\quad \text{and}\quad \xi(v)\in B(\delta_5^*(p,q),q) \tag{8}$$

for some $v>u$, then $\xi(u,v)$ is oriented ε-shadowed by an elementary pseudotrajectory. In particular, in this case, $V(p)>V(q)$.

Proof. Let d_4^* and δ_4^* be given by Lemma 2.3.

If $p\neq q$ and $\mathcal{L}(p,q)=+\infty$, let $d_5^*(p,q)=d_4^*$ and $\delta_5^*(p,q)=\delta_4^*$, then such a $d_5^*(p,q)$-pseudotrajectory cannot exist, and the statement of the lemma obviously holds. In what follows, we only consider $p\neq q$ such that $\mathcal{L}(p,q)<+\infty$ and a d_4^*-pseudotrajectory ξ satisfying relations (8) for some $u<v$. Note that in this case, $V(p)>V(q)$.

We prove the result using induction on $\mathcal{L}(p,q)$. Fix an $\varepsilon>0$ and take the balls $B(\delta_4^*,\pi_i)$ as the neighborhoods $U'(\pi_i)$ in Lemma 2.2 to get the corresponding values d^* and δ^* (assuming, in addition, that $d^*<d_4^*$ and $\delta^*<\delta_4^*$).

For any pair of distinct fixed points p and q with $\mathcal{L}(p,q)=1$ set $d_5^*(p,q)=d^*$ and $\delta_5^*(p,q)=\delta^*$. Assume that $\xi\subset\mathcal{A}$ is a d^*-pseudotrajectory satisfying (8).

If there exists a fixed point r different from p and q and such that $\xi(u,v)\cap B(\delta_4^*,r)\neq\emptyset$, then Lemma 2.3 implies that $\mathcal{L}(p,q)\ge\mathcal{L}(p,r)+\mathcal{L}(r,q)$, and we get a contradiction. Thus, our statement is proved by Lemma 2.2 in the case $\mathcal{L}(p,q)=1$.

Take a natural $l>1$ and assume that we have constructed all the desired values $d_5^*(p,q)$ and $\delta_5^*(p,q)$ for all fixed points p,q with $\mathcal{L}(p,q)<l$. Now we consider the case of points p,q with $\mathcal{L}(p,q)=l$.

Take $d_5'=\min\{d_5^*(p,q):\ \mathcal{L}(p,q)<l\}$ and $\delta_5'=\min\{\delta_5^*(p,q):\ \mathcal{L}(p,q)<l\}$. Applying Lemma 2.2 with the fixed ε and $U'(\pi_i)=B(\delta_5',\pi_i)$, we find the corresponding d^* and δ^* (assuming that $d^*<d_5'$ and $\delta^*<\delta_5'$) for which the conclusion of Lemma 2.2 holds.

For any $p\neq q$ with $L(p,q)=l$ set $d_5^*(p,q)=d^*$ and $\delta_5^*(p,q)=\delta^*$. Let $\xi\subset A$ be a $d^*(p,q)$-pseudotrajectory that satisfies relation (8) with indices $u<v$.

If there exists a fixed point $r \neq p, q$ such that $\xi(w) \in B(\delta_5', r)$ for some $w \in (u, v)$, then Lemma 2.3 impies the relation $l = \mathcal{L}(p, q) \geq \mathcal{L}(p, r) + \mathcal{L}(r, q)$. In this case, $\mathcal{L}(p, r), \mathcal{L}(r, q) < l$, and by the induction hypothesis, both $\xi(u, w)$ and $\xi(w, v)$ are oriented ε-shadowed by elementary pseudotrajectories. In this case, $\xi(u, v)$ is oriented ε-shadowed by an elementary pseudotrajectory by Statement 2.4.

Otherwise,

$$\xi(u, v) \cap \left(\bigcup_{r \neq p, q} B(\delta_2', r) \right) = \emptyset,$$

and the conclusion of our lemma follows from Lemma 2.2.

Now we prove the main result of the paper.

Theorem 2.1. *Under the conditions formulated in Section 1, for any $\varepsilon > 0$ there exists a $d > 0$ such that any d-pseudotrajectory $\xi \subset \mathcal{A}$ is oriented ε-shadowed by an elementary pseudotrajectory.*

Proof. Fix an $\varepsilon > 0$ and find the numbers $d_5^*(p, q)$ and $\delta_5^*(p, q)$ given by Lemma 2.4 for this ε and for pairs p, q of different fixed points. Let d_4^* and δ_4^* be given by Lemma 2.3.

Denote

$$d_5^* = \min\{d_5^*(p, q) : \ p \neq q \in \{\pi_1, \dots, \pi_N\}\} \quad \text{and} \quad \delta_5^* = \min\{\delta_5^*(p, q) : \ p \neq q \in \{\pi_1, \dots, \pi_N\}\}.$$

Take $U_2(\pi_i) = B(\varepsilon/2, \pi_i)$ in Statement 2.2 and find the corresponding d_2^* and δ_2^*. Set $\delta = \min(\delta_2^*, \delta_4^*, \delta_5^*)$. Take

$$U_1 = \bigcup_{k=1}^{N} B(\delta, \pi_k)$$

in Statement 2.1 and find the corresponding d_1^* and T_1^*. Let $d = \min(d_1^*, d_2^*, d_4^*, d_5^*)$. Let $\xi \subset \mathcal{A}$ be a d-pseudotrajectory. Denote by

$$p_1, \dots, p_s \in \{\pi_1, \dots, \pi_N\}, \quad 1 \leq s \leq N,$$

all the different fixed points π_i such that $\xi \cap B(\delta, \pi_i) \neq \emptyset$. By Statement 2.1, $s \geq 1$. Fix indices u_j such that

$$\xi(u_j) \in B(\delta, p_j), \quad j = 1, \dots, s.$$

Without loss of generality, we assume that $u_1 < u_2 < \cdots < u_s$. By Lemma 2.3, this implies that $V(p_1) > \cdots > V(p_s)$.

We consider the following two cases.

Case 1: $s = 1$. We claim that $\xi \subset B(\varepsilon/2, p_1)$. Otherwise, there exists an index v such that $\xi(v) \notin B(\varepsilon/2, p_1)$. First, we assume that $v > u_1$. By Statement 2.2, $\xi(k) \notin B(\delta, p_1)$ for $k \geq v$. Then, it follows from Statement 2.1 that there exists a fixed point q such that $\xi(v, +\infty) \cap B(\delta, q) \neq \emptyset$. By Lemma 2.3, $V(p_1) > V(q)$, which contradicts the assumption that $s = 1$. The case $v < u_1$ is considered similarly.

Case 2: $s \geq 2$. Similarly to the proof of Case 1, we can show that $\xi(u_s, +\infty)$ is oriented ε-shadowed by the trajectory of p_s and $\xi(-\infty, u_1)$ is oriented ε-shadowed by the trajectory of p_1. We prove only the second statement.

We claim that $\xi(-\infty, u_1) \subset B(\varepsilon/2, p_1)$ from which our statement follows. Otherwise, there exists a $v < u_1$ such that $\xi(k) \notin B(\delta, p_1)$. Then, it follows from Statement 2.2 that $\xi(k) \notin (\delta, p_1)$ for $k \leq v$. Now Statement 2.1 implies that there exists a fixed point q such that $\xi(-\infty, v) \cap B(\delta, q) \neq \emptyset$. Then, $V(q) > V(p_1)$ by Lemma 2.3, which contradicts the choice of p_1.

Applying Lemma 2.4, we conclude that $\xi(u_1, u_s)$ is oriented ε-shadowed by an elementary pseudotrajectory. To complete the proof, it remains to apply Statement 2.4.

3. ORIENTED SHADOWING BY ELEMENTARY PSEUDOTRAJECTORIES IN A NEIGHBORHOOD OF AN ATTRACTOR

Now we study how pseudotrajectories in a neighborhood of an attractor can be shadowed by trajectories belonging to the attractor.

Given an $\varepsilon > 0$, we define the ε-neighborhood of the attractor $\mathcal{A}$ in a standard way:

$$\mathcal{N}_\varepsilon(A) = \{x \in \mathcal{X} : \text{dist}(x, \mathcal{A}) < \varepsilon\}.$$

The set $\mathcal{A}$ is compact, so for any neighborhood U of $\mathcal{A}$ there is an $\varepsilon > 0$ such that $\mathcal{N}_\varepsilon(\mathcal{A}) \subset U$. Besides, all the sets $\mathcal{N}_\varepsilon(\mathcal{A})$ are bounded.

If the space $\mathcal{X}$ is finite-dimensional, any attractor $\mathcal{A}$ of an invertible mapping f has a neighborhood U_0 such that if $x \in U_0 \setminus \mathcal{A}$, then the negative semitrajectory of x leaves U_0. A similar statement holds true in many infinite-dimensional cases. Meanwhile, it is easy to construct pseudotrajectories that stay in a small neighborhood of $\mathcal{A}$.

Lemma 3.1. *Assume that the mapping f is uniformly continuous in a neighborhood of $\mathcal{A}$. Then, for any $\varepsilon > 0$ there exist $\delta > 0$ and $d > 0$ such that for any d-pseudotrajectory $\{x_k \in \mathcal{N}_\delta(\mathcal{A}) : k \in \mathbb{Z}\}$ there exists a sequence $\{y_k \in \mathcal{A} : k \in \mathbb{Z}\}$ such that*

$$\text{dist}(x_k, y_k) < \varepsilon \tag{9}$$

and

$$\text{dist}(f(y_k), y_{k+1}) < \varepsilon. \tag{10}$$

Proof. Let $\mathcal{N}_{\delta_0}(\mathcal{A})$ be a neighborhood of $\mathcal{A}$ in which f is uniformly continuous. Fix an $\varepsilon > 0$ and find a $\delta < \min(\varepsilon/4, \delta_0)$ such that for $x, y \in \mathcal{N}_{\delta_0}(\mathcal{A})$, the inequality $\text{dist}(x, y) < \delta$ implies that $\text{dist}(f(x), f(y)) < \varepsilon/4$.

Let $\{x_k \in \mathcal{N}_\delta(\mathcal{A}) : k \in \mathbb{Z}\}$ be a d-pseudotrajectory of f with $d < \delta$. Find for any point x_k a point $y_k \in \mathcal{A}$ such that $\text{dist}(x_k, y_k) < \delta$. Then, inequalities (9) obviously hold and

$$\text{dist}(f(y_k), y_{k+1}) \leq \text{dist}(x_{k+1}, y_{k+1}) + \text{dist}(f(x_k), x_{k+1}) + \text{dist}(f(y_k), f(x_k)) < \delta + d + \frac{\varepsilon}{4} < \varepsilon,$$

which proves (10).

To say more about the shadowing near an attractor, one has to impose some sharper conditions on the rate of convergence to $\mathcal{A}$. In fact, we follow the idea of Theorem 3.4.1 of [9].

Assume that there exists a positive number B and a nonnegative function $a(b, n)$ defined for $b \in [0, B)$ and $n \geq 0$ and having the following properties:

(a1) The function a is bounded, i.e., there exists an $\alpha > 0$ such that $a(b, n) \leq \alpha$ for $b \in [0, B)$ and $n \geq 0$;

(a2) if $x \in \mathcal{N}_B(\mathcal{A})$, then

$$\text{dist}(f^n(x), \mathcal{A}) \leq a((\text{dist}(x, \mathcal{A}), n), \quad n \geq 0;$$

and

(a3) for any $\delta > 0$ there exists an $n_0 = n_0(\delta)$ such that

$$a(b, n) \leq \delta, \quad b \in [0, B), \quad n \geq n_0.$$

In the often-studied case of so-called exponential attractors, one takes

$$a(b, n) = a_0 b \mu^n \quad \text{with} \quad a_0 \geq 1, \quad \mu \in (0, 1).$$

Lemma 3.2. *Assume that a mapping f is Lipschitz continuous in a the ε-neighborhood $\mathcal{N}_\varepsilon(\mathcal{A})$ of the attractor $\mathcal{A}$ and there exist a $B \in (0, \varepsilon)$ and a function a defined for $b \in [0, B)$ and $n \geq 0$ and having properties (a1)−(a3). Then, there exists a number $B_1 > 0$ such that for any $\delta > 0$ we can find a $d > 0$ with the following property: If Y is a d-pseudotrajectory of f belonging to $\mathcal{N}_{B_1}(\mathcal{A})$, then Y is 2δ-shadowed by a $(2\delta(l + 1) + d)$-pseudotrajectory belonging to $\mathcal{A}$, where l is a Lipschitz constant of f in $\mathcal{N}_\varepsilon(\mathcal{A})$.*

Proof. Take $B_1 = \min(B, B/(2\alpha))$. Clearly, if $y \in \mathcal{N}_{B_1}(\mathcal{A})$, then

$$\text{dist}(f^k(y), \mathcal{A}) \le \alpha B_1 < B, \quad k \ge 0. \tag{11}$$

Let $Y = \{y_n\}$ be a d-pseudotrajectory in $\mathcal{N}_{B_1}(\mathcal{A})$ (and hence in $\mathcal{N}_B(\mathcal{A})$). Take an arbitrary $\delta > 0$ and an index $n \in \mathbb{Z}$ and set $m = n - n_0$, where $n_0 = n_0(\delta)$. Then,

$$\text{dist }(f^{n_0}(y_m), \mathcal{A}) \le a(B, n_0) \le \delta.$$

Since $y_k \in \mathcal{N}_{B_1}(\mathcal{A})$ for $k \ge m$ and inequalities (11) hold

$$\text{dist}(f^{n_0}(y_m), y_n) \le d_1 := d\left(1 + l + \dots + l^{n_0-1}\right). \tag{12}$$

Then,

$$\text{dist}(y_n, \mathcal{A}) \le \text{dist}\left(f^{n_0}(y_m), \mathcal{A}\right) + \text{dist}\left(f^{n_0}(y_m), y_n)\right) \le \delta + d_1.$$

This inequality holds for any n. Since n_0 only depends on δ, it follows from (12) that we can find a d (also depending only on δ) such that

$$\text{dist}\,(y_n, \mathcal{A}) < 2\delta, \quad n \in \mathbb{Z}.$$

Find a point $x_n \in \mathcal{A}$ such that $|x_n - y_n| \le 2\delta$. Then,

$$|f(x_n) - x_{n+1}| \le |f(x_n) - f(y_n)| + |f(y_n) - y_{n+1}| + |y_{n+1} - x_{n+1}| \le \delta_1 := 2\delta(l+1) + d.$$

Thus, the pseudotrajectory Y is 2δ-shadowed by a δ_1-pseudotrajectory on the attractor.

Now the following statement is an obvious corollary of Theorem 2.1 and Lemma 3.2.

Theorem 3.1. *If a mapping f satisfies the assumptions of Theorem 2.1 and Lemma 3.2, then there is a neighborhood U of the attractor $\mathcal{A}$ with the following property: for any $\varepsilon > 0$ there exists a $d > 0$ such that any d-pseudotrajectory of f belonging to U is oriented ε-shadowed by an elementary pseudotrajectory.*

4. APPLICATION TO SEMIGROUPS GENERATED BY PARABOLIC PDES

As an application of our main result, we consider the infinite-dimensional semigroup generated by a parabolic PDE.

In this section, we refer to the results of the books [6, 7] and papers [3, 4]. We consider the boundary-value problem for a parabolic partial differential equation

$$u_t = u_{xx} + \mathcal{F}(u), \quad u \in \mathbb{R}, \quad t > 0, \quad x \in [0, 1], \tag{13}$$

with the Dirichlet boundary conditions

$$u(0, t) = u(1, t) = 0 \tag{14}$$

assuming that $\mathcal{F} \in C^2([0, 1])$ and that

$$u\mathcal{F}(u) \le C \tag{15}$$

for some $C > 0$.

Let $H^1 = H^1([0, 1])$ be the space of functions in $L^2([0, 1])$ having the derivative belonging to $L^2([0, 1])$ and endowed with the standard norm

$$||v||_{H^1} = \left(\int_0^1 |v|^2 dx\right)^{1/2} + \left(\int_0^1 |v_x|^2 dx\right)^{1/2}.$$

Let $\mathcal{H} := H_0^1$ be the closure in H_1 of the set of functions from $C_0^\infty([0, 1])$ (the C^∞ smooth functions v satisfying the condition $v(0) = v(1) = 0$).

It was shown in [7] that condition (15) implies the existence of a semigroup $\varphi : \mathbb{R}^+ \times \mathcal{H} \mapsto \mathcal{H}$ which represents the semiflow of the considered boundary value problem.

Moreover, that semiflow is gradient-like: the energy functional

$$\int\limits_0^1 \left(\frac{1}{2}u_x^2 - \mathcal{G}(\varphi(x))\right) dx$$

is decreasing along any non-constant trajectory of the flow φ. Here

$$\mathcal{G}(u) = \int\limits_0^1 \mathcal{F}(s)ds.$$

This is a Lyapunov function satisfying condition (LF).

We relate to the semigroup φ a mapping f of the space $\mathcal{H}$ by setting $f(v) = \varphi(1)v$ (the time-1 shift mapping). Since $\phi(t)$ is of class C^2 for $t > 0$ (which is well-known), f is of class C^2 as well.

If condition (15) is satisfied, the semigroup φ (and the mapping f) has a compact global attractor $\mathcal{A}$ in the space $\mathcal{H}$ [6]. It is also known that, under our conditions, $\varphi(t)u_0$ ($t > 0$, $u \in \mathcal{H}$) admits the Fréchet derivative with respect to u_0. Moreover, for any fixed $t > 0$ and any bounded set $X \subset \mathcal{H}$, there exists a $c(t, X)$ such that

$$|\varphi(t)v - \varphi(t)v_0| \leq c(t, X)|v - v_0|$$

for any $v, v_0 \in X$. Thus, the mapping f satisfies a Lipschitz condition in a neighborhood of the attractor.

To apply our previous results to the mapping f, let us recall some known statements.

A fixed point $p = p(x)$ of φ is a solution of the boundary-value problem

$$p_{xx} + \mathcal{F}(p) = 0, \quad p(0) = p(1) = 0. \tag{16}$$

Observe that p is also a fixed point for the mapping f. Clearly, all these fixed points of φ (and also of f) belong to the global attractor $\mathcal{A}$.

A fixed point p of φ is hyperbolic (in the sense of Section 1) if and only if 0 is not an eigenvalue of the linear variational operator

$$\Psi := \frac{d^2}{dx^2} + \mathcal{F}_u'(p(x))$$

with the Dirichlet boundary conditions (observe that all the eigenvalues of that operator are real, of multiplicity 1, and tend to $-\infty$). Clearly, a fixed point p of φ is hyperbolic if it is a hyperbolic fixed point of f.

Given a positive integer $k \geq 2$, denote by $\mathfrak{G}$ the space of C^k smooth functions from $\mathbb{R}$ to $\mathbb{R}$ with the strong Whitney C^k-topology.

Let us mention the result of Corollary 3.2 of [3] (note that we have slightly changed the original formulation).

Theorem 4.1. *There is a residual set $\mathfrak{G}_1$ in $\mathfrak{G}$ such that for any $\mathcal{F} \in \mathfrak{G}_1$, all fixed points of φ are hyperbolic.*

This result implies that the mapping f satisfies condition (MS1) for a residual subset of functions $\mathcal{F}$.

Let p be a hyperbolic fixed point of φ (and of f). The stable manifold $W_\varphi^s(p)$ and the unstable manifold $W_\varphi^u(p)$ for the semigroup φ obviously coincide with the corresponding manifolds $W_f^s(p)$ and $W_f^u(p)$ for the mapping f (let us denote them simply by $W^s(p)$ and $W^u(p)$, respectively).

It follows from the stability of the global attractor $\mathcal{A}$ that for any fixed point p of f, its unstable manifold $W^u(p)$ is a subset of $\mathcal{A}$.

The compactness of the global attractor $\mathcal{A}$ implies that for any fixed point p of f, its unstable manifold $W^u(p)$ is finite-dimensional.

The spectrum of the operator $Df(p)$ is

$$\left\{\cdots < e^{\lambda_2 t} < e^{\lambda_1 t} < e^{\lambda_0 t}\right\},$$

where $\{\lambda_0 > \lambda_1 > \lambda_2 > \dots\}$ are the eigenvalues of Ψ. In this case, only a finite number of eigenvalues of Ψ is positive, say N. Clearly, N coincides with the index of p, i.e., the dimension of $W^u(p)$.

The manifold $W^u(p)$ is diffeomorphic to $\mathbb{R}^N$, where $N = N(p)$, and the semiflow on $W^u(p)$ (the restriction of φ) is conjugate to that generated by a C^1 vector field on $\mathbb{R}^N$ denoted by F. Identify p with with the origin in $\mathbb{R}^N$, then the linear part of F at p, $F'(p)$, has eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_{N-1} > 0$.

Consequently, the flow φ can be reduced to a finite-dimensional flow generated by an autonomous system of ordinary differential equations on any unstable manifold $W^u(p)$. Evidently, such a flow is invertible, hence it is invertible on the whole attractor $\mathcal{A}$.

We have shown that all the assumptions (A1)–(A3), (MS1), (MS2), and (LF) of Section 1 are satisfied for mappings f related to a residual set of the functions $\mathcal{F}$. Therefore, for a typical function $\mathcal{F}$, the time-1 shift for the semigroup φ of the problem (13)–(14) satisfies the conditions of our Theorem 2.1 (and, hence, of Theorem 3.1).

At the end of this section, we make a short comment concerning the transversality condition (MS3). It is well-known that for the semiflow generated by the parabolic equation (13) (as well as by some more general equations) with scalar variable u, the stable and unstable manifolds of two hyperbolic fixed points are always transverse (see, for example, [1]).

At the same time, Poláčik have constructed in [13] an example of a parabolic equation $u_t = \Delta u + \mathcal{F}(u)$ with the Dirichlet boundary condition on a bounded domain $\Omega \subset \mathbb{R}^2$ such that its semiflow has two hyperbolic fixed points whose stable and unstable are not transverse. In this paper, we do not discuss this example in detail.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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