ISSN 0147-6882, Scientific and Technical Information Processing, 2024, Vol. 51, No. 6, pp. 526–534. © Allerton Press, Inc., 2024.
Russian Text © The Author(s), 2023, published in Iskusstvennyi Intellekt i Prinyatie Reshenii, 2023, No. 4, pp. 82–93.

Multicriteria Choice Based on Interval Fuzzy Information

V. D. Noghin*
Saint Petersburg State University, Saint Petersburg, Russia
**e-mail: noghin@gmail.com*
Received September 30, 2023

Abstract—In this paper, we examine a class of multicriteria choice problems where the decision maker's preferences are modeled using interval-valued second-order fuzzy relations. We formulate foundational axioms of reasonable choice that, among other things, enable us to justify the Edgeworth–Pareto principle within this context. We introduce the concept of a quantum of interval fuzzy information, as well as the notion of a consistent set of such quanta. We establish a criterion for the consistency of a set of quanta and present a framework for utilizing interval fuzzy information to reduce the Pareto set. To illustrate the proposed approach, we analyze a detailed example.

Keywords: multicriteria choice, interval fuzzy relation, quantum of interval fuzzy information, consistency of quanta

DOI: 10.3103/S0147688224700461

INTRODUCTION

Optimal choice problems involving multiple numerical criteria form a broad and practically significant class of decision-making tasks. The objective in solving a multicriteria selection problem is to identify the best option from a set of feasible alternatives or, alternatively, to determine a subset of such alternatives. A fundamental tool in solving this class of problems is the Edgeworth–Pareto principle, which posits that only Pareto-optimal solutions can be considered best. However, the Pareto set is typically extensive, and final selection within this set often proves challenging. This gives rise to the problem of narrowing the Pareto set [1], which cannot be resolved without additional information from the decision maker (DM).

The primary source of this additional information lies in the preferences of the DM, which are expressed in the form of so-called quanta of information [1]. In the simplest case, possessing such a quantum allows the DM to eliminate one of two compared alternatives, thereby simplifying the selection process. By adopting certain axioms that regulate the selection procedure, the use of a single quantum can significantly reduce the set of Pareto-optimal options. When a set of quanta is available, a substantial narrowing of the Pareto set becomes possible, thereby greatly simplifying the decision-making process.

The identification of quanta of information typically takes place through dialogue with the DM, but this process can present certain challenges. When multiple criteria are involved, determining which of two alternatives is preferable can often be difficult for the DM. On the one hand, one option may exhibit several advantages over the other, offering a strong basis for considering it a good choice within the pair. On the other hand, the same option may have drawbacks that render its selection questionable, relegating it to the category of bad options. In other words, for the DM, classifying potential choices as simply good or bad may be overly simplistic in such situations. A more flexible and useful approach involves fuzzy set theory, where the DM is asked to assign a specific value from the interval [0,1] to a pair of alternatives. This value represents the degree of confidence that one option is clearly preferable to the other.

Recently, second-order fuzzy sets have acquired significant traction in applied research, with a substantial body of work devoted to their study and application [2–6]. Of particular interest is a simpler class of these sets—interval second-order fuzzy sets [7–10]. First introduced in [7], these sets are frequently used in various applied studies [11–12].

In this paper, we assume that the DM's preference relation is modeled as an interval second-order fuzzy relation. Such a relation allows for the modeling of uncertain situations, where the DM's confidence that one option is preferable to another is blurred due to measurement inaccuracies or other factors and thus falls within a certain range.

Section 1 introduces the fundamental concepts of fuzzy set theory, including fuzzy sets and second-order fuzzy relations. In Section 2, we formulate the multicriteria choice problem with an interval second-order fuzzy preference relation, alongside the axioms of rational choice and the Edgeworth–Pareto principle for this class of problems. Section 3 introduces the

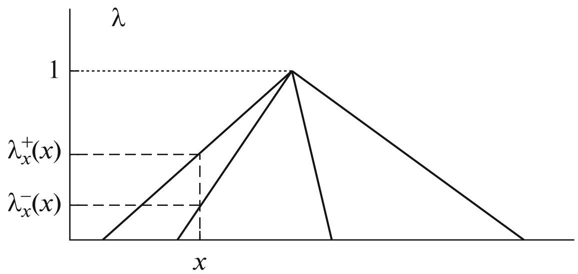

Fig. 1. Example of an IVFR2 on a numerical axis.

central concept of this work—the quantum of interval fuzzy information regarding the DM's preference relation. This section also outlines the process of using one or several quanta to narrow the Pareto set. Section 4 defines what constitutes a consistent set of quanta and provides the necessary and sufficient conditions for such consistency. The article concludes with Section 5, which presents an illustrative example of using quanta of interval fuzzy information to narrow the Pareto set according to the proposed framework. Proofs of the Edgeworth–Pareto principle and the consistency criterion are included in the Appendix.

1. FUZZY SET INFORMATION

A fuzzy set (first order) X within a universal set A is defined by its membership function, $\lambda_X : A \to [0,1]$. For each element $x \in A$, the value $\lambda_X(x)$ is interpreted as the degree of membership of x in the fuzzy set X. A fuzzy set can be viewed as a collection of ordered pairs $(x, \lambda_X(\cdot))$, where $x \in A$ and $\lambda_X(x) \in [0,1]$. This is often how fuzzy sets are formally defined.

When A corresponds to the set of real numbers $A = \mathbf{R}$, the fuzzy set is referred to as a *fuzzy number*. A fuzzy set is considered *normal* if the supremum of its membership function $\lambda_X(\cdot)$ over the set A equals 1. *A fuzzy number is,* by definition, a normal convex fuzzy set. *Convex fuzzy sets* are characterized by the property that their α-level sets, defined as $\{x \in \mathbf{R} \mid \lambda_X(x) \geqq \alpha\}$, form bounded convex subsets of the real line.

For a crisp set, unlike a fuzzy set, the membership degree is either $x \in X$ or $x \notin X$. Therefore, if $x \in X$ is a crisp set, then $\lambda_X(x) = 1$, and $\lambda_X(x) = 0$ if $x \notin X$. However, for a fuzzy set X, the membership degree can take any intermediate value between 0 and 1. The greater the membership degree $\lambda_X(x) \in (0,1)$, the less uncertainty that $\lambda_X(x)$, belongs to $x \in X$.

In practice, there may be situations where, due to a high level of uncertainty, it becomes difficult to assign a precise membership degree for any element X. In such cases, instead of assigning a specific number, one may use a first-order fuzzy number to represent the degree of membership. If the membership degree of an element x in set A is itself a fuzzy number defined on $[0,1]$, then X is called a second-order fuzzy set (2FS). Formally, 2FS X is defined as a set of triples $(x, u, \lambda_X(x,u))$, where $x \in A$, $u \in [0,1]$ and $\lambda_X(x,u) \in [0,1]$. The value u is called the primary membership degree, while $\lambda_X(x,u)$ is the secondary membership degree that is associated with element x. According to this definition, each element $x \in A$ is assigned a function $\lambda_X(x,\cdot)$ that is defined on $[0,1]$ and takes values within the same interval. This function describes the type of uncertainty related to the membership of x in the 2FS X.

An 2FS X is referred to as an *interval-valued fuzzy set* (IVFS) if, for each element $x \in A$ there exist two values $\lambda_X^-(x), \lambda_X^+(x) \in [0,1]$, denoted as the lower and upper bounds, such that the secondary membership degree $\lambda_X(x,\cdot)$ can be represented as:

$$\lambda_X(x,u) = \begin{Bmatrix} 1, & \text{if } \lambda_X^-(x) \leqq u \leqq \lambda_X^+(x), \\ 0, & \text{otherwise} \end{Bmatrix}$$

Another name for IVFS is an interval-valued fuzzy set [8]. An IVFS X is fully characterized by its *lower* and *upper bounds*, $\lambda_X^-(x)$ and $\lambda_X^+(x)$, which assign to each element $x \in A$ a range $[\lambda_X^-(x), \lambda_X^+(x)] \subset [0,1]$ of possible values representing the degree of membership of x to the set X (Fig. 1).

The greater the difference $\lambda_X^+(x) - \lambda_X^-(x)$, the higher the uncertainty associated with the second-order fuzziness of element x membership to set X. In the case of a first-order fuzzy set, there is no difference between the lower and upper bounds: $\lambda_X^-(x) \equiv \lambda_X^+(x)$.

Set-theoretic operations such as intersection and union, as well as the inclusion relation for IVFS, are defined as follows (for each $x \in A$):

- intersection $\lambda_{A\cap B}^-(x) = \min\{\lambda_A^-(x); \lambda_B^-(x)\}$, $\lambda_{A\cap B}^+(x) = \min\{\lambda_A^+(x); \lambda_B^+(x)\}$;
- union $\lambda_{A\cup B}^-(x) = \max\{\lambda_A^-(x); \lambda_B^-(x)\}$, $\lambda_{A\cup B}^+(x) = \max\{\lambda_A^+(x); \lambda_B^+(x)\}$;
- inclusion $A \subset B \Leftrightarrow \lambda_A^-(x) \leqq \lambda_B^-(x)$, $\lambda_A^+(x) \leqq \lambda_B^+(x)$.

It is easy to see that applying these operations results in another IVFS. In other words, the class of IVFS is closed under standard set-theoretic operations.

A *first-order fuzzy binary relation* is defined as a fuzzy subset of the Cartesian product $A \times A$. Its membership function takes two arguments: $\mu_A(\cdot,\cdot) : A \times A \to [0, 1]$.

An interval-valued fuzzy relation of the second order (IVFR2) s defined by two components—the upper bound $\mu_A^-(x,y)$ and the lower bound $\mu_A^+(x,y)$, which assign to each pair $(x,y) \in A \times A$ a set

$[\mu_A^-(x,y), \mu_A^+(x,y)] \subset [0,1]$ of possible confidence values that element x is in relation to element y. It is assumed that $\mu_A^-(x,y) \leqq \mu_A^+(x,y)$ for all pairs $(x,y) \in A \times A$.

IVFR2 is called

• irreflexive if there does not exist any $x \in A$, for which $\mu_A^+(x,x) = 0$;

• Asymmetric if $\mu_A^+(x,y) = 0$ for all $x, y \in A$, such that $\mu_A^-(y,x) > 0$;

• transitive if $\mu_A^-(x,z) \geqq \min\{\mu_A^-(x,y), \mu_A^-(y,z)\}$ and $\mu_A^+(x,z) \geqq \min\{\mu_A^+(x,y), \mu_A^+(y,z)\}$ for all $x, y, z \in A$.

It is easy to verify that every irreflexive and transitive interval-valued fuzzy relation is also asymmetric, just as in the crisp case.

2. MULTICRITERIA CHOICE PROBLEM WITH IVFR2

Let the DM be tasked with selecting one or more options from a crisp set of alternatives X of arbitrary nature. Let $C(X)$ denote the set of *selected alternatives*, which constitutes the solution to the multicriteria choice problem and is to be determined.

The distinctive feature of a multicriteria choice problem, as opposed to a single-criterion problem, is that there is no single, universally accepted idea of a solution. This distinction arises because the set $C(X)$ is usually formed in the decision-making process and depends not only on X, but also on various factors directly related to the DM. We do not introduce a strict definition of the set of selected alternatives because such a definition does not exist universally. Nevertheless, the following exposition remains mathematically consistent.

Each potential alternative in the set X is evaluated using a set of numerical criteria $f_1, f_2, \ldots, f_m$, that is defined on this set. Without loss of generality, we assume that each criterion should be maximized. It is clear that the best alternative, from a practical perspective, would be one that maximizes all m criteria simultaneously. However, such an alternative is rarely feasible, and the DM must typically reach a compromise to conclude the decision-making process. This compromise is entirely dependent on the DM and their individual preferences, which is why there is no single universally accepted solution concept for all potential DMs in a multicriteria selection problem.

To capture the preferences of the DM for the set X, we introduce an asymmetric binary relation $\succ_X$ called the DM's *preference relation*. In the case of a crisp preference relation, the expression $x' \succ_X x''$ for two alternatives x', x'' signifies that the DM would choose x' over x''. In this work, we consider the situation where the preference relation is represented by an interval-valued fuzzy relation of the second order (IVFR2). This allows us to formalize the DM's preferences by specifying bounds within which the degree of confidence that one option is preferable to another can vary. Such situations often arise due to inaccuracies or uncertainties in the DM's understanding of their own preferences.

Let $\mu^-_{\succ_X}(\cdot,\cdot)$ and $\mu^+_{\succ_X}(\cdot,\cdot)$ denote the lower and upper bounds of the IVFR2 preference relation $\succ_X$. Because the preference relation $\succ_X$, in the multicriteria selection problem IVFR2 is considered, it is natural to assume that the set of selected alternatives $C(X)$ also has an analogous nature, i.e., it is an interval-valued fuzzy set of the second order (IVFS2). We denote its lower and upper bounds by $\lambda^-_{C(X)}(\cdot)$ and $\lambda^+_{C(X)}(\cdot)$, respectively.

The following four axioms prescribe, in a certain sense, the rational behavior of the DM during the decision-making process:

Axiom 1. *The inequalities* $\lambda^-_{C(X)}(x) \leqq 1 - \mu^-_{\succ_X}(x', x)$ and $\lambda^+_{C(X)}(x) \leqq 1 - \mu^+_{\succ_X}(x', x)$ *hold for* $x, x' \in X$, $x \neq x'$.

In the crisp case, Axiom 1 requires that an alternative not selected from a pair (i.e., when $\mu^-_{\succ_X}(x,x') = \mu^+_{\succ_X}(x,x') = 1$) should not be selected from the entire set of possible alternatives (i.e., $\lambda^-_{C(X)}(x) = \lambda^+_{C(X)}(x) = 0$). In the fuzzy case, such categorical distinctions are softened due to the uncertainty that arises from the fuzziness of the preference relation.

For each element $x \in X$, its image $f(x) = (f_1(x), f_2(x), \ldots, f_m(x)) \in \mathbf{R}^m$ is called its feasible vector, and $Y = f[X] = \{f(x) \in \mathbf{R}^m \mid \text{for some } x \in X\}$ is the set of all feasible vectors. Here, $\mathbf{R}^m$ represents an arithmetic space of m-dimensional vectors with standard vector addition and scalar multiplication operations. This space is referred to as the *criterion space*.

The relation $\succ_X$ induces a corresponding preference relation $\succ_Y$ on the set Y as follows:

$$\mu^-_{\succ_Y}(f(x), f(x')) = \mu^-_{\succ_X}(x, x'),$$
$$\mu^+_{\succ_Y}(f(x), f(x')) = \mu^+_{\succ_X}(x, x'),$$

with $x, x' \in \tilde{X}$, where $\tilde{X}$ is the set of equivalence classes on X, induced by the equality relation on $\mathbf{R}^m$.

Axiom 2. *Across the entire space* $\mathbf{R}^m$, *there exists an irreflexive and transitive IVFR2 relation with components* $\mu^-(\cdot)$, $\mu^+(\cdot)$, *whose restriction to the set Y coincides with* c $\succ_Y$.

This axiom asserts that the fuzzy irreflexive and transitive relation $\succ_Y$ can be extended to the entire criterion space $\mathbf{R}^m$ while preserving the properties of irreflexivity and transitivity.

Axiom 3. *For any pair of vectors* $y, y' \in \mathbf{R}^m$, *such that* $y_i > y_i'$ *for some* $i \in \{1,2,\ldots,m\}$ *and* $y_j = y_j'$ *for all other* $j \neq i$, *it holds that* $\mu^-(y,y') = \mu^+(y,y') = 1$.

This axiom reflects the DM's interest in maximizing each of the given criteria.

Axiom 4. *The IVFR2 relation* $\succ$ *is invariant under any positive linear transformation, i.e.,*

$$\mu^-(\alpha x + c, \alpha y + c) = \mu^-(x,y),$$
$$\mu^+(\alpha x + c, \alpha y + c) = \mu^+(x,y);$$

for all $x, y, c \in \mathbf{R}^m$ *and every* $\alpha > 0$.

Recall that the Pareto relation $\geq$ on the space $\mathbf{R}^m$ is defined by the equivalence:

$$y \geq y' \Leftrightarrow y_i \geqq y_i', \quad i = 1,2,\ldots,m, y \neq y',$$

The (crisp) set of Pareto-optimal alternatives is then:

$$P_f(X) = \{x^* \in X \mid \text{does not exist } x \in X \text{ such that } f(x) \geq f(x^*)\}.$$

Pareto Axiom. *The inequality* $f(x') \geq f(x)$ *implies* $\mu^-(x',x) = \mu^+(x',x) = 1$.

It is easy to see that the Pareto Axiom implies Axiom 3, though the converse is not true.

Edgeworth–Pareto Principle. *Acceptance of Axiom 1 and the Pareto Axiom guarantees that for any IVFS2* $C(X)$ *the inclusion* $C(X) \subset P_f(X)$ *holds. In terms of membership functions, this is expressed by the inequality:*

$$\lambda_C^+(x) \leqq \lambda_P(x) \text{ for all } x \in X, \tag{1}$$

where $\lambda_P(x)$ *is the characteristic function of the Pareto set*.

For fuzzy second-order relations, this principle was established in [13]. Here, it is formulated and proven (see the Appendix) for the case of IVFR2. This principle plays a crucial role in multicriteria choice problems. According to it, for a sufficiently broad class of multicriteria delete problems—specifically, those that comply with Axiom 1 and the Pareto Axiom—any selection must lie within the Pareto set. It is noteworthy that transitivity of the preference relation is not required for the Edgeworth–Pareto principle to hold.

3. QUANTA OF FUZZY INFORMATION AND THE SCHEME OF THEIR USE

Assuming Axioms 1–4 are satisfied, we can now introduce the key concept of this work.

Definition 1. Let A and B be two non-empty, non-overlapping subsets of the index set of criteria $I = \{1,2,\ldots,m\}$. We say that a *quantum of IVFR2 information* is specified with these groups of criteria and two sets of positive parameters w_i for all $i \in A$ and w_j for along with a confidence interval $[\mu^-,\mu^+] \subset (0,1]$, $\mu^- \leqq \mu^+$, if $\mu^-(u,\mathbf{0}) = \mu^-$, $\mu^+(u,\mathbf{0}) = \mu^+$, where the vector $u \in \mathbf{R}^m$ has components:

$$u_i = w_i, \quad u_j = -w_j, \quad u_s = 0 \quad \text{for all } i \in A, \quad j \in B, \quad s \notin A \cup B. \tag{2}$$

In this definition, group $f_i, i \in A$ is considered to be more important than group B $f_j, j \in B$.

As noted, IVFR2 plays a central role in this definition, with its existence guaranteed by Axiom 2. The lower and upper components of this relation are denoted by $\mu^-(\cdot,\cdot)$ and $\mu^+(\cdot,\cdot)$.

According to Definition 1, any vector

$$u \in \mathbf{N}^m = \{u \in \mathbf{R}^m \mid \exists u_i, u_j \text{ such that } u_i > 0, u_j < 0\},$$

with components that are defined by equation (2) can form a quantum of IVFR2 information, provided that $\mu^-(u,\mathbf{0}) = \mu^-$ and $\mu^+(u,\mathbf{0}) = \mu^+$ hold for some values of $\mu^-,\mu^+ \in (0,1]$, $\mu^- \leqq \mu^+$. For instance, $\mu^-((2,-1),\mathbf{0}) = 0.6$ and $\mu^+((2,-1),\mathbf{0}) = 0.8$ define the simplest quantum of interval fuzzy information can specify that criterion 1 is more important than criterion 2 for the DM, with parameters $w_1 = 2$, $w_2 = 1$ and a confidence range $[0.6, 0.8]$.

Let us discuss the potential use of quanta of interval fuzzy information to reduce the Pareto set. In the case of a single quantum, we rely on the following result, which addresses the use of first-order fuzzy preference information. For this, we must define the set of selected vectors $C(Y)$ with a membership function

$$\lambda_Y^C(y) = \begin{cases} \lambda_X^C(x), & \text{if } y = f(x) \text{ for some } x \in X, \\ 0, & \text{otherwise,} \end{cases}$$

and the set of Pareto-optimal vectors $P(Y) = f(P_f(X))$ with a characteristic function:

$$\lambda_Y^P(y) = \lambda_X^P(x) \text{ for all } y = f(x), x \in \tilde{X},$$

where $\tilde{X}$ denotes the set of equivalence classes that are induced by the equality relation on $\mathbf{R}^m$.

Theorem 1 [1]. *Let a quantum of first-order fuzzy preference information be given, with non-empty, non-overlapping groups of criteria indices* A, $B \subset I$, *positive parameters* w_i, w_j *for all* $i \in A$, $j \in B$ *and a confidence level* $\mu \in (0,1]$. *Then, for any first-order fuzzy set of selected vectors with a membership function* $\lambda_Y^C(\cdot)$ *the following inequalities hold*:

$$\lambda_Y^C(\cdot) \leqq \lambda_M(y) \leqq \text{ for all } y \in Y,$$

where $\lambda_Y^P(\cdot)$ *is the characteristic function of the Pareto-optimal set, and* $\lambda_M(\cdot)$ *is the membership function, defined by:*

$$\lambda_M(y) = 1 - \sup_{z \in Y} \varsigma(z, y) \text{ for all } y \in Y,$$

$$\varsigma(z, y) = \begin{cases} 1, \text{ if } z - y \in \mathbf{R}_+^m, \\ \mu, \text{ if } \hat{z} - \hat{y} \in \mathbf{R}_+^p, \ z - y \notin \mathbf{R}_+^m, \\ 0, \text{ in other cases,} \end{cases}$$

$$\text{for all } y, z \in Y, \ y \neq z,$$

Here, $\mathbf{R}_+^p = \{y \in \mathbf{R}^p \mid y \geq \mathbf{0}\}$, $p = m - |B| + |A| \cdot |B|$, *the vector* $\hat{y}$ *(and likewise* $\hat{z}$*) consists of components* y_i, $i \in \{1, 2, \ldots, m\} \backslash B$, *(correspondingly,* z_i, $i \in \{1, 2, \ldots, m\} \setminus B$*), while the remaining components are* $w_j y_i + w_i y_j$ *(correspondingly,* $w_j z_i + w_i z_j$*) for all* $i \in A$, $j \in B$.

The formulation of Theorem 1 demonstrates that, to construct a first-order fuzzy set with membership function $\lambda_M(y)$, and thereby narrow down the Pareto set based on a quantum of first-order fuzzy information, two multicriteria problems need to be solved. More precisely, we need to determine the Pareto sets in two multicriteria problems. We begin by constructing the Pareto set for a multicriteria problem with a vector function f and a set of possible alternatives X. Once the Pareto set is identified, each vector in the set should be assigned a membership degree of 1, while all other vectors receive a membership degree of 0. Next, we solve another multicriteria problem on the same set X, but this time with a new (recalculated) p-dimensional vector function whose components are f_i for all $i \in I \backslash B$ and $w_j f_i + w_i f_j$ for all $i \in A$, $j \in B$. Any vector from the old Pareto set that does not belong to the new Pareto set will now receive a membership degree of $1 - \mu$. In this way, we achieve a reducing of the Pareto set, where the final set of selected vectors will lie.

If the set of frasible vectors Y is infinite, constructing the function λ_M as per Theorem 1 can be computationally challenging. However, for the finite set Y, the task is considerably simplified. The Pareto set can be found by exhaustively comparing all pairs of feasible vectors and evaluating them according to the Pareto relation.

Based on the formulas that define the function $\lambda_M(\cdot)$, it is evident that the consideration of a single quantum of interval-valued fuzzy information of the second order can be reduced to a two-step application of Theorem 1. Specifically, one first applies it to one component of the interval fuzzy relation, for instance, using $\mu = \mu^-$, in Theorem 1, and then to the other component, using $\mu = \mu^+$. As a result, each element of the set of feasible vectors is assigned two values, λ_M^- and λ_M^+, where the former provides the lower bound of the membership degree, and the latter provides the upper bound. This process yields an upper estimate for the set of selected vectors $C(Y)$ in the form of an interval-valued fuzzy set of the second order, characterized by the components λ_M^-, λ_M^+.

If this upper estimate turns out to be overly broad, making the final selection for the DM difficult, the resulting interval-valued fuzzy set of the second order can be defuzzified to a first-order fuzzy set. This can be done by taking, for example, the average of the values λ_M^- and λ_M^+ as the degree of membership. Afterward, a heuristic approach as presented in [16] is recommended. This approach involves appending the membership function of the resulting first-order fuzzy set to the existing set of criteria, followed by applying a suitable scalarization method for the multicriteria problem. This process allows for the completion of the final selection.

When the DM has access to a set of quanta of interval-valued fuzzy information, the aforementioned considerations remain applicable. However, instead of using Theorem 1, one should employ a theorem suitable for this context from [1] or the general algorithm described in [17].

4. CONSISTENCY OF A SET OF FUZZY INFORMATION QUANTA

According to Definition 1, each vector $u \in \mathbf{R}^m$ of the form (2) constitutes a quantum of interval-valued fuzzy information if it is deemed preferable to a null vector with a certain degree of confidence. When considering a set of such vectors individually, each clearly represents a corresponding quantum. However, analyzing them collectively may reveal inconsistencies with the foundational system of Axioms 1–4. In such cases, the sets of vectors cannot be used for narrowing the Pareto set and will be referred to as inconsistent sets.

Suppose there is a set of vectors $u^1, u^2, \ldots, u^k \in \mathbf{N}^m$, $k \geqq 1$, along with a set of pairs of numbers $\mu_i^-, \mu_i^+ \in (0, 1]$ such that $\mu^-(u^i, \mathbf{0}) = \mu_i^-$, $\mu^+(u^i, \mathbf{0}) = \mu_i^+$, $i = 1, 2, \ldots, k$, for each $(\mu_1^-, \mu_2^-, \ldots, \mu_k^-) \leq (\mu_1^+, \mu_2^+, \ldots, \mu_k^+)$. Without loss of generality for subsequent discussions, let $1 \geqq \mu_1^+ \geqq \mu_2^+ \geqq \ldots \geqq \mu_k^+ > 0$.

Definition 2. A set of vectors $u^1, u^2, \ldots, u^k \in \mathbf{N}^m$ together with a set of numbers $\mu_i^-, \mu_i^+ \in (0, 1]$, $i = 1, 2, \ldots, k$, $(\mu_1^-, \mu_2^-, \ldots, \mu_k^-) \le (\mu_1^+, \mu_2^+, \ldots, \mu_k^+)$, forms *a consistent set of interval-valued fuzzy information quanta* if there exists an IVFR2 relation with components $\mu^-(\cdot,\cdot)$, $\mu^+(\cdot,\cdot)$, that satisfies Axioms 2–4 and also meets the conditions $\mu^-(u^i, \mathbf{0}) = \mu_i^-$, $\mu^+(u^i, \mathbf{0}) = \mu_i^+$, $i = 1, 2, \ldots, k$. Otherwise, the set of vectors is termed *inconsistent*.

Definition 2 is a direct generalization of the definition of a consistent set of first-order fuzzy information quanta provided in [1].

Let e^i represent a unit vector in space $\mathbf{R}^m$, $i = 1, 2, \ldots, m$. We introduce precise polyhedral cones C_i, $i = 1, 2, \ldots, k$, generated by the set of vectors $e^1, e^2, \ldots, e^m$ along with vectors u^j, $j = 1, 2, \ldots, i$. It is evident that $C_1 \subset C_2 \subset \ldots \subset C_k$.

The following theorem establishes the criterion for the consistency of a set of interval-valued fuzzy information quanta.

Theorem 2. *Suppose Axioms 2–4 hold, and* $1 \geqq \mu_1^- \geqq \mu_2^- \geqq .. \geqq \mu_k^- > 0$. *The set of vectors* $u^1, u^2, \ldots, u^k \in \mathbf{N}^m$ *with the set of numbers* $\mu_i^-, \mu_i^+ \in (0, 1]$, $i = 1, 2, \ldots, k$, $(\mu_1^-, \mu_2^-, \ldots, \mu_k^-) \le (\mu_1^+, \mu_2^+, \ldots, \mu_k^+)$ *is consistent if and only if the system of linear equations*

$$\lambda_1 e^1 + \ldots + \lambda_m e^m + \xi_1 u^1 + \ldots + \xi_k u^k = \mathbf{0}, \qquad (3)$$

has no solution $\lambda_1, \ldots, \lambda_m, \xi_1, \ldots, \xi_k \geqq 0$, $\sum_{i=1}^m \lambda_i + \sum_{j=1}^k \xi_j > 0$ *and additionally, each cone* C_l, $l \in \{1, \ldots, k-1\}$, *contains only vectors* u^i, *for which* $\mu_i^- \geqq \mu_l^-$ *and* $\mu_i^+ \geqq \mu_l^+$.

The proof of this result, which can be found in the Appendix, generalizes the consistency condition for first-order fuzzy relations, previously derived by the author [1]. As in the case of precise relations, standard linear programming algorithms can be applied to verify this condition (see Theorem 4.3 [1]). The second consistency condition in Theorem 2, related to the cone C_l, can also be reduced to solving a specific linear programming problem since fulfilling this condition means representing a given vector as a non-negative linear combination of the set of vectors that form the cone C_l.

For problems of small dimensionality, one can verify the consistency of a given set of vectors without resorting to linear programming techniques.

It is worth noting that in [14], the concept of a consistent set of information quanta was introduced for the general case of second-order fuzzy relations, including a general approach outlined for using it to narrow the Pareto set [15]. The definition introduced here differs from that in [14] as it takes into account the specificity of interval-valued fuzzy relations, which, in the author's view, makes it more suitable for the considered problems involving IVFR2.

5. ILLUSTRATIVE EXAMPLE

Let us consider an example involving the application of a set of interval-valued fuzzy information quanta, assuming that Axioms 1–4 hold. Suppose $Y = \{y^1, y^2, \ldots, y^5\} \subset \mathbf{R}^3$, where:

$$y^1 = (4,3,5), \quad y^2 = (5,2,3), \quad y^3 = (4,3,3),$$
$$y^4 = (5,2,7), \quad y^5 = (2,5,5).$$

In this case, the set of Pareto-optimal vectors consists of three elements, with the characteristic function $\lambda_Y^P(y^1) = \lambda_Y^P(y^4) = \lambda_Y^P(y^5) = 1, \lambda_Y^P(y^2) = \lambda_Y^P(y^3) = 0$. Vectors that are not Pareto-optimal can be excluded from further consideration, as the Edgeworth–Pareto principle dictates that they should not be selected.

Suppose the DM provides the information that the first criterion is more important than the second with parameters $w_1' = 0.4$, $w_2' = 0.6$ and with a confidence degree in the range [0.6, 0.8]. Additionally, the first criterion is considered more important than the third with parameters $w_1'' = 0.5$, $w_3'' = 0.5$ and a confidence interval of [0.4, 0.7].

First, we verify the consistency of this information. Here, $u^1 = (0.4, -0.6, 0)$, $u^2 = (0.5, 0, -0.5)$, and the system of linear equations (3) takes the form:

$$\lambda_1 + 0.4\xi_1 + 0.5\xi_2 = 0;$$
$$\lambda_2 - 0.6\xi_1 = 0;$$
$$\lambda_3 - 0.5\xi_2 = 0.$$

If we assume that this system is consistent, then considering the non-negativity of all variables, the first equation implies that $\lambda_1 = \xi_1 = \xi_2 = 0$. Then, from the other two equations, we find that the remaining variables are also equal to zero: $\lambda_2 = \lambda_3 = 0$. Therefore, the system (3) does not have any non-zero nonnegative solutions.

Next, we check the fulfillment of the second condition in Theorem 2. We have $1 > 0.8 > 0.6$ and

$1 > 0.7 > 0.4$, and also $(1, 0.7, 0.4) \le (1, 0.8, 0.6)$. In this case, the cone C_1 is generated by vectors e^1, e^2, e^3, u^1. It does not contain the vector u^2 since the linear system of equations:

$$\lambda_1 + 0.4\xi_1 = 0.5;$$
$$\lambda_2 - 0.6\xi_1 = 0;$$
$$\lambda_3 = -0.5;$$

has no nonnegative solutions.

To incorporate the given fuzzy information, Theorem 2 is not applicable, as we are dealing with two quanta. Instead, we use Theorem 7.6 from [1], which addresses similar situations involving two quanta. According to this theorem, and using its notations for the new second criterion, we obtain: $\overline{y}_2 = 0.6y_1 + 0.4y_2$. Thus, the representatives of the new set of Pareto-optimal vectors are:

$$\overline{y}^1 = (6,\ 4.8,\ 5.5),\quad \overline{y}^4 = (5,\ 3.8,\ 6),\quad \overline{y}^5 = (2,\ 3.2,\ 5).$$

Here, the Pareto set consists of two vectors—the first and the fourth. According to the recommendations of Theorem 7.6 [1], for the lower bound, we assign:

$$\lambda^M(y^1) = \lambda^M(y^4) = 1,\quad \lambda^M(y^5) = 1 - 0.8 = 0.2.$$

Next, following the same theorem, we introduce a pair of new (second and third) criteria $\hat{y}_2 = 0.6y_1 + 0.4y_2$, $\hat{y}_3 = 0.5y_1 + 0.5y_2$, and also form an additional fourth criterion $\hat{y}_4 = 0.3y_1 + 0.2y_2 + 0.3y_3$. We find the corresponding new image (excluding $\hat{y}^5$):

$$\hat{y}^1 = (6,\ 4.8,\ 4.5,\ 4),\quad \hat{y}^4 = (5,\ 3.8,\ 3.5,\ 3.7).$$

Here, the Pareto-optimal vector is $\hat{y}^4$. As a result, we derive the following upper estimate for the lower bound of the unknown fuzzy set of selected vectors:

$$\lambda_C^-(y^1) = 1 - 0.6 = 0.4, \lambda_C^-(y^4) = 1,$$
$$\lambda_C^-(y^5) = 0.2, \lambda_C^-(y^2) = \lambda_C^-(y^3) = 0.$$

Similarly, for the upper bound, we get:

$$\lambda_C^+(y^1) = 1 - 0.4 = 0.6,\quad \lambda_C^+(y^4) = 1,$$
$$\lambda_C^+(y^5) = 1 - 0.7 = 0.3,\quad \lambda_C^+(y^2) = \lambda_C^+(y^3) = 0.$$

The resulting IVFR2 set $C(X)$ is a narrowed version of the initial Pareto set, using the specified two quanta of interval-valued fuzzy information. It is evident that the best candidate for selection is the fourth vector, as its degree of membership equals 1. The first vector follows, then the fifth. The remaining vectors should not be part of the selected set under any circumstances since they are not Pareto-optimal.

CONCLUSIONS

Information regarding the interval-valued fuzzy preference relation of the DM in the form of corresponding quanta can be utilized when solving multicriteria decision-making problems. For this purpose, the DM must accept four axioms of reasonable choice, for which the corresponding fuzzy variant of the Edgeworth–Pareto principle holds. Further narrowing of the Pareto set is proposed using quanta of interval-valued fuzzy information of the second order, following the described scheme, which allows leveraging previously obtained results based on first-order fuzzy quanta.

A criterion for the consistency of a finite set of interval-valued fuzzy quanta has been established, and its verification, in general, can be performed using linear programming methods. The illustrative example of a low-dimensional multicriteria decision-making problem demonstrates the possibility of accounting for multiple quanta of interval-valued fuzzy information without resorting to complex calculations or employing linear methods of programming.

Appendix

PROOF OF THE EDGEWORTH-PARETO PRINCIPLE

Assume that inequality (1) is violated for some $x \in X$ meaning $\mu_C^+(x) > \lambda_P(x)$. In the right-hand side of the inequality, the characteristic function of the Pareto set is written. It can only take two values—0 or 1. The second value is not possible in this case, so we arrive at the equality $\lambda_P(x) = 0$, indicating that the element x is not Pareto-optimal. Consequently, $\mu_C^+(x) > 0$ and for x here exists an element such $x' \in X$ that $f(x') \ge f(x)$. From this, according to the Pareto Axiom, we obtain $\mu_\succ(x', x) = 1$, and subsequent application of Axiom 1 immediately leads to equality $\mu_C^+(x) = 0$, which contradicts the previously obtained inequality $\mu_C^+(x) > 0$. This contradiction establishes the validity of inequality (1).

PROOF OF THEOREM 2

Necessity. Let the set of vectors $u^1, u^2, \ldots, u^k \in \mathbf{N}^m$ together with the set of numbers $\mu_i^- \in (0, 1]$, $i = 1, 2, \ldots, k$, form a consistent set of interval-valued fuzzy information quanta in the sense of Definition 2. Consider a first-order fuzzy relation with membership function $\mu^+(\cdot,\cdot)$. This relation satisfies Axioms F2–F4 [1] and, together with the numbers, defines a consistent set in the sense of Definition 7.2 [1]. Considering the inequalities $1 \geqq \mu_1^+ \geqq \mu_2^+ \geqq \ldots \geqq \mu_k^+ > 0$, we apply Theorem 7.2 [1]. According to this theorem, the system of linear equations (3) has no solution $\lambda_1,\ldots,\lambda_m,\xi_1,\ldots,\xi_k \geqq 0, \sum_{i=1}^m \lambda_i + \sum_{j=1}^k \xi_j > 0$. Moreover, each cone C_l, $l \in \{1,\ldots,k-1\}$ does not contain vectors u^i for which $\mu_i^+ < \mu_l^+$. This means that each such cone contains only vectors u^i for which $\mu_i^+ \geqq \mu_l^+$.

Consider a first-order fuzzy relation with membership function $\mu^-(\cdot,\cdot)$. Due to $1 \geqq \mu_1^- \geqq \mu_2^- \geqq \ldots \geqq \mu_k^- > 0$, each cone C_l contains only those vectors u^i for which $\mu_i^- \geqq \mu_l^-$, $l \in \{1,\ldots,k-1\}$.

Sufficiency. Suppose the system of linear equations (3) has no solution $\lambda_1,\ldots,\lambda_m,\xi_1,\ldots,\xi_k \geqq 0$, $\sum_{i=1}^m \lambda_i + \sum_{j=1}^k \xi_j > 0$. and each cone C_l, $l \in \{1,\ldots,k-1\}$ contains only those vectors u^i for which $\mu_i^- \geqq \mu_l^-$ and $\mu_i^+ \geqq \mu_l^+$.

Consider a first-order fuzzy relation with membership function $\mu^+(\cdot,\cdot)$. According to Theorem 7.2 [1], taking into account the inequalities $1 \geq \mu_1^+ \geqq \mu_2^+ \geqq \ldots \geqq \mu_k^+ > 0$, the set of vectors $u^1,u^2,\ldots,u^k$ together with the set of numbers $\mu_i^+ \in (0, 1]$, $i = 1,2,\ldots,k$ form a consistent set of first-order fuzzy information quanta in the sense of Definition 7.2 [1].

Similarly, one can consider a first-order fuzzy relation with membership function $\mu^-(\cdot,\cdot)$ and conclude that the set of vectors $u^1,u^2,\ldots,u^k$ together with the set of numbers $\mu_i^- \in (0,1]$, $i = 1,2,\ldots,k$, also form a consistent set of quanta in the sense of Definition 7.2 [1].

Since both first-order fuzzy relations satisfy Axioms F2–F4 [1], the IVFR2 with components $\mu^-(\cdot,\cdot)$, $\mu^+(\cdot,\cdot)$ satisfies Axioms 2–4, with $\mu^-(u^i,\mathbf{0}) = \mu_i^-$, $\mu^+(u^i,\mathbf{0}) = \mu_i^+$, $i = 1,2,\ldots,k$. Therefore, the set of vectors $u^1,u^2,\ldots,u^k$ together with the set of numbers $\mu_i^-,\mu_i^+ \in (0,1]$, $i = 1,2,\ldots,k$, form a consistent set of quanta in the sense of Definition 2.

FUNDING

This work was funded by the institution's budget. No additional grants were received for conducting or managing this specific research.

CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

REFERENCES

1. Noghin, V.D., *Reduction of the Pareto Set: An Axiomatic Approach*, Studies in Systems, Decision and Control, vol. 126, Cham: Springer, 2018. https://doi.org/10.1007/978-3-319-67873-3
2. Karnik, N.N., Mendel, J.M., and Liang, Q., Type-2 fuzzy logic systems, *IEEE Trans. Fuzzy Syst.*, 1999, vol. 7, no. 6, pp. 643–658. https://doi.org/10.1109/91.811231
3. Mendel, J.M., Advances in type-2 fuzzy sets and systems, *Inf. Sci. (N. Y.)*, 2007, vol. 177, no. 1, pp. 84–110. https://doi.org/10.1016/j.ins.2006.05.003
4. Mendel, J.M., Type-2 fuzzy sets and systems: A retrospective, *Informatik-Spektrum*, 2015, vol. 38, no. 6, pp. 523–532. https://doi.org/10.1007/s00287-015-0927-4
5. Baskov, O.V. and Noghin, V.D., Type-2 fuzzy sets and their application in decision-making: General concepts, *Sci. Tech. Inf. Process.*, 2022, vol. 49, no. 5, pp. 283–291. https://doi.org/10.3103/s014768822205001x
6. Bustince, H., Barrenechea, E., Pagola, M., Fernandez, J., Xu, Z., Bedregal, B., Montero, J., Hagras, H., Herrera, F., and De Baets, B., A historical account of types of fuzzy sets and their relationships, *IEEE Trans. Fuzzy Syst.*, 2016, vol. 24, no. 1, pp. 179–194. https://doi.org/10.1109/tfuzz.2015.2451692
7. Liang, Q. and Mendel, J.M., Interval type-2 fuzzy logic systems: Theory and design, *IEEE Trans. Fuzzy Syst.*, 2000, vol. 8, no. 5, pp. 535–550. https://doi.org/10.1109/91.873577
8. Bustince, H., Montero, J., Pagola, M., Barrenechea, E., and Gomez, D., A survey of interval-valued fuzzy sets, *Handbook of Granular Computing*, Pedrycz, W., Skowron, A., and Kreinovich, V., Eds., Wiley, 2008, pp. 489–515. https://doi.org/10.1002/9780470724163.ch22
9. Bustince, N., Interval-valued fuzzy sets in soft computing, *Int. J. Comput. Intell. Syst.*, 2010, vol. 3, no. 2, pp. 215–222. https://doi.org/10.2991/ijcis.2010.3.2.9
10. Huidobro, P., Alonso, P., Janiš, V., and Montes, S., Convexity and level sets for interval-valued fuzzy sets, *Fuzzy Optim. Decis. Making*, 2022, vol. 21, pp. 553–580. https://doi.org/10.1007/s10700-021-09376-7
11. Runkler, T., Coupland, S., and John, R., Interval type-2 fuzzy decision making, *Int. J. Approximate Reasoning*, 2017, vol. 80, no. 80, pp. 217–224. https://doi.org/10.1016/j.ijar.2016.09.007
12. Bustince, H., Pagola, M., Jurio, A., Barrenechea, E., Fernández, J., Couto, P., and Melo-Pinto, P., A survey of applications of the extensions of fuzzy sets to image processing, *Bio-Inspired Hybrid Intelligent Systems for*

Image Analysis and Pattern Recognition, Melin, P., Kacprzyk, J., and Pedrycz, W., Eds., Studies in Computational Intelligence, Berlin: Springer, 2009, pp. 3–32. https://doi.org/10.1007/978-3-642-04516-5_1

13. Baskov, O.V. and Noghin, V.D., The Edgeworth–Pareto principle in the case of IT2F preference relation, *J. Phys.: Conf. Ser.*, 2021, vol. 1801, no. 1, p. 012021. https://doi.org/10.1088/1742-6596/1801/1/012021
14. Baskov, O.V., Consistency of information about type-2 fuzzy preference relation, *Int. J. Inf. Technol. Decis. Making*, 2022, vol. 22, no. 6, pp. 1797–1811. https://doi.org/10.1142/s0219622022500900
15. Baskov, O.V., Pareto set reduction based on information about type two fuzzy preference relation. Algorithm description, *Sci. Tech. Inf. Process.*, 2023, vol. 50, no. 6, pp. 520–526. https://doi.org/10.3103/s0147688223060023
16. Noghin, V.D., Multicriteria choice on a fuzzy set as a problem of searching for compromise, *Sci. Tech. Inf. Process.*, 2019, vol. 46, no. 6, pp. 397–403. https://doi.org/10.3103/s0147688219060054
17. Noghin, V.D., Algorithm for the reduction of the Pareto set using a collection of fuzzy information quanta, *Sci. Tech. Inf. Process.*, 2023, vol. 50, no. 6, pp. 610–617. https://doi.org/10.3103/s0147688223060126

Translated by V. Tereshchenko