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MARINE PHYSICS

Mixed Topographic-Planetary Waves in an Exponentially Stratified Ocean

V. S. Travkin[a, c, *], **V. G. Gnevyshev**[a, b], **and T. V. Belonenko**[a]

[a] *St. Petersburg University, St. Petersburg, 199034 Russia*

[b] *Shirshov Institute of Oceanology, Russian Academy of Sciences, Moscow, 117997 Russia*

[c] *Zubov State Oceanographic Institute, Roshydromet, Moscow, 119034 Russia*

**e-mail: vtravkin99@gmail.com*

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Abstract—On the continental slope, a dispersion relation in terms of Bessel functions is found for mixed topographical-planetary waves in the approximation of a rigid-lid and an exponential stratification profile. It is numerically analyzed for both positive and negative meridional slopes of the topography. It is shown that taking into account exponential stratification, in comparison with a constant stratification profile, has a twofold effect. On the one hand, indeed, the exponential stratification profile leads to smaller vertical eigenvalues and, as a result, higher propagation velocities of topographical and planetary waves than on a constant stratification profile. On the other hand, the influence of topography on eigenvalues is negligible. If, during the transition from the baroclinic mode to the surface mode for a constant stratification profile, the topography can lead to a fourfold increase in the wave propagation velocity for the first baroclinic mode, then for the exponential profile such an effect is limited to 30%. As a result, mixed topographical-planetary waves do not possess hypersensitivity to the stratification profile, as previously established. The transition from the baroclinic mode to the surface mode with an exponential stratification profile, as with a constant stratification profile, occurs in the range of angles of inclination 10^{-4}~10^{-2} for 100-km wavelengths.

Keywords: planetary waves, stratified ocean, exponential stratification profile, barotropic and baroclinic modes, Bessel functions

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1. INTRODUCTION

The effect of stratification (baroclinicity) on topographical Rossby waves for the open ocean was first considered by Rhines [13–15]. In [18], the following definition of types of topographies is given: if $L \gg a$, where L is the wavelength scale and a is the topography variability scale, then such a *bottom topography* is called *roughness* (cf. [1, 2, 16]). If $L \sim a$, such a topography is called *undulations*. For $L \ll a$, such topography is called a *slope* [2, 13, 14, 17]. This paper considers slope-type topography with exponential stratification.

In a barotropic ocean, regardless of the magnitude of the slope and wave number, the β-effect acts as a meridional slope, or, in other words, the meridional slope is equivalent to the β-effect. These two physical factors in the problem are added algebraically. The gradient of absolute planetary vorticity has the form $(f/H)_y \sim (\beta - fH_y/H)$, from which we find that the slope becomes equivalent to the β-effect for slope values $H_y \sim H/R \cot\varphi = 0.6 \times 10^{-3} \cot\varphi$, where $H =$ 4 km is the average ocean depth and R is the radius of the Earth.

In the presence of stratification, such direct arithmetic addition does not generally occur. The arithmetic addition of the topographical slope and the β-effect for stratification occurs only asymptotically in special cases, either in the long-wavelength limit for the barotropic mode [14], or with small stratification for the topographical mode [12], or for baroclinic modes with small slopes in the long-wavelength approximation [5, 6].

The spectral problem of topographical waves on a slope with stratification has anisotropy to the direction of the meridional slope. For positive slopes (water shoaling towards the pole in the Northern Hemisphere), the barotropic mode disappears, replaced by a topographical mode (localized in the vicinity of the bottom). For negative slopes, the barotropic mode exists, but its behavior depends on the wavelength and magnitude of the topographical slope. For long wavelengths or low topographical slopes, this is a classic fast barotropic mode. At the same time, for short wavelengths or large topographical slopes, it becomes similar to a slow baroclinic mode (shown below). Baroclinic modes are transformed into so-called surface modes, the vertical structure of the solution is localized in the upper layer of the ocean. However, the influence of topography is finite, and the frequency of Rossby waves is limited from above by a maximum fourfold increase [14]. It is important to note that the restructuring of the vertical mode with constant stratification and the change in the vertical eigenvalue occur in the range of wave numbers $0 < |k| < 10$ and bottom slopes from 0.1 to 10 in units of the beta parameter. The restructuring of solutions with a constant stratification profile does not go far from the parity barotropic range in terms of the magnitude of topographical slopes and the β-effect ($\alpha \sim 10^{-3}$, $\delta \sim 10^{-2}$). In works by Rhines [13–15] and subsequent studies, the stratification profile was assumed to be constant. At the same time, Rhines believed that if the actual stratification profile is taken, and not a constant value, then the influence of stratification on topographical waves will be even smaller. Subsequently, analytical solutions also appeared for the exponential stratification profile: e.g., in [19], an exponential stratification profile without taking topography into account; in works [9, 10], an exponential stratification profile taking into account the topographical meridional slope. These studies were preceded by well-known work [3].

However, in [10], for an exponential stratification profile, a somewhat unexpected, in our opinion, result was obtained: the restructuring of the solution from the baroclinic to the surface mode occurs at significantly smaller values of the topographical slope, compared to a constant stratification profile. A certain supersensitivity of the solution to the type of stratification profile appears, or, conversely, a supersensitivity of the baroclinic mode to topography, which may indicate a certain structural instability of the classical Rhines solution.

In this article, based on the results of [19], we obtained an eigenvalue equation that differs from the equation in [10] and performed numerical calculations for both positive and negative meridional slopes. In this case, we obtained results that differ from the results of [10] in the magnitude of the slopes for which the transition from the baroclinic to the surface mode occurs [6, 10].

Our results support Rhines' assumption that the type of stratification profile is unimportant for estimating Rossby wave frequencies on a topographical slope. Similar results, indicating the insignificance of the influence of the stratification profile on the boundary value problem, were obtained earlier in works on the numerical calculation of Rossby waves on real topography profiles [7, 8].

2. FORMULATION OF THE PROBLEM

On Rhines' dimensionless scale [13]

$$(x,y) = L(x',y'), \quad z = H_0 z', \quad t = f_0^{-1} t',$$
$$(u,v) = U_0(u',v'), \quad w = (H_0 U_0 / L) w', \qquad (1)$$
$$p = \rho_* U_0 f_0 L p', \quad \rho = (\rho_* U_0 f_0 L / g H_0) \rho'$$

the equations for planetary waves in the β-plane approximation in the presence of stratification and topography are as follows:

for density,

$$\rho_t - S^2 w = 0, \qquad (2)$$

hydrostatics,

$$p_z = -\rho. \qquad (3)$$

Here, u and v are the velocity components in x and y directions, H is depth, p is pressure, ρ is the density of water, and f is the Coriolis parameter. The classical right-hand coordinate system is adopted, the x, y axes are directed to the east and north, respectively, and the z axis is directed upwards; $S^2 = N^2 H^2 / f^2 L^2$ is Burger's number, which we assume to be constant of order 1; N is the Brunt–Väisälä frequency. Indeed, for $N = O(10^{-3}$ rad/s), $H_0 = 5 \times 10^3$ m and $L = 10^5$ m, we find $S \approx 1$.

We determine the topography as follows (see [11, 13, Eq. (20.24)]):

$$z = -H(y) = -H_0(1 - \delta y / L), \qquad (4)$$

where δ is the actual slope of the bottom. In this formulation, there are two wave-forming parameters: the β-parameter (the change in the Coriolis parameter with latitude) and the topography, the gradient of which is specified by the parameter $\alpha = dz/dy = H_0 \delta / L$. Following Rhines, we take into account both factors, considering them to be of the same order, assuming $\alpha \sim 10^{-3}$. Taking $H_0 = 5 \times 10^3$ m, $L = 10^5$ m, we obtain $\delta \sim 10^{-2}$.

For the upper kinematic condition, we take the rigid-lid condition:

$$w = 0, \quad z = 0. \qquad (5)$$

The lower boundary kinematic condition is the no-flow condition:

$$w = \delta v, \quad z = -1 + \delta y. \qquad (6)$$

We seek a solution in the form of a plane wave $\exp[+i(kx + ly - \omega t)]$, where k and l are the zonal and meridional wave numbers; for certainty the standard assumption of positive frequency is adopted: $\omega > 0$ [4]. Next, following [13], we construct an asymptotic expansion with respect to the small topographical parameter δ. We adopt a low-pass approximation that filters out internal waves and the Kelvin wave [13]:

$$\omega = 0 + \delta \omega^{(1)} + \ldots \qquad (7)$$

Next, applying the classical procedure of velocity field reduction to the equations, we obtain the following equation for pressure:

$$\omega^{(1)} \left[\left\{ S^{-2}(z) p_z \right\}_z - (k^2 + l^2) p \right] - \hat{\beta} k p = 0, \qquad (8)$$

where $\hat{\beta} = (\beta / f_0)(L / \delta) = \beta L / f_0 \delta$. In this case, below, a dimensionless expression for the meridional component of velocity through pressure is also be used:

$$v = ikp. \tag{9}$$

The upper boundary condition is the rigid lid:

$$p_z = 0, \quad z = 0. \tag{10}$$

The lower boundary condition is the no-flow condition:

$$\omega^{(1)} p_z = S^2 kp, \quad z = -1. \tag{11}$$

The sought frequency $\omega^{(1)}$ is included both in the differential equation and in the lower boundary condition.

3. EXPONENTIAL STRATIFICATION: ANALYTICAL SOLUTION

Let us rewrite (8) as

$$\left(S^{-2}(z)\, p_z\right)_z + r_n^2 p = 0, \tag{12}$$

where r_n is the separation variable [19]. Then,

$$r_n^2 = -\left(k^2 + l^2 + \frac{\hat{\beta} k}{\omega^{(1)}}\right) \tag{13}$$

or

$$\omega^{(1)} = -\frac{\hat{\beta}\, k}{k^2 + l^2 + r_n^2}. \tag{14}$$

In the case of exponential stratification for a flat bottom, analytical solutions in terms of Bessel functions for Rossby waves were found in [9, 19]. In dimensionless form

$$N = N_0 \exp az, \quad S = S_0 \exp az, \quad S_0 = \frac{N_0 H}{fL}, \tag{15}$$

where a is the the so-called e-parameter (e-folding). Physically a is a value reciprocal to depth (in our case it is a dimensionless value, dimensionless to an ocean depth of 5 km, at which stratification decreases by e (2.71) times). Since the dimensionless ocean depth varies in the range $-1 \le z \le 0$, we take the e-parameter as a typical dimensional value of 1 km, if $a = 5$ is taken into account in numerical calculations (do not confuse this with a—the topography scale in [13], see section 1). S_0 is the dimensionless value of stratification on the surface. In the model with constant stratification, $N \sim 10^{-3}$ is assumed; then for a wave length of 100 km $S \sim 1$ and for the surface stratification value $N \sim 10^{-2}$, we obtain $S \sim 10$ [10].

The expression for the vertical velocity $w = i\omega^{(1)} p_z^0 / S^2$, satisfying the upper boundary condition of the rigid lid (see [19], Appendix A) up to a numerical normalizing factor, is expressed in terms of Bessel functions:

$$\begin{gathered} w = i\omega^{(1)} G_n(z) \\ = i\omega^{(1)} \left[J_0(\xi) Y_0(\xi_0) - J_0(\xi_0) Y_0(\xi)\right], \\ \xi_0 = \frac{S_0 r_n}{a}, \quad \xi = \xi_0 \exp az. \end{gathered} \tag{16}$$

Otherwise,

$$\begin{gathered} W = A_w \left[J_0(\xi) - \frac{J_0(\xi_0)}{Y_0(\xi_0)} Y_0(\xi)\right], \\ \xi_0 = \frac{S_0 r_n}{a}, \quad \xi = \xi_0 \exp az, \end{gathered} \tag{17}$$

where A_w is the normalizing factor (see [19], Appendix A). Then the pressure can be written as

$$p = \frac{1}{r_n^2} a\xi \left[Y_0(\xi_0) J_1(\xi) - J_0(\xi_0) Y_1(\xi)\right]. \tag{18}$$

Otherwise,

$$p = A_p \xi \left[J_1(\xi) - \frac{J_0(\xi_0)}{Y_0(\xi_0)} Y_1(\xi)\right], \tag{19}$$

where A_p is the normalizing factor ([19], Appendix A).

The Appendix presents relations to find pressure through vertical velocity (solution method presented in [19]), and vice versa, to find vertical velocity through pressure (solution method [9]), which indicates the identity of the two solutions in [10, 19].

3.1. Exponential Stratification: Flat Bottom

Let us consider the flat bottom solution first. For the lower boundary condition $(w = 0, z = -1)$, we obtain the equation

$$\begin{gathered} J_0(\xi_{-1}) Y_0(\xi_0) - J_0(\xi_0) Y_0(\xi_{-1}) = 0, \\ \xi_{-1} = \xi_0 \exp(-a). \end{gathered} \tag{20}$$

Numerically we obtain the following result: assuming $a = 5$, we find the roots of the first three baroclinic modes: $S_0 r_1 / a = 2.765; S_0 r_2 / a = 5.96; S_0 r_3 / a = 9.14$. We can numerically verify that, when passing to the limit of constant vertical stratification $(a \to 0, S \to S_0)$, we obtain the classical results of the baroclinic problem: $S_0 r_1 \to \pi; S_0 r_2 \to 2\pi; S_0 r_3 \to 3\pi$. The relationship between the baroclinic Rossby radius R_n and r_n is inversely proportional: $R_n \sim r_n^{-1}$. The vertical structure of the obtained solutions in terms of Bessel functions is shown in Fig. 1. Instead of the barotropic topography mode r_0 (constant along the vertical), Fig. 1 shows its asymptotic state (for large negative slopes) during the transition to a slow baroclinic mode.

3.2. Exponential Stratification: Meridional Slope of Topography

In the presence of topography, the lower boundary condition in dimensionless form $w = \alpha v, z = -1$ takes the form

$$\begin{gathered} \omega_d^1 r_n^2 \left[J_0(\xi_{-1}) Y_0(\xi_0) - J_0(\xi_0) Y_0(\xi_{-1})\right] \\ = ka\xi_{-1} \left[Y_0(\xi_0) J_1(\xi_{-1}) - J_0(\xi_0) Y_1(\xi_{-1})\right]. \end{gathered} \tag{21}$$

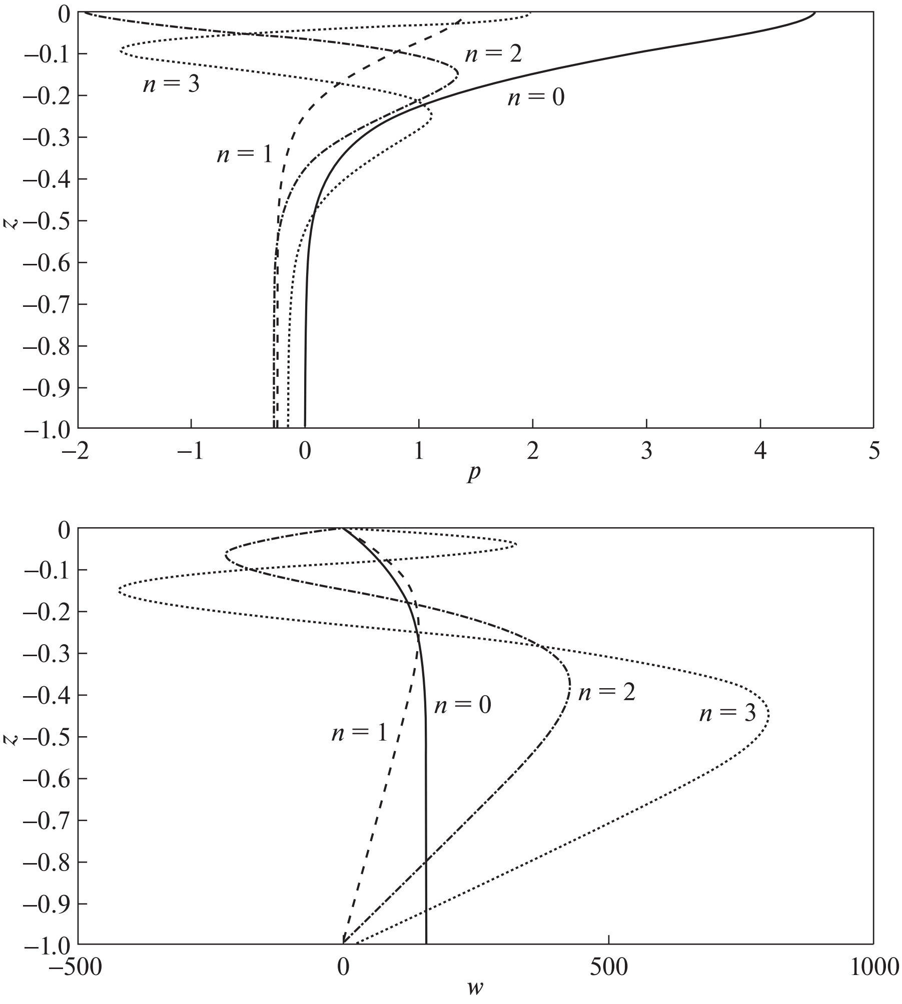

Fig. 1. Vertical eigenfunctions for exponential stratification for three baroclinic modes for $a = 5$ and no topography: $n = 1$, $S_0 r_1/a = 2.765$; $n = 2$, $S_0 r_2/a = 5.96$; $n = 3$, $S_0 r_3/a = 9.14$. Barotropic mode is modified by negative topographical slope (slow baroclinic mode): $n = 0$, $S_0 r_2/a = 2.4048$. Pressure dependence p (top) and vertical component of the velocity w (down).

Substituting (14) into (21), we find

$$J_0(\xi_{-1})Y_0(\xi_0) - J_0(\xi_0)Y_0(\xi_{-1}) = -\exp(-a)\frac{S_0\alpha}{\beta}\left(\frac{k^2+l^2}{r_n}+r_n\right) \times [Y_0(\xi_0)J_1(\xi_{-1}) - J_0(\xi_0)Y_1(\xi_{-1})]. \quad (22)$$

This Eq. (22) differs from Eq. (10) from [10] by a factor $\exp(-a)$ on the right-hand side of the equality.

3.3. Numerical Calculation

We found a numerical solution to Eq. (22) for both positive and negative slopes. The solutions are shown in Fig. 2.

Analysis of Fig. 2 yields interesting conclusions. Clearly, there is no barotropic mode for positive slopes. The spectrum begins with the first baroclinic mode. For positive slopes, the first baroclinic mode decreases its vertical eigenvalue with an increasing topographical slope. Thus, the topography contributes to an increase in the propagation velocity of of baroclinic waves.

However, most surprising is the transformation of the barotropic mode for negative slopes. This mode helps to cover the entire possible range of Rossby wave velocities in a stratified ocean [14]. The transformation of the barotropic mode for negative slopes begins from the fast barotropic mode (small topographical slopes) to a slow baroclinic mode. Figure 1 shows the vertical structure of this barotropic mode as it enters the slow baroclinic regime. Note that, qualitatively, our calculations for exponential stratification are similar to the results of [13] for constant stratification, but the result of [10] lies much to the left, in the region of too small topographical slopes.

It is also evident from Fig. 2 that for positive slopes only half of the baroclinic modes are realized, which

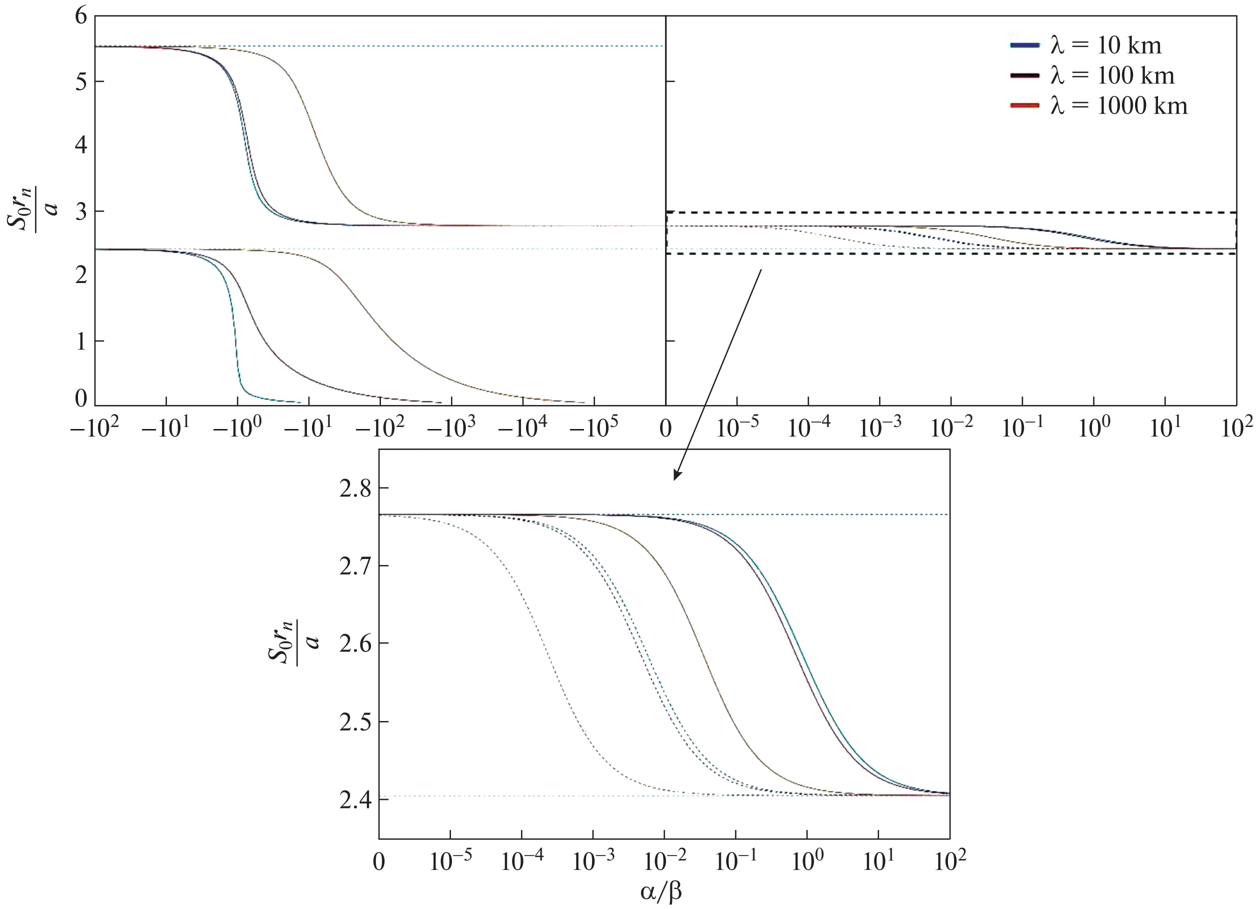

Fig. 2. Spectral problem of Rossby waves on meridional slope with negative (left) and positive (right) slopes for different wavelengths λ (numerical solution of Eq. (22)). The figure below shows graphs for positive slopes. Dotted lines show solutions of Eq. (22) without exponential factor on right-hand side. *X* axis shows α/β values, where topographical parameter α is taken in units of β-parameter. All calculations are done in dimensionless form. To pass to dimensional quantities, α should be multiplied by $0.6 \times 10^{-3} \cot \varphi$ (by approximately 10^{-3}).

was noted by Rhines [13]. However, the second half of the modes does not disappear, but is realized for negative angles of inclination.

4. CONCLUSIONS

With constant stratification for the first baroclinic mode on a positive meridional slope, the vertical eigenvalue changes according to the following law: $Sr_1 = \pi$ tends to $\pi/2$; i.e., the influence of topography leads to a twofold decrease in the vertical wave number and, as a consequence, the frequency and phase velocity of long Rossby waves can increase fourfold.

For the exponential stratification profile with the adopted dimensionless parameters $a = 5$, $S_0 = 10$ for the first baroclinic mode, we obtain: $r_1 = 2.765/2 = 1.38$ (without topography) and $r_1 = 2.4048/2 = 1.20$ (infinite slope). Consequently, on an exponential topography profile, Rossby waves propagate faster. It should be emphasized that the influence of topography on these waves is small: the increase in frequencies and velocities is no more than 30%. In this article, the horizontal asymptotes for positive slope coincide with the result of [10].

Along the *y* axis, we have the dimensionless value $S_0 r_n / a$. Our results do not confirm the hypersensitivity of the boundary value problem of topographical Rossby waves to the stratification profile. The transition from the baroclinic to the surface mode, according to our calculations, occurs in the classical range—the same as with constant stratification, when the topography is compared with the β-parameter. In dimensional quantities, the transition occurs in the range 10^{-4}–10^{-2}, while, according to the results of [10], everything takes place at much smaller slopes. In [10], for 10^{-4} mode restructuring has already ended, while according to our calculations, it is only just beginning. From the physical standpoint, in our opinion, this is more justified and expected. Moreover, this corresponds to the Rhines' idea [10] on the irrelevance of the type of stratification profile on the spectral problem of Rossby waves, as well as numerical calculation in [7, 8].

There is one more aspect that we would like to draw attention to: the normalization of proper functions. For a flat bottom, such normalizations are presented in [19], and we used them in plotting the first three baroclinic modes in Fig. 1. However, note that when taking into account the topography, the eigenfunctions and eigenvalues of the boundary value problem depend on the angle of inclination. Therefore, the normalization will also depend on the angle of inclination of the topography. Since we do not know the normalizations taking into account topography, it is not entirely correct to plot the eigenfunctions and claim that vertical velocities at the bottom decay faster in a model with exponential stratification.

The main result of this study can be formulated as follows: taking into account exponential stratification, in comparison with models with constant stratification, qualitatively improves the results in terms of constructing vertical eigenfunctions. At the same time, the problem remains structurally stable in the sense that the transition from baroclinic to surface modes occurs in the same range of topographical slopes, and there is no hypersensitivity with an exponential stratification profile.

APPENDIX

If we proceed from the formulas of [19], which allow us to find the solution for the vertical velocity, we should differentiate the vertical velocity in order to find the pressure. Then we use the following well-known relation for the derivative of the Bessel function:

$$[J_0(\xi)]_\xi = -J_1(\xi). \tag{23}$$

Due to the simplicity of the calculations, we omit the details of the transformations. Note that if we proceed from the solution in [10], in which it is constructed in terms of Bessel functions, we need to differentiate the pressure in order to find the vertical velocity:

$$w = \frac{i\omega^1 p_z}{S^2}, \quad \xi = Sr_n/a, \quad \xi^2 a^2 = S^2 r_n^2. \tag{24}$$

Now we use the following recurrence relation:

$$[Y_1(\xi)]_\xi = Y_0(\xi) - \frac{1}{\xi}Y_1(\xi), \tag{25}$$

in which instead of Y we can also use the Jth Bessel function [18]. If

$$p = a\xi(J_1 - BY_1), \tag{26}$$

then by differentiation we obtain

$$p_z = a^2\xi(J_0 - BY_0), \tag{27}$$

where we find the vertical velocity

$$w = \frac{i\omega^1 p_z}{S^2} = i\omega^1 r_n^2 (J_0 - BY_0), \tag{28}$$

which, up to a normalizing factor, coincides with the results of [10, 19] and indicates the identity of the solutions in these two works.

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ETHICS APPROVAL AND CONSENT TO PARTICIPATE

This work does not contain any studies involving human and animal subjects.

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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