Criticality conditions in the Derrida–Retaux model with a random number of terms

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Abstract

The article considers the Derrida–Retaux model with a random number of terms, i.e. a sequence of integer random variables defined by the relations $X_{n+1} = (X_n^{(1)} + X_n^{(2)} + ... + X_n^{(N_n)} - a)^+$, $n \geq 0$, where X_n^j are independent copies of X_n, the values of N_j are independent and identically distributed, a is a positive integer. The energy in the model is defined as $Q := \lim\limits_{n\to\infty} \frac{\mathbb{E}(X_n)}{(\mathbb{E}N_1)^n}$. We present sufficient conditions (in terms of distributions of X_0 and N_1) for subcritical ($Q = 0$) and supercritical ($Q > 0$) regimes of model behavior.

1 Introduction

1.1 Derrida — Retaux model

The theory of fluctuations in chaotic systems is a branch of physics that studies the behavior of systems in which chaos and randomness are present. The purpose of the theory is to understand how fluctuations or small changes in the properties of a system can lead to global changes in its overall behavior.

In particular, two states may be distinguished in the system and transitions between them are studied. The Derrida — Retaux model[3] in a simplified form describes such transitions (depinning transition) between certain states called "pinned" and "unpinned", which we do not need to describe. Similar relationships have also been studied in the context of physical [4] and mathematical studies [1].

The Derrida — Retaux model with a random number of terms, which is studied in this paper, is formulated as follows. The model parameters are an integer $a > 0$ and two integer random variables: $X_0 \geq 0$ – the initial value, and $N \geq 1$ – the number of terms. It is assumed that X_0 has a finite first moment and is not a constant, as well as $\mathbb{P}(N > 1) > 0$. The functioning of the system is determined by the recurrence relation:

$$X_{n+1} = (X_n^{(1)} + X_n^{(2)} + ... + X_n^{(N_{n+1})} - a)^+, \tag{1}$$

where $X_n^{(1)}$ and $X_n^{(2)}$ are independent copies of X_n, and $\forall\, x \in \mathbb{R}$, $x^+ := \max(x, 0)$ is the positive part of x. Let us note that

$$\mathbb{E}(\sum_{j=1}^{N_{n+1}} X_n^{(j)} - a) \leq \mathbb{E}(\sum_{j=1}^{N_{n+1}} X_n^{(j)} - a)^+ \leq \mathbb{E} \sum_{j=1}^{N_{n+1}} X_n^{(j)}.$$

Also $\mathbb{E} \sum\limits_{j=1}^{N_{n+1}} X_n^{(j)} = \mathbb{E}N_{n+1} \cdot \mathbb{E}X_n = \mathbb{E}N \cdot \mathbb{E}X_n$, therefore

$$\mathbb{E}N \cdot \mathbb{E}X_n - a \leq \mathbb{E}X_{n+1} \leq \mathbb{E}N \cdot \mathbb{E}X_n,$$

$$\frac{\mathbb{E}(X_{n+1})}{(\mathbb{E}N)^{n+1}} \leq \frac{\mathbb{E}(X_n)}{(\mathbb{E}N)^n} \quad \text{and} \quad \frac{\mathbb{E}(X_n) - \dfrac{a}{\mathbb{E}N - 1}}{(\mathbb{E}N)^n} \leq \frac{\mathbb{E}(X_{n+1}) - \dfrac{a}{\mathbb{E}N - 1}}{(\mathbb{E}N)^{n+1}}.$$

Thus, the following limit is well-defined

$$Q = \lim_{n\to\infty} \downarrow \frac{\mathbb{E}(X_n)}{(\mathbb{E}N)^n} = \lim_{n\to\infty} \uparrow \frac{\mathbb{E}(X_n) - \dfrac{a}{\mathbb{E}N - 1}}{(\mathbb{E}N)^n}. \tag{2}$$

The parameter Q is commonly referred to as free energy, and the recurrence relation described above was first introduced in the article by Derrida and Retaux [3] and serves to describe the Depinning transition process. Studying this process is important in both mathematics and physics. Of particular interest is the dependence of the energy value Q on the initial data — the random variable X_0 and a. The parameter a is referred to as the tax.

Depending on the value of Q, two cases are distinguished: supercritical ($Q > 0$) and subcritical ($Q = 0$). A critical case is also distinguished, which essentially «separetes» the situations described above. The most interesting questions regarding this model arise in critical cases or cases close to them.

The recurrence relation (1) with parameter $a = 1$ and determined numbers of terms was considered in the article [2]. This article provides necessary and sufficient conditions on the random variable X_0 under which supercritical, subcritical, and critical cases occur, respectively.

We attempted to generalize the results obtained in this article to arbitrary $a \geq 1$ and random numbers of terms. We found sufficient conditions on the random variable X_0 guaranteeing subcriticality or supercriticality.

2 Main result

The main result of our work is the following theorem.
Let F_0, G be the moment generating function of X_0 and N respectively.

Theorem 1. *Let it be* $D_0(s,m) = (m-1)sF_0'(s) - aF_0(s)$.
1) If $D_0(\mathbb{E}N^{\frac{1}{a}}, \mathbb{E}N) > 0$, *then* $Q > 0$.
2) Let $\exists M : \mathbb{P}(N \leq M) = 1$. *If* $D_0(1 + \frac{M-1}{a}, M) < 0$, *then* $Q = 0$.

Let us emphasize that the first point is true without assuming that the random variable N is bounded.
The result of Theorem 1 is interesting even if the number of terms is fixed, i.e. $\mathbb{P}(N = n) = 1$ for some n. In this case, Theorem 1 reduces to the following result.

Theorem 2. *Let it be* $D_0(s) = (n-1)sF_0'(s) - aF_0(s)$.
1) If $D_0(n^{\frac{1}{a}}) > 0$, *then* $Q > 0$.
2) If $D_0(1 + \frac{n-1}{a}) < 0$, *then* $Q = 0$.

If $a = 1$, then the conditions of points 1 and 2 stick together, and we get the result from [2]. If $a > 1$, then Theorem 2 is a new result.

3 Proof of the main result

3.1 Moment generating functions and their evolution

Let $F_n(s)$ be the moment generating function of X_n.
Let us rewrite the recurrence relation (1) in terms of generating functions:

$$\begin{aligned} F_{n+1}(s) &= \frac{G(F_n(s)) - \sum_{p=0}^{a-1} \frac{s^p}{p!}(G(F_n)^{(p)}(0))}{s^a} + \sum_{p=0}^{a-1} \frac{G(F_n)^{(p)}(0)}{p!} \\ &= \frac{G(F_n(s))}{s^a} + \sum_{p=0}^{a-1} (G(F_n))^{(p)}(0) \frac{1}{p!}\left(1 - \frac{1}{s^{a-p}}\right). \end{aligned} \tag{3}$$

By differentiating equality (3) we get

$$F_{n+1}'(s) = \frac{G'(F_n(s))F_n'(s)}{s^a} - a\frac{G(F_n(s))}{s^{a+1}} + \sum_{p=0}^{a-1} G(F_n)^{(p)}(0) \cdot \frac{a-p}{p!} \cdot \frac{1}{s^{a-p+1}}. \tag{4}$$

We will use these formulas a lot.

3.2 Proof of the first point of Theorem 1

The proof relies on two lemmas. First, we will prove the result using them, and the proof of the lemmas themselves will be presented below.

Lemma 1. *If the assumption of the first statement of the theorem is true, then there exists* $1 < s < (\mathbb{E}N)^{\frac{1}{a}}$, *such that*

$$(\mathbb{E}N - 1)\mathbb{E}(X_n s^{X_n}) - a\mathbb{E}(s^{X_n}) \to \infty \text{ as } n \to \infty.$$

In terms of generating functions, Lemma 1 can be represented as follows: there is $1 < s < (\mathbb{E}N)^{\frac{1}{a}}$ such that

$$(\mathbb{E}N - 1)sF_n'(s) - aF_n(s) \to \infty \text{ as } n \to \infty.$$

Lemma 2. *If $Q = 0$, then $\sup_{n\geq 0} \mathbb{E}(X_n s^{X_n}) < \infty$ with $0 < s < (\mathbb{E}N)^{\frac{1}{a}}$.*

Now let us apply these lemmas to prove the statement.

Let $a\mathbb{E}(((\mathbb{E}N)^{\frac{1}{a}})^{X_0}) < (\mathbb{E}N - 1)\mathbb{E}(X_0((\mathbb{E}N)^{\frac{1}{a}})^{X_0})$; we will show that $Q > 0$.

By Lemma 1, there is $1 < s < (\mathbb{E}N)^{\frac{1}{a}}$, such that $(\mathbb{E}N - 1)\mathbb{E}(X_n s^{X_n}) - a\mathbb{E}(s^{X_n}) \to \infty$, as $n \to \infty$, therefore $(\mathbb{E}N - 1)\mathbb{E}(X_n s^{X_n}) \to \infty$ as $n \to \infty$.

But if $Q = 0$, then by Lemma 2 $\sup_{n\geq 0} \mathbb{E}(X_n s^{X_n}) < \infty$. Hence $Q > 0$.
Now we will provide proof of the lemmas.

Proof of Lemma 1

By (4)

$$s(\mathbb{E}N-1)F_{n+1}'(s) = \frac{(\mathbb{E}N-1)sG'(F_n(s))F_n'(s)}{s^a} - a(\mathbb{E}N-1)\frac{G(F_n(s))}{s^a} + \sum_{p=0}^{a-1}(\mathbb{E}N-1)G(F_n)^{(p)}(0)\cdot\frac{a-p}{p!}\cdot\frac{1}{s^{a-p}}.$$

Let us give the lower bound for $G'(F_n(s))$.
Let us prove the inequality: for any $v \geq 1$, $vG'(v) \geq \mathbb{E}N \cdot G(v)$. In terms of generating functions, it means: $\mathbb{E}(Nv^N) \geq \mathbb{E}N \cdot E(v^N)$.
For any independent copies N_1, N_2 of the random variable N the following is true:
If $(N_1 - N_2)(v^{N_1} - v^{N_2}) \geq 0$, then $0 \leq \mathbb{E}[(N_1 - N_2)(v^{N_1} - v^{N_2})] = 2(\mathbb{E}(Nv^N) - \mathbb{E}N\mathbb{E}v^N)$.
We note that for any $v \geq 1$: $(N_1 - N_2)(v^{N_1} - v^{N_2}) \geq 0$.

Thus, for any $v \geq 1$, $vG'(v) \geq \mathbb{E}N \cdot G(v)$. Therefore, $F_n(s)G'(F_n(s)) \geq \mathbb{E}N \cdot G(F_n(s))$. Let us substitute this relation into the previous inequality:

$$s(\mathbb{E}N-1)F_{n+1}'(s) \geq \frac{\mathbb{E}N(\mathbb{E}N-1)sG(F_n(s))F_n'(s)}{F_n(s)s^a} - a(\mathbb{E}N-1)\frac{G(F_n(s))}{s^a} + \sum_{p=0}^{a-1}(\mathbb{E}N-1)G(F_n)^{(p)}(0)\cdot\frac{a-p}{p!}\cdot\frac{1}{s^{a-p}}.$$

Therefore,

$$s(\mathbb{E}N-1)F_{n+1}'(s) - aF_{n+1}(s)$$
$$\geq \frac{\mathbb{E}N \cdot G(F_n(s))}{F_n(s)s^a}\cdot(s(\mathbb{E}(N)-1)F_n'(s) - aF_n(s)) + \sum_{p=0}^{a-1}G(F_n)^{(p)}(0)\cdot\frac{1}{p!}\cdot\frac{(a-p)(\mathbb{E}N-1)+a-as^{a-p}}{s^{a-p}}. \quad (5)$$

Let us prove that $(a-p)(\mathbb{E}N-1)+a-as^{a-p} \geq 0$ for $p = 0, 1, \ldots, a-1$ and $s \leq \mathbb{E}N^{\frac{1}{a}}$.

Rewrite this as:

$$y(\mathbb{E}N-1)+a \geq as^y, \text{ where } y = a-p > 0. \quad (6)$$

Let us introduce two functions $f_1(y) = a(\mathbb{E}N)^{\frac{y}{a}}$ and $f_2(y) = y(\mathbb{E}(N)-1)+a$.
These functions are equal at $y = 0$ and $y = a$ and the function f_1 is concave, while f_2 is linear.
Therefore,

$$f_2(y) \geq f_1(y), \quad 0 \leq y \leq a,$$

namely,

$$y(\mathbb{E}N-1)+a \geq a(\mathbb{E}N)^{\frac{y}{a}} \geq as^y.$$

We have proved inequality (6), hence the sum in (5) is non-negative.
We conclude that

$$(\mathbb{E}N-1)sF_{n+1}'(s) - aF_{n+1}(s) \geq \frac{\mathbb{E}N \cdot G(F_n(s))}{F_n(s)s^a}[(\mathbb{E}N-1)sF_n'(s) - aF_n(s)].$$

Let $G(v) = \sum_{k=1}^{\infty} a_k v^k$, where $a_k \geq 0$ $(N > 0)$. So $\frac{G(v)}{v} = \sum_{k=1}^{\infty} a_k v^{k-1} \geq \sum_{k=1}^{\infty} a_k 1^{k-1} = 1$.

This means that

$$(\mathbb{E}N-1)sF'_{n+1}(s)-aF_{n+1}(s)\geq\left(\frac{\mathbb{E}N}{s^a}\right)^n[(\mathbb{E}N-1)sF'_0(s)-aF_0(s)].$$

Let $p=\frac{\mathbb{E}N}{s^a}>1$, $C_1=(\mathbb{E}N-1)sF'_0(s)-aF_0(s)$, then

$$(\mathbb{E}N-1)sF'_{n+1}(s)-aF_{n+1}(s)\geq p^nC_1,$$

which goes to $+\infty$, if $C_1>0$. As by assumption $D_0(\mathbb{E}(N)^{\frac{1}{a}},\mathbb{E}N)>0$, hence, for s close to $\mathbb{E}N^{\frac{1}{a}}$ we have $D_0(s,\mathbb{E}N)>0$.

Proof of Lemma 2

Let us fix $k\geq 1$ and $n\geq 0$. Consider the following inequality

$$X_{n+k}\geq\sum_{i=1}^{T_k}\mathbb{1}_{X_n^{(i)}\geq ak+1},$$

where $X_n^{(i)}, i\geq 1$, are independent copies of X_n, and T_k is a random variable equal to the number of ancestors, when building our counting tree, at depth k. It follows, from independence and properties of mathematical expectation that $\mathbb{E}T_k=(\mathbb{E}N)^k$.

Therefore,

$$\mathbb{E}(X_{n+k})\geq(\mathbb{E}N)^k\mathbb{P}(X_n\geq ak+1).$$

On the other hand, because $Q=0$, we may use (2), hence $\mathbb{E}(X_{n+k})\leq\frac{a}{\mathbb{E}N-1}$ for any $n\geq 0,k\geq 1$. Therefore,

$$\mathbb{P}(X_n\geq ak+1)\leq\frac{a}{(\mathbb{E}N-1)(\mathbb{E}N)^k}.$$

Let us sum it up and get the desired inequality.

$$\mathbb{E}(X_ns^{X_n})=\sum_{k=0}^{\infty}\mathbb{P}(X_n=k)ks^k\leq\sum_{k=1}^{\infty}\mathbb{P}\Big(ak+1\geq X_n>a(k-1)+1\Big)(ak+1)s^{ak+1}$$

$$\leq\sum_{k=1}^{\infty}\mathbb{P}(X_n\geq a(k-1)+1)(ak+1)s^{ak+1}\leq\sum_{k=1}^{\infty}\frac{a}{(\mathbb{E}N-1)(\mathbb{E}N)^{k-1}}(ak+1)s^{ak+1}$$

$$=s\left(\sum_{k=1}^{\infty}\frac{a}{(\mathbb{E}N-1)(\mathbb{E}N)^{k-1}}s^{ak+1}\right)'=C_{s,a,\mathbb{E}N}<\infty,$$

where $C_{s,a,\mathbb{E}N}$ is some constant that depends on $s,\mathbb{E}N$ and a.
Thus,

$$\mathbb{E}(X_ns^{X_n})\leq C_{s,a,\mathbb{E}N}.$$

3.3 Proof of the second point of Theorem 1

Let us define the sequence of functions D_n:

$$D_n(s)=(M-1)sF'_n(s)-aF_n(s).$$

As in Section 3, we formulate two lemmas, which will be proved below.

Lemma 3.

For any $s\geq 1+\dfrac{M-1}{a}$, *and* $n\geq 0$ *it is true that* $D_{n+1}(s)\leq\dfrac{MG(F_n(s))}{F_n(s)s^a}D_n(s)$.

Lemma 4. *If* $s>1$, *then*

$$\mathbb{E}(X_ns^{X_n})\geq\mathbb{E}(X_n)\mathbb{E}(s^{X_n}).$$

Lemma 3 implies that if $D_0(s_0) < 0$ for some $s_0 \geq 1 + \dfrac{M-1}{a}$, then for any $n > 0$ inequality $D_n(s_0) < 0$ holds.
Let us express D_n in terms of mathematical expectation:

$$D_n(s) = (M-1)sF_n'(s) - aF_n(s) = (M-1)\mathbb{E}(X_n s^{X_n}) - a\mathbb{E}(s^{X_n}) \leq 0,$$

Therefore,

$$a\mathbb{E}(s^{X_n}) \geq (M-1)\mathbb{E}(X_n s^{X_n}).$$

By Lemma 4 we have:

$$a\mathbb{E}(s^{X_n}) \geq (M-1)\mathbb{E}(X_n)\mathbb{E}(s^{X_n}).$$

We get that $\mathbb{E}(X_n) \leq \frac{a}{M-1}$, which means that $Q = 0$.

Proof of Lemma 3

For $p = 0, 1, \ldots, a-1$ and $s \geq 1 + \frac{M-1}{a}$ we will prove the following:

$$(a-p)(M-1) + a - as^{a-p} \leq 0.$$

Let us denote $y = a - p \geq 1.$ and rewrite the required fact as

$$y(M-1) + a \leq as^y.$$

Let us use Bernoulli's inequality:

$$(1+x)^n \geq 1 + nx, \text{ for any } n \in \mathbb{N}, x > 0.$$

We note that $as^y = a(1 + (s-1))^y$.
By assumption $y \geq 1, s \geq 1 + \frac{1}{a}$; it means $s - 1 \geq \frac{M-1}{a} > 0$. Therefore, we can apply Bernoulli's inequality to $x = s-1$ and $n = y$. It gives us:

$$(1 + (s-1))^y \geq 1 + (s-1)y \geq 1 + \frac{y(M-1)}{a}.$$

Hence,

$$as^y \geq a\left(1 + \frac{y(M-1)}{a}\right) = (M-1)y + a.$$

Let us prove the auxiliary inequality:

$$vG'(v) \leq MG(v), \text{ namely } F_n(s)G'(F_n(s)) \leq MG(F_n(s)).$$

As $\mathbb{E}N \leq M$ is true, $vG'(v) \leq \mathbb{E}(Nv^N) \leq \mathbb{E}N \cdot \mathbb{E}v^N \leq M \cdot \mathbb{E}v^N = MG(v)$ is true.
It follows that

$$s(\mathbb{E}(N)-1)F_{n+1}'(s) - aF_{n+1}(s)$$

$$\leq \frac{MG(F_n(s))}{F_n(s)s^a} \cdot (s(M-1)F_n'(s) - aF_n(s)) + \sum_{p=0}^{a-1} G(F_n)^{(p)}(0) \cdot \frac{1}{p!} \cdot \frac{(a-p)(M-1) + a - as^{a-p}}{s^{a-p}}. \quad (7)$$

All terms in the corresponding sum are non-positive, because as proven, $(a-p)(M-1) + a - as^{a-p} \leq 0$.
Therefore,

$$D_{n+1}(s) \leq \frac{MG(F_n(s))}{F_n(s)s^a} D_n(s).$$

Proof of Lemma 4

Let Y_1, Y_2 be independent copies X_n.

At $s \geq 1$ we have:

$$(Y_1 - Y_2)(s^{Y_1} - s^{Y_2}) \geq 0.$$

Therefore,

$$\mathbb{E}[(Y_1 - Y_2)(s^{Y_1} - s^{Y_2})] \geq 0.$$

Y_1 and Y_2 are independent, hence Y_1 and s^{Y_2} are independent.

Therefore,

$$\mathbb{E}(Y_1 s^{Y_1}) + \mathbb{E}(Y_2 s^{Y_2}) - \mathbb{E}Y_1 \mathbb{E}s^{Y_2} - \mathbb{E}Y_2 \mathbb{E}s^{Y_1} \geq 0.$$

Y_1, Y_2 are independent copies X_n, hence

$$2\mathbb{E}(X_n s^{X_n}) - 2\mathbb{E}X_n \mathbb{E}s^{X_n} \geq 0,$$

$$\mathbb{E}(X_n s^{X_n}) - \mathbb{E}X_n \mathbb{E}s^{X_n} \geq 0.$$

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