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TO THE 300th ANNIVERSARY OF SPbU

On the Role of St. Petersburg in the Development of Mathematical Analysis

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Abstract—The main stages in the development of mathematical analysis in St. Petersburg in the 18th–19th centuries are considered. We present brief biographical information and a description of the most important results of outstanding representatives of the St. Petersburg mathematical school: L. Euler, M.V. Ostrogradsky, V.Ya. Bunyakovsky, P.L. Chebyshev, Yu.V. Sokhotski, and A.A. Markov.

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In the second half of the 17th century, a major event occurred in world science: I. Newton and G.W. Leibniz laid the foundation for differential and integral calculus. Soon, fruitful collaboration between Leibniz and brothers Jacob and Johann Bernoulli began. However, the further development of this extremely important direction took place most intensively not in London, not in Berlin, and not in Paris; it took place in a completely “unsuitable” place, in the youngest European capital, St. Petersburg.

In the pre-Petrine times in Russia, systematic science did not exist and, in particular, mathematics did not exist. However, after the Tsar’s decree of 1724 on the establishment of the Academy of Sciences and the University under it, the situation began to change rapidly. These changes had the greatest impact on the exact sciences and especially on mathematics. The arrival in St. Petersburg of the 20-year-old Swiss Leonhard Euler, a student of Johann Bernoulli, played a decisive role in this. A unique talent, a great memory, and an amazing work capacity allowed Euler to create a word-level mathematical center in the shortest possible time. By a happy coincidence, he found himself in conditions that allowed him to fully demonstrate his genius. Euler highly appreciated the opportunities that life in St. Petersburg gave him. He wrote about this to the head of the Chancellery of the St. Petersburg Academy, I.D. Schumacher:

“*As far as I am concerned, in the absence of such an excellent circumstance I would be compelled, mainly, to turn to other activities in which, for all intents and purposes, could only be engaged in mere trifle. When His Royal Majesty*[1] *asked me recently where I learned what I know, I, in truth, answered, that I owe everything to my stay at the St. Petersburg Academy.*”

It is therefore not surprising that our brief review of the development of mathematics in St. Petersburg in the 18th–19th centuries begins with the works of Euler. His role in the creation of Russian science is unique. After him, some stagnation occurred in the development of Russian mathematics. The situation began to improve at the beginning of the next century, the 19th century, when interest in studying sciences increased not only in St. Petersburg and Moscow, but also in other major Russian cities, and European textbooks became available. This influenced the choice of life path of P.L. Chebyshev and Yu.V. Sokhotski. The other two heroes of our overview, M.V. Ostrogradsky and V.Ya. Bunyakovsky, chose education at the best universities in Europe, which became possible and prestigious. Over time, the level of

[1] Frederick II, King of Prussia.

Russian mathematics and its teaching increased. Brothers A.A. Markov and V. A. Markov graduated from St. Petersburg University.

We want to give the reader an opportunity to evaluate the contribution of these mathematicians associated with St. Petersburg to the formation and further development of mathematics. Our review, like any other, of this large topic cannot in any way claim to be complete. The reader can find a lot of additional interesting information in the book [1].

LEONHARD EULER (1703–1783)

“*Read Euler*, *read Euler*, *he is the master of us all.*”
(Pierre-Simon Laplace)

Leonhard Euler worked in St. Petersburg in 1724–1741 and 1766–1783, and in the interim, during his work in Berlin, while remaining a member of the St. Petersburg Academy of Sciences, he continued to publish his articles in the Proceedings of the Academy [2–4]. We will describe his main achievements in only one, but for him most important, branch of mathematics: the foundations of differential and integral calculus (now this branch is called mathematical analysis).

Basel problem (calculate the sum of inverse squares)**.** Jacob Bernoulli called in his book “Arithmetic Propositions on Infinite Series” (1689): “*If someone succeeds in finding what has so far defied our efforts, and if he lets us know about it, then we will be very much obliged to him.*” But during the life time of Bernoulli, the solution never appeared.

Euler, relying on the decomposition of the sine into an infinite product that he found (see below), came [5] to the equation

$$\sum_{n==1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

This result aroused great interest and immediately brought Euler into the circle of world mathematical leaders.

Euler’s reasoning, extremely witty and too bold for the 18th century, caused justified criticism from other mathematicians, in particular from his friends, the brothers Nikolaus and Daniel Bernoulli. Therefore, Euler later repeatedly returned to the Basel problem. In particular, he calculated the sum of a series with the correct 20 decimal places[2] and was convinced that the approximate values of π known at that time gave the same result. In addition, Euler considered it necessary to give formal (by the standards of the 18th century) proofs. He obtained one of them, based on the Taylor decomposition of the arcsine, in 1741. It is short and elegant:[3]

$$\frac{\pi^2}{8}=\int_0^{\frac{\pi}{2}}x\,dx=\int_0^{\frac{\pi}{2}}\arcsin(\sin x)\,dx=\int_0^{\frac{\pi}{2}}\sum_{k=0}^{\infty}\frac{(2k-1)!!}{(2k)!!(2k+1)}\sin^{2k+1}x\,dx$$

$$=\sum_{k=0}^{\infty}\frac{(2k-1)!!}{(2k)!!(2k+1)}\frac{(2k)!!}{(2k+1)!!}=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}=\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Therefore, $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{6}\pi^2$. For other ways to calculate this sum, see [6, 7].

[2] Apparently, the Euler transformation of slowly converging series and the Euler–Maclaurin formula are related to this.
[3] For the sake of simplicity, hereafter we assume that $(-1)!!=0!!=1$.

Later, Euler significantly enhanced his achievement by finding the sums of inverse even powers: for any positive integer m the equality

$$\zeta(2m) = \sum_{n=1}^{\infty} \frac{1}{n^{2m}} = (-1)^{m+1} \frac{(2\pi)^{2m}}{2(2m)!} B_{2m}$$

is true, where B_{2m} are the Bernoulli numbers, which are easily calculated using the recurrence formula $B_n = -\frac{1}{n+1} \sum_{k=1}^{n} \binom{n+1}{k+1} B_{n-k}$, $B_0 = 1$, $B_1 = -\frac{1}{2}$.

Decomposition of trigonometric functions into the sum of prime fractions. In 1735, Euler obtained a decomposition of the sine into an infinite product of linear factors, i.e., an analogue of Bezout's theorem for polynomials:

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right).$$

Taking the logarithm of this equality and then passing to derivatives, we can obtain the decompositions

$$\pi \cot \pi x = \sum_{n=-\infty}^{\infty} \frac{1}{n+x}, \qquad \frac{\pi}{\sin \pi x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n+x}$$

(the sums of these series are understood in the sense *of the principal value*, i.e., as the limit of symmetric partial sums $\sum_{n=-N}^{N} \ldots$).

Euler transform of numerical series. If the series $\sum_{n=0}^{\infty} a_n$ converges, then

$$\sum_{n=0}^{\infty} a_n = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \left(a_0 + k a_1 + \ldots + \frac{k!}{j!(k-j)!} a_j + \ldots + a_k \right).$$

For the convergence of a series on the right-hand side, the convergence of the original series is sufficient, but not necessary. Therefore, if the original series diverges and the transformed series converges, its sum is called the generalized Euler sum of the original series (*Euler's method of summation of series*). This is one of the first methods of the generalized summation of a numerical series.

Euler substitutions. The integration of functions of the form $R(x, y)$, where R is a rational function and $y(x) = \sqrt{ax^2 + bx + c}$, can be reduced to the integration of some rational function using substitutions (hereafter t is the new variable of integration)

I. if $a > 0$, then $t = y(x) \pm \sqrt{a}x$;

II. if $c > 0$, then $tx = y(x) \pm \sqrt{c}$;

III. if $y(x) = (x - \alpha)(x - \beta)$, then $t(x - \alpha) = y(x)$.

The Euler–Poisson integral.

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}$$

has become one of the most famous integrals in the world. It plays a huge role in probability theory and other areas of mathematics.

Euler integrals (Euler gamma and beta functions).

Following A.-M. Legendre, this is the name given to functions defined by the equations

$$\Gamma(x) = \int_0^{\infty} t^{x-1} \frac{dt}{e^t} \qquad (x > 0);$$

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt \qquad (x, y > 0).$$

Euler successfully applied them in solving many problems. The famous algebraist E. Artin speaks about the important role of these functions in mathematics in the preface to his short book [8]: "*the gamma-function can be classified in all respects as an elementary function*."

The relationship between the gamma and beta functions discovered by Euler

$$\mathrm{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (x, y > 0)$$

allows one to easily find the values of the gamma function at half-integer points, as well as to explicitly express the trigonometric integrals $\int_0^{\frac{\pi}{2}} \cos^p \varphi \sin^q \varphi d\varphi$ in terms of values of the gamma function.

Let us present some more formulas discovered by Euler.

• Infinite product for the gamma function

$$\frac{1}{\Gamma(x)} = x\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)\left(1+\frac{1}{n}\right)^{-x}.$$

• Euler–Gauss formula

$$\frac{1}{\Gamma(x)} = \lim_{n\to\infty} n^{-x} x(1+x)\cdots\left(1+\frac{x}{n}\right).$$

• Reflection formula for the gamma function

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

• The sum of trigonometric series

$$\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots = \frac{\pi - x}{2} \quad (0 < x < 2\pi)$$

is the first decomposition of an algebraic function into a trigonometric series (later called the Fourier series).

• Euler (1732)–Maclaurin (1735) formula

$$\sum_{k=m}^{n} f(k) = \int_m^n f(x)dx + \frac{f(n)+f(m)}{2} + \frac{f'(n)-f'(m)}{12} + \ldots + \frac{f^{(2j-1)}(n) - f^{(2j-1)}(m)}{(2j)!}B_{2j} + \ldots$$

(B_{2j} is the Bernoulli number).

• Euler–Mascheroni constant C (now it is often denoted by the letter γ):

$$C = \sum_{k=1}^{\infty}\left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right),$$

so

$$\sum_{k=1}^{n}\frac{1}{k} = \ln n + C + o(1) \quad \text{with } n \to +\infty.$$

Euler introduced the most important symbols for mathematics. After he began to use the notation π (from περιφερια) for the length of a circle with unit diameter, this notation, first used by William Jones in 1706, became generally accepted. Euler also introduced the symbols e to represent the Napier's number $\lim\left(1+\frac{1}{n}\right)^n$ (1727), $f(x)$ (1734), $i = \sqrt{-1}$ (1777), $\sum$ (1755), Δ (to denote finite differences).

Euler theorem for homogeneous functions. A function f given in $\mathbb{R}^m \backslash \{0\}$ is called homogeneous of degree p if $f(tx) = t^p f(x)$ for all x and all $t > 0$.

In order for the differentiable function to be homogeneous of degree p, it is necessary and sufficient that

$$pf(x) \equiv \langle x, \operatorname{grad} f(x)\rangle$$

(the scalar product of the argument x and the gradient of the function $f(x)$ is on the right-hand side).

Application of analysis to combinatorics, generating functions.[4]

• Euler identity (1737) representing the ζ function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

as

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}$$

(the product is taken over prime indices p) demonstrated the possibility of applying analysis to number theory (the precursor of analytic number theory).

• A partition of a natural number n is a representation of the number n as a sum of positive integers; $n = \lambda_1 + \lambda_2 + \ldots + \lambda_m$ (the order of the summands is not important). Let $p(n)$ denote the number of such partitions.

In 1740, Euler found the generating function[5] for the sequence $p(n)$:

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}.$$

• Euler's pentagonal theorem (1737):

$$\prod_{k=1}^{\infty} (1 - x^k) = \sum_{k=-\infty}^{\infty} (-1)^k x^{(3k^2-k)/2}.$$

The degree exponents on the right-hand side are the so-called pentagonal numbers.

• The Euler number E_n, $n \geqslant 0$, is defined as the number of alternating *up-down* permutations[6] of a segment of the natural series $\{1, 2, \ldots, n+1\}$. For example, $E_4 = 5$ is the number of *up-down* permutations (13254), (14253), (24154), (25341), and (25143) of fifth order. They can be calculated recursively using an analog of Pascal's triangle and are involved in many formulas, the most famous of which is:

$$\tan x = \sum_{k=0}^{\infty} \frac{E_{2k+1}}{(2k+1)!} x^{2k+1}, \quad \sec x = \sum_{k=0}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k}.$$

Continued fractions.

• Euler theorem on continued fractions: the decomposition of quadratic irrationality into continued fractions is eventually periodic.[7]

• Decomposition of the Napier's number into a continued fraction (1744):

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1 \ldots] = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \ldots}}}}}$$

• Euler related the partial sums of a series to convergents of a continued fraction (*Euler continued fraction*, 1744, [9]). Let $\{c_k\}_{k=0}^{\infty}$ be a sequence of nonzero complex numbers, $c_0 = b_0 = 1$, $b_k = c_k / c_{k-1}$ $(k \in \mathbb{N})$. Then

$$\sum_{k=0}^{n} c_k = \sum_{k=0}^{n} b_0 b_1 \cdots b_k = \cfrac{b_0}{1 - \cfrac{b_1}{1 + b_1 - \cfrac{b_2}{1 + b_2 - \cfrac{b_3}{1 + b_3 - \ldots - \cfrac{b_n}{1 + b_n}}}}}.$$

This technique allowed him to obtain many interesting formulas.

Euler also applied continued fractions to the calendar problem (1748).

4 Here series are understood in a formal sense as polynomials of infinite degree.
5 Euler did not use the term "generating function."
6 That is, there are such permutations σ that $\sigma(1) < \sigma(2) > \sigma(3) < \ldots$.
7 The converse statement belongs to Lagrange.

Differential equations. The simplest method for numerically solving the Cauchy problem for systems of ordinary differential equations (*Euler's broken lines*) was proposed by Euler in 1768.

Euler invented methods for solving linear equations with constant coefficients and ordinary differential equations of the second order with variable coefficients. He also proposed a method for solving equations in the form of power series, a method of varying constants, and introduced integrating factors.

Variational calculus, Euler–Lagrange equation. The term and the first systematic presentation of variational calculus were proposed by Euler in 1766. In the problem of finding the extremum of the functional

$$J(f) = \int_a^b F\left(x, f(x), f'(x)\right) dx,$$

where F is a twice continuously differentiable function, and f runs through the space of smooth functions on the segment $[a, b]$, Euler and Lagrange in the 1750s obtained a necessary condition for an extremum, reducing the problem to solving an ordinary differential equation

$$\frac{\partial F}{\partial f} - \frac{d}{dx}\frac{\partial F}{\partial f'} = 0.$$

Complex analysis. Euler dealt with complex analysis in various aspects. The most wonderful, according to many scientists, mathematical formula is the manifestation of the unity of algebra (i), geometry (π), and analysis (e):

$$e^{i\pi} + 1 = 0$$

is a consequence of Euler's formulas relating the exponent to trigonometric functions ($z \in \mathbb{C}$):

$$e^{iz} = \cos z + i \sin z,$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Euler derived equations, first considered by Alembert in connection with the equations of hydrodynamics, equivalent to the analyticity of a function. In order for the function $w = f(z)$, defined in some domain D of the complex plane, to be differentiable as a function of the complex variable $z = x + iy$, it is necessary and sufficient that its real and imaginary parts u and v are differentiable as functions of the real variables x and y and that, in addition, the conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied (later they came to be called Cauchy–Riemann conditions.)

Euler's four-square identity. The equality

$$(a^2 + b^2 + c^2 + d^2)(p^2 + q^2 + r^2 + s^2) = x^2 + y^2 + z^2 + t^2,$$

is true, where

$$x = ap + bq + cr + ds, \quad y = aq - bp \pm cs \mp dr,$$

$$z = ar \mp bs - cp \pm dq, \quad t = as \pm br \mp cq - dp.$$

This formula is important for the theory of quaternions.

MIKHAIL VASILYEVICH OSTROGRADSKY (1801–1861)

M.V. Ostrogradsky received his initial mathematical education in Kharkiv University, after which he continued his studies in mathematics in Paris, in the Sorbonne and the College de France. He attended lectures by the most famous French scholars: P.-S. Laplace, J. Fourier, A.-M. Ampere, S.D. Poisson, and A.L. Cauchy (who highly praised the works by Ostrogradsky, submitted to the Paris Academy of Sciences). In 1828 Ostrogradsky returned to Russia, where in 1830 he was elected as an Extraordinary Academician of the St. Petersburg Academy of Sciences. In addition to his work in the Academy, in 1831–1860 Ostrogradsky was a professor in many higher educational institutions of St. Petersburg and lectured on analysis, geometry, and mechanics. Contemporaries spoke of him as a remarkable lecturer.

The most famous achievement of Ostrogradsky is the formula connecting the surface and volume integrals, currently known as the Gauss–Ostrogradsky formula. We present here its classical formulation.

Let V be a compact subset in $\mathbb{R}^3$ with piecewise smooth boundary $S = \partial V$, and $\mathbf{F} = (P, Q, R)$ be a vector field continuously differentiable in some neighborhood of the set V. Then the equality

$$\int_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV = \int_S (P\cos\alpha + P\cos\beta + R\cos\gamma)\, dS \tag{1}$$

holds. Here dV and dS denote differentials of volume and surface area, that is, in fact, Lebesgue measures in $\mathbb{R}^3$ and on the surface S, and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the components of the unit external normal $\mathbf{n}$ to the boundary S. The formula (1) can be written as follows:

$$\int_V \operatorname{div} \mathbf{F}\, dV = \int_S \langle \mathbf{F}, \mathbf{n} \rangle\, dS.$$

The same formula is valid in any space $\mathbb{R}^n$, $n \geq 2$ (in the case $n = 2$ it is equivalent to Green's formula).

The first special cases of formula (1) (known in English publications as the "divergence theorem") were noted by Lagrange in his work "New Investigations on the Nature and Propagation of Sound" (1762). Special cases of this formula were also considered by C. Gauss (in 1813, and then in 1830 and 1839). However, Ostrogradsky was the first to formulate and prove the general version of the formula. He presented this result to the Paris Academy of Sciences in February 1826. The proof was first published in 1828 in the Proceedings of the St. Petersburg Academy of Sciences ("Note on the Theory of Heat", ([10], X)). In 1834, Ostrogradsky generalized the formula to the case of an arbitrary number of variables in the paper ([11], V). In it, using this result, he found an expression for the parameter derivative of a multiple integral with variable limits and obtained a formula for the variation of a multiple integral.

Of course, the Gauss–Ostrogradsky formula was later included as a special case in the Stokes formula for differential forms:

$$\int_M d\omega = \int_{\partial M} \omega.$$

For the history of the Stokes formula and its predecessors, see [12].

A number of works by Ostrogradsky are devoted to the theory of integration (see [11]). Here his name is given to a method for integrating a rational function that has multiple roots in the denominator ([11], XVIII). As noted by Katz [13], in the paper [11, X] "Ostrogradsky gave the first rigorous proof of the change of variables formula in the double integral using the concept of infinitesimals."

Important results were obtained by Ostrogradsky in the field of ordinary differential equations. Thus, for example, he considered a method for solving nonlinear equations using decomposition into a power series with respect to the parameter. Similar results were obtained in the same years by Liouville. In particular, in 1838 Ostrogradsky ([11], XII) and Liouville simultaneously and independently obtained a formula for the Wronskian $W(x)$ for the system of solutions of a homogeneous differential equation of order n. Namely, if the equation

$$y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_{n-1}(x)y' + P_n(x)y = 0$$

is given, then

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} P_1(t)dt\right).$$

A similar formula is valid for the determinant of the fundamental solution Φ of the system of n linear homogeneous equations $y' = A(x)y$, where A is the coefficient matrix:

$$\det\Phi(x) = \det\Phi(x_0)\exp\left(\int_{x_0}^{x} \mathrm{Tr}\, A(t)dt\right).$$

VICTOR YAKOVLEVICH BUNYAKOVSKY (1804–1889)

The entire long-term scientific and pedagogical activity of the outstanding Russian mathematician V.Ya. Bunyakovsky is connected with St. Petersburg. Bunyakovsky, like Ostrogradsky, received his mathematical education mainly in France, where he attended lectures by such great mathematicians as Laplace, Poisson, Fourier, Cauchy, Legendre. In 1826, after he was awarded the degree of Doctor of Mathematical Sciences by the University of Paris for a thesis written under the supervision of Cauchy, Bunyakovsky came to St. Petersburg. From 1826 to 1860 he was a professor at a number of higher educational institutions of St. Petersburg, in particular, from 1846 to 1860 he was a professor at St. Petersburg University. In 1828 Bunyakovsky became an adjunct of the St. Petersburg Academy of Sciences, in 1830 he became an Extraordinary Academician of the St. Petersburg Academy of Sciences, and in 1836 he became an Ordinary Academician. From 1864 to 1889 Bunyakovsky was a vice president of the St. Petersburg Academy of Sciences.

Bunyakovsky made a very significant contribution to the teaching of mathematics in Russia. For many years he taught a course in differential and integral calculus, following the "strict" presentation by Cauchy. Having reworked the lectures of Laplace that he had attended in Paris, Bunyakovsky wrote the best textbook of that time, "Fundamentals of the Mathematical Theory of Probability."

However, the most famous contribution of Bunyakovsky to analysis is the integral inequality that bears his name. In 1821, Cauchy proved an inequality for sums

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leqslant \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2.$$

In 1859, Bunyakovsky [14] proved its analog for the integral of the product of two real functions, continuous on the interval $[a, b]$:

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \leqslant \int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx.$$

This inequality (often called the Cauchy–Schwarz inequality) was rediscovered much later (in 1888) by H. Schwarz.

Currently, the name "Cauchy–Bunyakovsky–Schwarz inequality" (also Cauchy–Bunyakovsky or Cauchy–Schwarz) bears also a more general inequality between the scalar product and the norm in an abstract Hilbert space: $|\langle x, y\rangle| \leqslant \|x\| \cdot \|y\|$.

Bunyakovsky also obtained a number of important results in number theory. He formulated a remarkable hypothetical criterion that among the values of the polynomial at integer points there are infinitely many prime numbers. With the exception of the case of first-degree polynomials (Dirichlet's theorem on arithmetic progressions), this hypothesis remains completely open.

PAFNUTY LVOVICH CHEBYSHEV (1821–1894)

P.L. Chebyshev studied and defended his master's thesis at Moscow University, and his dissertations "On integration using logarithms" (for the position of adjunct professor, 1847) and "Theory of comparisons" (for doctorate, 1849) were defended at St. Petersburg University. Chebyshev was an adjunct professor in 1847—1853, then he was an extraordinary professor in 1853—1857, after which he was an ordinary professor of St. Petersburg University in 1857—1882. In 1859, he was elected an Ordinary Academician of the St. Petersburg Academy of Sciences.

The source of the information given below is the collection of works by Chebyshev [15—18].

Integration theory.

• Chebyshev's fundamental result in the theory of indefinite integrals is as follows: the integral $\int x^m (a + bx^n)^p \, dx$ "is not expressed in finite form," that is, it is not reduced to integrals from rational functions, except for the three Euler cases

$$1)\ p \in \mathbb{Z}, \quad 2)\ \frac{m+1}{n} \in \mathbb{Z}, \quad 3)\ \frac{m+1}{n} + p \in \mathbb{Z}.$$

• The development of the theory of indefinite integrals, which can be expressed in finite form, led Chebyshev to unexpected formulations. For example, in order for an integral

$$\int \frac{dx}{\sqrt[3]{x^3 + ax + b}}$$

to be expressed in elementary functions, it is necessary that at least for one of the roots of the equations

$$x^3 = \frac{27}{4}\frac{b^2}{a^3} + 1, \quad 3\left(\frac{b^2}{a^3}\right)^2 x^4 + 6\frac{b^2}{a^3}(x^2 + 2x) = 1,$$

the condition $\frac{a^3}{b^2} x \in \mathbb{Q}$ be satisfied.

• **Integral inequality for monotone functions.** Let the functions u and v be nondecreasing in $[0,1]$. Then

$$\int_0^1 u(x)v(x)dx \geqslant \int_0^1 u(x)dx \int_0^1 v(x)dx.$$

If u and v are strictly increasing, then the inequality is strict.

• **Chebyshev inequality.** Chebyshev also obtained the inequality for the distribution function (in probability theory it is sometimes called A.A. Markov's inequality). Here we give its modern formulation. Let $(X, \mathfrak{A}, \mu)$ be a measure space. Then for any integrable function f on X and any $t > 0$ the inequality

$$\mu(\{x \in X : |f(x)| > t\}) \leq \frac{1}{t}\int_X |f| \, d\mu$$

is true. Despite its simplicity, the Chebyshev inequality plays a fundamental role in analysis (for example, for weak-type estimates in the theory of singular integral operators) and in probability theory.

Orthogonal polynomials. The algebraic polynomials T_n and U_n introduced by Chebyshev have, as was noted by him and as further development of the analysis showed, a number of important properties.

• The polynomials $\{T_n\}_{n\geqslant 0}$, $T_n(x) = \frac{1}{2^{n-1}}\cos(n\arccos x)$ at $|x| \leqslant 1$, form an orthogonal system with the weight $\frac{1}{\sqrt{1-x^2}}$:

$$\int_{-1}^{1} T_j(x)T_k(x)\frac{dx}{\sqrt{1-x^2}} = 0 \quad \text{for} \quad j \neq k.$$

• Polynomials $\{U_n\}_{n\in\mathbb{N}}$, $U_n(x) = \frac{1}{2^n}\frac{\sin((n+1)\arccos x)}{\sin\arccos x}$ at $|x| \leqslant 1$ form an orthogonal system with the weight $\sqrt{1-x^2}$:

$$\int_{-1}^{1} U_j(x)U_k(x)\sqrt{1-x^2}\,dx = 0 \quad \text{for} \quad j \neq k.$$

Chebyshev systematically used the decomposition of functions into continued fractions as a powerful analytical tool. In a continued fraction

$$\frac{1}{\sqrt{z^2-1}} = \cfrac{1}{z - \cfrac{1}{2z - \cfrac{1}{2z - \cfrac{1}{2z - \ldots}}}}$$

polynomials U_{n-1} and T_n are the numerator and denominator of the nth convergent.

Approximation theory. Chebyshev laid the foundations of the theory of the best approximation of a function by polynomials (now the term "*Chebyshev approximations*" is commonly used). In formulating the results of this theory, it is convenient to use the notations that have become standard: $\mathcal{P}_n$ is the set of all algebraic polynomials of degree no greater than n; for a function f continuous in the interval $[a,b]$, its *uniform* (or *Chebyshev*) *norm* $\|f\|_C$ is determined by the equality

$$\|f\|_C = \max_{x\in[a,b]}|f(x)|.$$

• The following theorem is one of the most important results in this field. For a real function $f \in C([a,b])$ and $n \in \mathbb{Z}_+$, we set

$$E_n(f) = \min_{P\in\mathcal{P}_n}\|f - P\|_C.$$

Let this minimum be achieved for the *best approximation polynomial* $P_n \in \mathcal{P}_n$. Then the set

$$Z_n = \{x \in [a,b] : |f(x) - P_n(x)| = E_n(f)\}$$

consists of at least $n+2$ points, and the values of the difference $f - P_n$ have opposite signs at neighboring points (*Chebyshev alternance*).

An important property of polynomials $\{T_n\}$ on the interval $[-1,1]$ is the following: for $n \in \mathbb{N}$ the equality

$$\min_{P\in\mathcal{P}_{n-1}}\left\|x^n - P(x)\right\|_C = \frac{1}{2^{n-1}}$$

holds and the minimum is attained at the polynomial $x^n - T_n(x)$.

In other words, among the polynomials of $\mathcal{P}_n$ with the higher coefficient equal to one, the polynomial $T_n(x)$ has the smallest Chebyshev norm on the interval $[-1,1]$ (the polynomial that deviates the least from zero).

• As established by Chebyshev, the polynomials $\{U_n\}$ are extremal in the metric $L^1([-1,1])$ in the following sense: the smallest value of the integral $I(P) = \int_{-1}^{1} \left| x^n - P(x) \right| dx$ on $\mathcal{P}_{n-1}$ is equal to $\frac{1}{2^{n-1}}$ and is attained[8] at $P(x) = x^n - U_n(x)$.

• The development of the theory of best approximation sometimes led Chebyshev to complex expressions. Let $h > 0$ and $H \notin [-h,h]$. The polynomial P_{n-1} of $\mathcal{P}_{n-1}$, for which the norm $\left\| P_{n-1}(x) - \frac{1}{H-x} \right\|_C$ is minimal, is given by the formula

$$P_{n-1}(x) = \frac{1}{H-x} + \frac{1}{2}\left(\frac{\left(x + \sqrt{x^2 - h^2}\right)^{n-1} \left(Hx - h^2 + \sqrt{\left(H^2 - h^2\right)\left(x^2 - h^2\right)}\right)}{\left(H^2 - h^2\right)\left(H + \sqrt{H^2 - h^2}\right)^{n-1} (H - x)} \right.$$
$$\left. + \frac{\left(x - \sqrt{x^2 - h^2}\right)^{n-1} \left(Hx - h^2 - \sqrt{\left(H^2 - h^2\right)\left(x^2 - h^2\right)}\right)}{\left(H^2 - h^2\right)\left(H + \sqrt{H^2 - h^2}\right)^{n-1} (H - x)} \right).$$

• The following theorem is important for the theory of moments.

Let D be a domain in $\mathbb{R}^{2m}$ and $C = (c_0, c_1, \ldots, c_{2m-1}) \in D$. Assume that

$$\begin{vmatrix} c_0 & \ldots & c_{k-1} \\ c_1 & \ldots & c_k \\ \ldots & \ldots & \ldots \\ c_{k-1} & \ldots & c_{2k-2} \end{vmatrix} > 0 \quad \text{for } k = 1, \ldots, m$$

and positive numbers $x_1(C), \ldots, x_m(C)$ are the roots of the polynomial

$$P_m(x, C) = \begin{vmatrix} c_0 & c_1 & \ldots & c_m \\ c_1 & c_2 & \ldots & c_{m+1} \\ \ldots & \ldots & \ldots & \ldots \\ c_{m-1} & c_m & \ldots & c_{2m-1} \\ 1 & x & \ldots & x^m \end{vmatrix}.$$

Then $x_1(C), \ldots, x_m(C)$ are increasing functions of $c_1, c_3, \ldots, c_{2m-1}$ and decreasing functions of $c_0, c_2, \ldots, c_{2m-2}$.

The distribution of prime numbers. The problem of finding asymptotically exact formulas for the value $\pi(x)$, i.e., the number of prime numbers not exceeding x, has been of interest to mathematicians for many years. At the beginning of the 19th century, Legendre conjectured that $\pi(x)$ is well approximated by the function $\frac{x}{\ln x}$, and, in 1838, P.G.L. Dirichlet proposed the integral logarithm $\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}$ to approximate the function $\pi(x)$.

In two works (1848 and 1850), Chebyshev proved that for all sufficiently large x the inequality holds:

$$0.89 \frac{x}{\ln x} \leqslant \pi(x) \leqslant 1.11 \frac{x}{\ln x}.$$

[8] This result was rediscovered in the works by E.I. Zolotarev–A.N. Korkin, T.J. Stiltjes, M. Fujiwara, and S.N. Bernstein.

He also showed that if the limit $\lim\limits_{x\to+\infty}\dfrac{\pi(x)\ln x}{x}$ exists, then it is equal to 1. The latter statement follows, in particular, from the following profound result by Chebyshev: for any numbers $\alpha > 0$ and $n > 0$, each of the inequalities

$$\pi(x) > \int_2^x \frac{dt}{\ln t} - \frac{\alpha x}{\ln^n x}, \quad \pi(x) < \int_2^x \frac{dt}{\ln t} + \frac{\alpha x}{\ln^n x}$$

has infinitely many solutions.

The results obtained by Chebyshev allowed him to prove Bertrand's postulate (a hypothesis formulated in 1840 by French mathematician J. Bertrand): for any $n \geqslant 2$ there is a prime number lying strictly between numbers n and $2n$.

In 1896, J. Hadamard and C.-J. de la Vallée-Poussin independently proved the exact asymptotics $\pi(x) \sim \dfrac{x}{\ln x}$, $x \to +\infty$. However, their methods (as well as Chebyshev's methods) were essentially analytic. An elementary (but not simple) proof of the theorem on the distribution of prime numbers using number theory methods was found only in 1949 by A. Selberg and P. Erdös.

YULIAN VASILIEVICH SOKHOTSKI (1842–1927)

Yu.V. Sokhotski received his initial mathematical education at the Warsaw gymnasium, he graduated from its physics and mathematics department in 1860. In the same year he moved to St. Petersburg. In 1866, Sokhotski successfully passed his master's examinations at the mathematics department of the Physics and Mathematics Faculty of St. Petersburg University (the examiner was Chebyshev). Two years later he defended his master's thesis (the opponents were Chebyshev and O.I. Somov) and since the autumn of the same year Sokhotski was a privat-docent of St. Petersburg University. In 1873 he defended his doctoral thesis. From 1883, Sokhotski was an ordinary professor, and in 1893 he become an honorary professor of St. Petersburg University. In 1890, Sokhotski become one of the founders (and deputy president) of the St. Petersburg Mathematical Society, and from 1892 to 1917 he was the president of this society.

The main works of Sokhotski are in the field of the theory of functions of a complex variable and the theory of elliptic functions. His name is associated with at least two remarkable results in complex analysis. The first is the Sokhotski–Weierstrass theorem, which states that *a holomorphic function in each neighborhood of an essential singularity takes values as close as desired to an arbitrary given complex number*.

Theorem. *If the set $G \subset \mathbb{C}$ is open, $a \in G$, the function f is holomorphic in $G_a = G\backslash\{a\}$ and a is an essential singularity of the function f, then the set $f(G_a)$ is everywhere dense in $\mathbb{C}$.*

This theorem was derived by Sokhotski in his master's thesis (1868). Interestingly, the same result was simultaneously and independently proved by other mathematicians. In the same year, 1868, this result was obtained by Italian mathematician F. Casorati, and in 1876 it was published by K. Weierstrass (that is why the Sokhotski–Weierstrass theorem in foreign publications is sometimes called the Casorati–Weierstrass theorem). However, a similar result was first mentioned in a book by the French mathematicians C. Briot and J.-C. Bouquet (1859).

As early as in 1879, C.E. Picard obtained a much stronger result (the so-called Picard's Great Theorem): *a holomorphic function in each neighborhood of an essential singularity takes all complex values, with at most one exception*.

However, the Sokhotski–Weierstrass theorem still has "methodological value." Unlike Picard's theorem, the proof of which is quite complicated, the Sokhotski–Weierstrass theorem is included in basic courses in complex analysis as a useful application of the theorem that the boundedness of an analytic function near a singularity implies that this singularity is removable. Indeed, suppose that $A \notin \operatorname{Clos} f(G_a)$. Then the function $g(z) = \dfrac{1}{f(z) - A}$ is bounded in G_a, and hence the singularity a is removable. Thus, $f(z) = \dfrac{1}{g(z)} + A$, and the singularity a of the function f is removable or a pole.

The most important result of Sokhotski, which retains its significance to this day, relates to the behavior of Cauchy-type integrals on a curve when approaching this curve. This result was also obtained by Sokhotski in his master's thesis and published in 1873 [19].

Let γ be a smooth closed Jordan curve in $\mathbb{C}$, and let f be a sufficiently good (say, continuous) function on γ. We consider the Cauchy integral

$$F(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{C}\backslash\gamma. \tag{2}$$

The formula (2) defines the analytic functions F_i and F_e, respectively, in the bounded region G bounded by the curve γ, and in its complement $\widehat{G} = \mathbb{C}\backslash(G \cup \gamma)$. We will also consider the Cauchy transform (the integral in the sense of the principal value)

$$(Cf)(z) = \lim_{\delta\to 0} \int_{\gamma\backslash B_\delta(z)} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \gamma,$$

where $B_\delta(z) = \{\zeta \in \mathbb{C} : |\zeta - z| < \delta\}$. This integral makes sense, for example, if the function f lies in some Lipschitz class Lip_α, $\alpha > 0$, on the curve γ. In this case, for all $z \in \gamma$, the following formulas connecting F_i, F_e, and Cf take place:

$$\begin{aligned} \lim_{w\to z} F_i(w) &= (Cf)(z) + \frac{1}{2} f(z), \\ \lim_{w\to z} F_e(w) &= (Cf)(z) - \frac{1}{2} f(z). \end{aligned} \tag{3}$$

Specifically,

$$f(z) = \lim_{w\to z}\left(F_i(w) - F_e(w)\right).$$

Formulas (3) are known as the Sokhotski–Plemelj formulas. Unfortunately, as in many other cases, Sokhotski's results were not noticed by the mathematical community. In 1908, J. Plemelj independently proved these formulas in his paper [20], devoted to solving the Riemann–Hilbert problem. We note that in the same paper, Plemelj announced a solution (positive) to Hilbert's 21st problem on the existence of linear differential equations with a given monodromy group. Plemelj's proof was considered correct for more than 60 years. Only in the 1970s was an error found in Plemelj's reasoning, and in 1989, A.A. Bolibrukh solved Hilbert's 21st problem in the negative sense by constructing a counterexample.

It is easy to show that for the validity of the Sokhotski–Plemelj formulas (3) it is sufficient to assume that the curve γ is rectifiable. A much deeper question is whether it is possible to weaken the conditions on the function f. I.I. Privalov showed that if we restrict ourselves to nontangential (angular) boundary values, then the finite limits $\lim_{w\to z} F_i(w)$, $\lim_{w\to z} F_e(w)$, and $(Cf)(z)$, where $z \in \gamma$, exist or do not exist simultaneously, and the corresponding limits satisfy the Sokhotski–Plemelj formulas. Thus, the problem is reduced to studying the existence of the Cauchy transform on a curve. This question became one of the origins of the most important branches of twentieth-century analysis: the Calderón–Zygmund theory singular integrals. Using Calderón's results that the Cauchy transform on Lipschitz curves with a sufficiently small Lipschitz constant has weak type (1,1) [21], we can prove the following result.

Theorem. *If the curve* γ *is rectifiable and* $f \in L^1(\gamma)$, *then for almost all* $z \in \gamma$ *there exist finite values* $\lim_{w\to z} F_i(w)$, $\lim_{w\to z} F_e(w)$, *and* $(Cf)(z)$, *and for them the Sokhotski–Plemelj formulas are true*.

Reducing the case of a rectifiable curve to the case of a curve with a sufficiently small Lipschitz constant can be done using one result obtained by V.P. Khavin in 1965 [22], another prominent representative of the St. Petersburg school of analysis (but already of the 20th century). A more detailed discussion of the Sokhotski–Plemelj formulas and the development of related ideas can be found in the book ([1], pp. 91–98).

ANDREY ANDREEVICH MARKOV (1856–1922)

Unlike the other heroes of our review, A.A. Markov received a mathematical education in St. Petersburg University, where he attended lectures by Chebyshev, who became his scientific supervisor, as well as A.N. Korkin and E.I. Zolotarev. Thus, Markov is a representative of the St. Petersburg school of mathematics in the proper sense of the word. In 1880, he defended his master's thesis "On the binary quadratic forms of a positive determinant," in which he obtained outstanding results in number theory. In 1884, he defended his doctoral thesis "On some applications of algebraic continued fractions." From 1880 until the end of his life A.A. Markov taught at St. Petersburg University (from 1886 as a professor). In 1896, he was elected an Ordinary Academician of the St. Petersburg Academy of Sciences.

Andrei Markov's younger brother, Vladimir Andreevich Markov (1871–1897), was also a talented mathematician. He is the author of a remarkable generalization of the inequality for the derivative of an algebraic polynomial obtained by A.A. Markov. We will discuss this inequality below. Unfortunately, V.A. Markov died tragically early with tuberculosis, without having time to complete his master's thesis "On positive triple quadratic forms," published under the supervision of A.A. Markov after the author's death.

The most famous results of A.A. Markov relate to probability theory. He created theories of random processes (later called Markov processes) and Markov chains, and made significant progress in the study of the law of large numbers and the central limit theorem. Another direction of Markov's research is related to number theory, where he obtained fundamental results on the theory of quadratic forms and Diophantine equations. Here we will limit ourselves to discussing several outstanding results of A.A. Markov in mathematical analysis.

Polynomial inequalities. In 1889, A.A. Markov published a note entitled "On a Question of D.I. Mendeleev" [23] (see also [24]). In this, the following question was studied: how can we estimate the derivative of a polynomial of a given degree on a segment, provided that the polynomial itself on this segment does not exceed a given number in modulus? D.I. Mendeleev was interested only in the special case $n = 2$, which has applications in the theory of aqueous solutions. As a result, the classical Markov inequality appeared for a polynomial p of degree n on the interval $[-1,1]$:

$$\|p'\|_C \leqslant n^2 \|p\|_C. \tag{4}$$

The inequality (4) becomes an equality for Chebyshev polynomials of the first kind T_n. Markov also studied in detail the question of estimating $|p'(x)|$ at a given point $x \in \mathbb{R}$ (not necessarily from the segment $[-1,1]$) and obtained exact estimates, also in terms of Chebyshev polynomials.

Iterating the inequality (4), it is easy to show that

$$\left\|p^{(k)}\right\|_C \leqslant (n(n-1)\dots(n-k+1))^2 \|p\|_C,$$

however, this inequality is not exact (because the derivative of the Chebyshev polynomial will no longer be a Chebyshev polynomial). The exact form of the Markov inequality for higher derivatives was established in 1892 by Vladimir Markov [25] (in 1916, its translation into German was reprinted in Mathematische Annalen with a preface by Bernstein):

$$\left\|p^{(k)}\right\|_C \leqslant \frac{n^2(n^2-1)(n^2-2^2)\dots(n^2-(k-1)^2)}{(2k-1)!!}\|p\|_C. \tag{5}$$

Inequality (5) is already exact, and the extremal polynomials for it are again the Chebyshev polynomials T_n.

Along with Bernstein's inequality $\|h'\|_C \leq n\|h\|_C$ for a trigonometric polynomial h of a degree no greater than n (1912), the Markov inequality is a one of the most important tools of approximation theory. It served as a starting point for a whole number of studies; at present, a huge number of different generalizations and analogs of this inequality are known (see, for example, [26–28] and the classical monographs by Borwein and Erdélyi [29] and Rahman and Schmeisser [30]).

Orthogonal polynomials. A series of papers by Markov is devoted to the behavior of the zeros of orthogonal polynomials. In the studies by Markov [31], the following important result about the monotonic behavior of zeros of orthogonal polynomials depending on the parameter is established.

Theorem. *Let $\{p_n(x,t)\}_{n\geqslant 0}$ be a sequence of polynomials of the variable x, orthogonal on the interval (a, b) with the weight $w(x,t)$ depending on the parameter $t \in (c,d)$. Suppose that the weight $w(x,t)$ is positive and has a continuous first derivative for t at $(x,t) \in (a,b)\times(c,d)$. Suppose also that the integrals*

$$\int_a^b x^k \frac{\partial w}{\partial t}(x,t)\,dx, \qquad k = 0,1,\dots,2n-1,$$

converge uniformly on t on every compact subinterval of the interval (c, d). *If the function* $\ln w(x,t)$ *is convex as a function of* $t \in (c, d)$, *then the* kth *root of the polynomial* $p_n(x,t)$ *is an increasing function of t for every fixed* $k \leqslant n$.

As an application of these results, Markov showed that the zeros of Jacobi polynomials orthogonal on the interval $(-1,1)$ with the weight $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha, \beta > -1$, decrease as functions of α and increase as functions of β. Also in studies by Markov [31], the bounds for zeros of Legendre polynomials are obtained. The results and methods of Markov were developed in the works of a number of specialists on orthogonal polynomials (G. Szegö, G. Freud, M. E. H. Ismail, A. Kroh, F. Peherstorfer, etc.).

The moment problem. Markov made a significant contribution to the study of the moment problem. Here we present one extremely important result, the so-called Chebyshev–Markov inequalities or Chebyshev–Markov–Stieltjes inequalities (the inequalities were formulated by Chebyshev without a proof, they were proved by Markov and, independently and a little later, by Stieltjes). The proof was published by Markov [32] (see also ([33], Chapter IV, Section 3)). Informally speaking, the Chebyshev–Markov–Stieltjes inequalities give exact estimates of the measures from above and below in terms of their first moments (more precisely, first orthogonal polynomials).

Let μ be a measure on a line such that $\int_{\mathbb{R}} |x|^{2n-1}\, d\mu(x) < +\infty$, and let p_k, $k = 0, \ldots, n$, be polynomials orthogonal with respect to the measure μ, that is, $\int_{\mathbb{R}} p_j(x) p_k(x)\, d\mu(x) = \delta_{jk}$, $0 \leqslant j \leqslant n-1$, and $0 \leqslant k \leqslant n$. Let us denote by $x_1, \ldots, x_n$, $x_1 < x_2 < \ldots < x_n$, the zeros of the polynomial p_n and put

$$\rho_{n-1}(x) = \left(\sum_{k=0}^{n-1} p_k^2(x) \right)^{-1}$$

(the reciprocal value to the Christoffel–Darboux kernel). Then for $j = 1, 2, \ldots, n$ the following inequalities

$$\mu\left((-\infty, x_j]\right) \leqslant \rho_{n-1}(x_1) + \cdots + \rho_{n-1}(x_j) \leqslant \mu\left((-\infty, x_{j+1})\right)$$

are valid (of course, for $j = n$ only the left-hand side inequality makes sense).

Further information about Markov's works on the moment problem and the development of the ideas of Chebyshev and Markov can be found in the monograph [33].

CONCLUSIONS

In this brief review we have touched upon only some of the works and results of our famous predecessors of the 18th–19th centuries. The seeds they sowed gave abundant sprouts in the 20th century all over the world, but above all in St. Petersburg/Leningrad, where their knowledge and traditions were passed through a chain of direct contacts from teachers to students. Let us list here several names that have made the Department of Mathematical Analysis of St. Petersburg/Leningrad University famous: S.N. Bernstein, H.M. Müntz, V.I. Smirnov, G.M. Fikhtengol'ts, I.P. Natanson, G.M. Goluzin, G.G. Lorentz, L.V. Kantorovich, B.Z. Vulikh, S.M. Lozinsky, N.A. Lebedev, G.P. Akilov, D.A. Vladimirov, G.I. Natanson, M.Z. Solomyak, B.M. Makarov, V.P. Khavin, A.M. Vershik, B.S. Pavlov, and S.A. Vinogradov.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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