Exact lock-in range for a second-order PLL with discontinuous sawtooth phase detector characteristic

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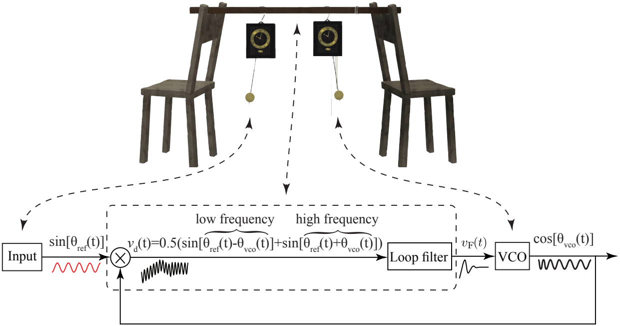
Operating principle of phase-locked loops: synchronization

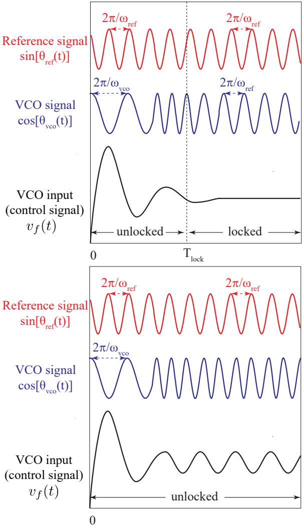
mutual synchronization

C. Huygens, *"Horologium Oscillatorium"*, **1673**.



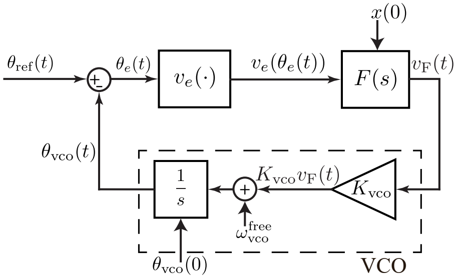
"master"–"slave" synchronization

E.V. Appleton, *"Automatic synchronization of triode oscillators"*, **1923** (Nobel Prize winner, 1947).



Mathematical model:

$$\dot{x} = Ax + Bv_e(\theta_e),$$
$$\dot{\theta}_e = \omega_e^{\text{free}} - K_{\text{vco}}(Cx + Dv_e(\theta_e)).$$

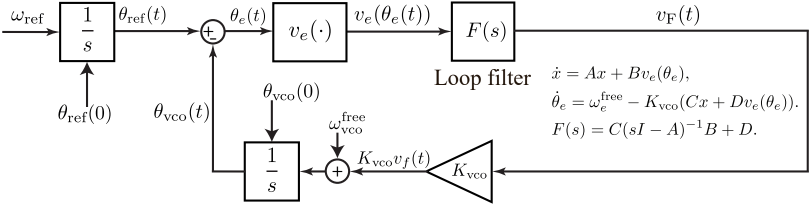


Baseband model of an analog PLL.

$x(t) \in \mathbb{R}^n$ — filter state
$\theta_e(t) \in \mathbb{R}$ — phase error
$A \in \mathbb{R}^{n\times n}$ — constant matrix
$B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, $D \in \mathbb{R}$
$F(s) = C(sI - A)^{-1}B + D$ — loop filter transfer function

$K_{\text{vco}} > 0$ — VCO gain
$\omega_e^{\text{free}} = \omega_{\text{ref}} - \omega_{\text{vco}}^{\text{free}}$ — frequency error
$\omega_{\text{ref}} \equiv \dot{\theta}_{\text{ref}}(t)$ — reference frequency
$\omega_{\text{vco}}^{\text{free}}$ — free-running frequency of VCO
$v_e(\theta_e)$ — phase detector characteristic (nonlinear periodic function)

Main characteristics of PLL dynamics



Since system is invariant with respect to $\big(\omega_e^{\rm free},\ x,\ \theta_e\big) \to \big(-\omega_e^{\rm free},\ -x,\ -\theta_e\big)$, we can study it for $\omega_e^{\rm free} \geq 0$ only and introduce the concept of *frequency deviation*:

$$|\omega_e^{\rm free}| = |\omega_{\rm ref} - \omega_{\rm vco}^{\rm free}|.$$

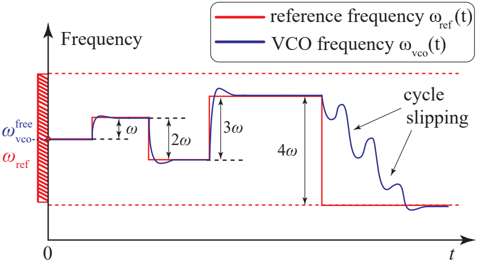
A *hold-in range* ($\approx$ local stability) — the largest interval of frequency errors $|\omega_e^{\rm free}| \in [0, \omega_h)$ such that an asymptotically stable locked state exists and varies continuously while $\omega_e^{\rm free}$ varies continuously within the interval.

A *pull-in range* ($\approx$ global stability) — the largest interval $|\omega_e^{\rm free}| \in [0, \omega_p)$ from the hold-in range such that a locked state is acquired for an arbitrary initial state.

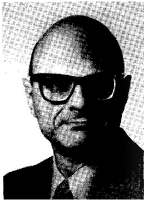
The locked states of the model correspond to the equilibria of the system.

G.A. Leonov, N.V. Kuznetsov, M.V. Yuldashev, R.V. Yuldashev, *Hold-in, pull-in, and lock-in ranges of PLL circuits: rigorous mathematical definitions and limitations of classical theory*, IEEE TCAS-I, 2015.

A *lock-in range* — the largest interval $|\omega_e^{\text{free}}| \in [0, \omega_l) \subset [0, \omega_p)$ such that the PLL after any change of ω_e^{free} within the interval re-establishes an asymptotically stable locked state without cycle slipping ($\sup\limits_{t>0} |\theta_e(0) - \theta_e(t)| < 2\pi$).



The Gardner problem (2005): "There is no natural way to define exactly any unique lock-in frequency", "despite its vague reality, lock-in range is a useful concept".



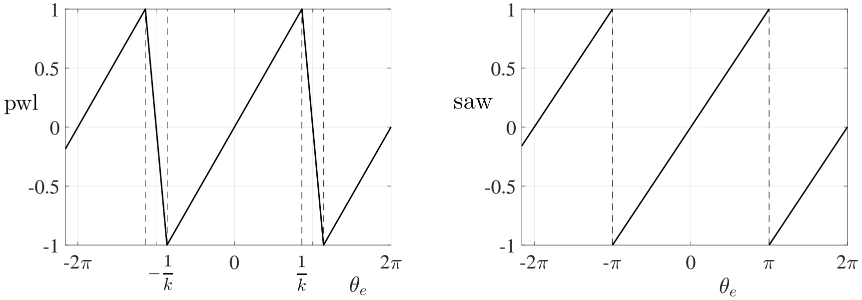
G.A. Leonov, N.V. Kuznetsov, M.V. Yuldashev, R.V. Yuldashev, *Hold-in, pull-in, and lock-in ranges of PLL circuits: rigorous mathematical definitions and limitations of classical theory*, IEEE TCAS-I, 2015.

Second-order PLL with proportionally-integrating loop filter

Consider the second-order PLL with proportionally-integrating filter $F(s) = \frac{1+s\tau_2}{s\tau_1}$:

$$\dot{x} = v_e(\theta_e),$$
$$\dot{\theta}_e = \omega_e^{\mathrm{free}} - \frac{K_{\mathrm{vco}}}{\tau_1}\left(x + \tau_2 v_e(\theta_e)\right).$$

Consider piecewise-linear $v_e(\theta_e) = \mathrm{pwl}(\theta_e, k)$, $k > \frac{1}{\pi}$ and sawtooth $v_e(\theta_e) = \mathrm{saw}(\theta_e) = \lim\limits_{k \to +\frac{1}{\pi}} \mathrm{pwl}(\theta_e, k)$ phase detector characteristics:



Second-order PLL with proportionally-integrating loop filter

$$\dot{x} = \mathrm{saw}(\theta_e),$$
$$\dot{\theta}_e = \omega_e^{\mathrm{free}} - \frac{K_{\mathrm{vco}}}{\tau_1}\left(x + \tau_2 \mathrm{saw}(\theta_e)\right).$$

The PLL system is discontinuous on discontinuity surfaces

$$S_m = \{(x,\ \theta_e) \mid x \in \mathbb{R},\ \theta_e = \pi + 2\pi m\},\ m \in \mathbb{Z},$$

Surfaces S_m have a zero Lebesgue measure.

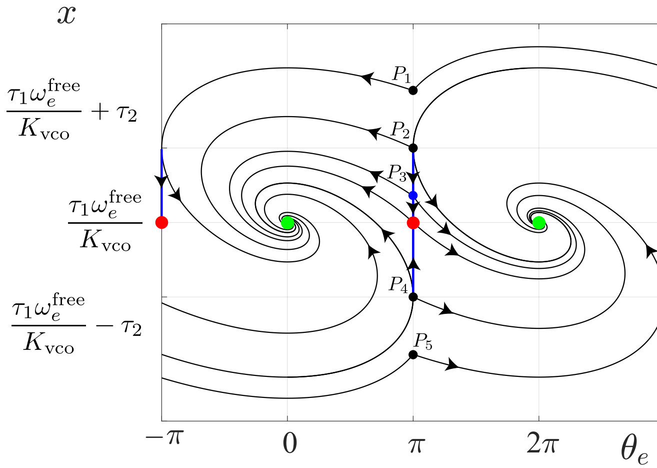
For definition of solutions we use the Filippov's approach and consider the differential inclusion

$$\dot{x} \in \psi(\theta_e),$$
$$\dot{\theta}_e \in \omega_e^{\mathrm{free}} - \frac{K_{\mathrm{vco}}}{\tau_1}\left(x + \tau_2 \psi(\theta_e)\right),$$

where

$$\psi(\theta_e) = \begin{cases} \mathrm{saw}(\theta_e), & \theta_e \neq \pi + 2\pi m, \\ [-1,\ 1], & \theta_e = \pi + 2\pi m. \end{cases}$$

Phase portrait (Filippov approach)



Figure

Sliding bands are $D_m = \left\{ (x,\ \theta_e) \mid \left| x - \frac{\tau_1 \omega_e^{\text{free}}}{K_{\text{vco}}} \right| \leq \tau_2,\ \theta_e = \pi + 2\pi m \right\},\ m \in \mathbb{Z}.$

Second-order PLL with lead-lag loop filter: hold-in range

$$\dot{x} = v_e(\theta_e),$$
$$\dot{\theta}_e = \omega_e^{\text{free}} - \frac{K_{\text{vco}}}{\tau_1}\left(x + \tau_2 v_e(\theta_e)\right).$$

Stationary set is

$$\Lambda = \left\{(x,\ \theta_e) \mid x = \frac{\tau_1 \omega_e^{\text{free}}}{K_{\text{vco}}},\ \theta_e = \pi m,\ m \in \mathbb{Z}\right\}.$$

- equilibria $\left(\frac{\tau_1 \omega_e^{\text{free}}}{K_{\text{vco}}},\ 2\pi m\right)$ are asymptotically stable
- equilibria $\left(\frac{\tau_1 \omega_e^{\text{free}}}{K_{\text{vco}}},\ \pi + 2\pi m\right)$ are unstable

Since an asymptotically stable equilibrium exists for any frequency error ω_e^{free}, and the hold-in range is infinite

$$[0,\ \omega_h) = [0,\ +\infty).$$

for any $K_{\text{vco}} > 0,\ \tau_1 > 0,\ \tau_2 > 0$.

Global stability theorem for pull-in range estimation

$$\dot{x} \in \psi(\theta_e),$$
$$\dot{\theta}_e \in \omega_e^{\text{free}} - \frac{K_{\text{vco}}}{\tau_1}\left(x + \tau_2\psi(\theta_e)\right).$$

Leonov theorem on global stability of periodic systems

If there is a continuous function $V(x,\ \theta_e) : \mathbb{R}^2 \to \mathbb{R}$ such that:
(i) $V(x,\ \theta_e + 2\pi) = V(x,\ \theta_e) \quad \forall x \in \mathbb{R},\ \forall \theta_e \in \mathbb{R}$,
(ii) for any solution $(x(t),\ \theta_e(t))$ of the inclusion the function $V(x(t),\ \theta_e(t))$ is non-increasing,
(iii) $V(x(t),\ \theta_e(t)) \equiv V(x(0),\ \theta_e(0))$ implies $(x(t),\ \theta_e(t)) \equiv (x(0),\ \theta_e(0))$,
(iv) $V(x,\ \theta_e) + \theta_e^2 \to +\infty$ as $|x| + |\theta_e| \to +\infty$,
then any solution of the differential inclusion tends to a stationary set of this inclusion.

The following continuous Lyapunov function

$$V(x,\ \theta_e) = \frac{K_{\text{vco}}}{2\tau_1}\left(x - \frac{\tau_1\omega_e^{\text{free}}}{K_{\text{vco}}}\right)^2 + \int_0^{\theta_e} \text{saw}(\sigma)d\sigma,$$

allows to prove the infiniteness of the pull-in range: $[0,\ \omega_p) = [0,\ +\infty)$ for any $K_{\text{vco}} > 0,\ \tau_1 > 0,\ \tau_2 > 0$.

Lock-in range

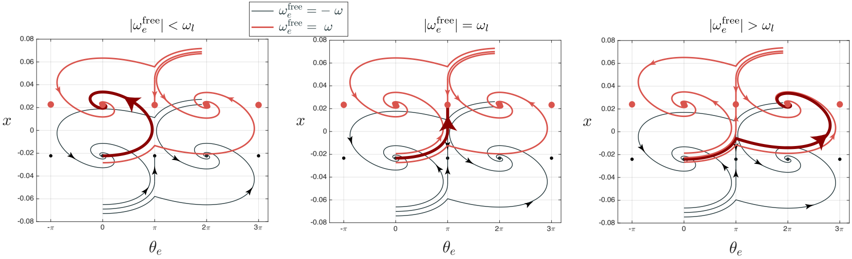


Figure: The trajectories of the model with negative $\omega_e^{\text{free}} = -\omega < 0$ are in black, the trajectories of the model with positive $\omega_e^{\text{free}} = \omega > 0$ are in red. Left subfigure: $\omega = 88 < \omega_l$; middle subfigure: $\omega = \omega_l \approx 92.27$; right subfigure: $\omega = 95 > \omega_l$.

The lock-in frequency of the PLL model with the sawtooth PD characteristic is

$$\omega_l = \begin{cases} \frac{a\sqrt{\pi}}{2\tau_2}\left(\frac{a+b}{a-b}\right)^{\frac{a}{2b}}, & a^2 > 4\pi, \\ \frac{\pi e}{\tau_2}, & a^2 = 4\pi, \\ \frac{a\sqrt{\pi}}{2\tau_2}\exp\left(\frac{a}{b}\arctan\frac{b}{a}\right), & a^2 < 4\pi, \end{cases}$$

where

$$a = \sqrt{\frac{K_{\text{vco}}}{\tau_1}}\tau_2, \quad b = \sqrt{|a^2 - 4\pi|}.$$

Comparison with triangle PD characteristic

$$\dot{x} = v_e(\theta_e),$$
$$\dot{\theta}_e = \omega_e^{\text{free}} - \frac{K_{\text{vco}}}{\tau_1}\left(x + \tau_2 v_e(\theta_e)\right).$$

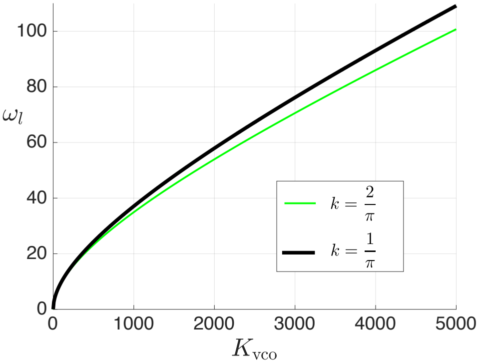


Figure: Comparison of the lock-in frequency of PLL model with sawtooth PD characteristic $v_e(\theta_e) = \text{saw}(\theta_e) = \text{pwl}(\theta_e,\ \frac{1}{\pi})$ and the lock-in frequency of PLL model with triangular characteristic $v_e(\theta_e) = \text{tri}(\theta_e) = \text{pwl}(\theta_e,\ \frac{2}{\pi})$. Parameter $\tau_2 = 0.0225$.

List of publications

- ✓ Leonov, G., Kuznetsov, N., Yuldashev, M., and Yuldashev, R. (2015). Hold-in, pull-in, and lock-in ranges of PLL circuits: rigorous mathematical definitions and limitations of classical theory. IEEE Transactions on Circuits and Systems-I: Regular Papers, 62(10), 2454–2464.
- ✓ Best, R., Kuznetsov, N., Leonov, G., Yuldashev, M., and Yuldashev, R. (2016). Tutorial on dynamic analysis of the Costas loop. IFAC Annual Reviews in Control, 42, 27–49.
- ✓ Kuznetsov, N., Leonov, G., Yuldashev, M., and Yuldashev, R. (2017). Hidden attractors in dynamical models of phase-locked loop circuits: limitations of simulation in MATLAB and SPICE. Communications in Nonlinear Science and Numerical Simulation, (51), 39–49.
- ✓ Kuznetsov, N., Lobachev, M., Yuldashev, M., and Yuldashev, R. (2019). On the Gardner problem for phase-locked loops. Doklady Mathematics, 100(3), 568–570.
- ✓ Kuznetsov, N., Lobachev, M., Yuldashev, M., Yuldashev, R., Kudryashova, E., Kuznetsova, O., Rosenwasser, E., and Abramovich, S. (2020). The birth of the global stability theory and the theory of hidden oscillations. In 2020 European Control Conference Proceedings, 769–774.
- ✓ Kuznetsov, N., Lobachev, M., Yuldashev, M., and Yuldashev, R. (2021). The Egan problem on the pull-in range of type 2 PLLs. IEEE Transactions on Circuits and Systems II: Express Briefs, 68(4), 1467–1471.
- ✓ Kuznetsov, N., Matveev, A., Yuldashev, M., and Yuldashev, R. (2021). Nonlinear analysis of charge-pump phase-locked loop: The hold-in and pull-in ranges. IEEE Transactions on Circuits and Systems I: Regular Papers, 68(10), 4049–4061.
- ✓ Kuznetsov, N., Arseniev, D., Blagov, M., Lobachev, M., Wei, Z., Yuldashev, M., and Yuldashev, R. (2022). The Gardner problem and cycle slipping bifurcation for type-2 phase-locked loops. International Journal of Bifurcation and Chaos, 32(9). art. num. 2250138.
- ✓ Kuznetsov, N., Lobachev, M., Yuldashev, M., Yuldashev, R., and Tavazoei, M. (2023). The Gardner problem on the lock-in range of second-order type 2 phase-locked loops. IEEE Transactions on Automatic Control.
- ✓ Kuznetsov, N., Lobachev, M., and Mokaev, T. (2023). Hidden boundary of global stability in a counterexample to the Kapranov conjecture on the pull-in range. Doklady Mathematics. 108(4):300–308.