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MECHANICS

Fundamental Frequencies of Long Prismatic Shells

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Abstract—The impact of the length of a shell on the fundamental frequencies of natural vibrations of a prismatic thin shell is studied in the paper. A thin cylindrical shell, the cross section of which is a regular polygon, is analyzed. Provided that the perimeter of the cross section is preserved, the influence of the length of the structure and the number of faces on the natural frequencies is examined. For limiting cases of the number of faces (shells with a square cross section and circular cross section), the results of numerical analysis in COMSOL and the analytical results are in close agreement with each other.

Keywords: prismatic thin shell, vibrations of thin shells, FEM analysis

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1. INTRODUCTION

The vibrations of a prismatic thin shell with a cross section in the form of a regular polygon are considered. The work continues the publication of research results, begun in work [1], by discussing the effect of the length of a prismatic thin shell on the spectrum of natural-vibration frequencies. As was shown in work [1], for shells of medium length, the lower part of the spectrum consists of frequencies of "plate" or "shell" vibrations. In the first case, the rectangular faces of the shell perform natural vibrations with frequencies and shapes close to the vibrations of hinged plates. In the second case, the vibration shapes resemble those of a circular cylindrical shell with the formation of m waves in the circumferential and one wave in the axial directions.

However, as the length of the shell increases, beam frequencies begin to appear in the lower part of the spectrum, that is, those at which the cross-section of the shell is not deformed, and the shell behaves like a long beam.

In work [2], in which solutions based on the generalized theory of beams were used to study the vibrations of polygonal shells, primary attention was paid to studying vibrational modes and their interaction. For a long shell with a hexagonal cross section, an approximate solution based on the equations of semi-momentless theory was obtained in [3].

In this study, we analyze the behavior of the fundamental (lowest natural) frequencies of small bending vibrations of prismatic thin shells with a change in the number of faces and an increase in length, using analytical solutions and the finite-element method.

2. PROBLEM FORMULATION

We consider the natural transverse vibrations of a prismatic isotropic thin shell of length l and thickness
1 h with a cross section in the form of a regular n-gon with the side length a. The shell material is characterized by Young's modulus E, Poisson's ratio ν, and density ρ. Such shells can be used as models in studying the vibrations of structural elements in the form of prismatic thin-walled pipes (Fig. 1).

The faces of the prismatic shell consist of thin plates. In the plane of the ith plate, we introduce local rectangular coordinates (x, y) (see Fig. 2).

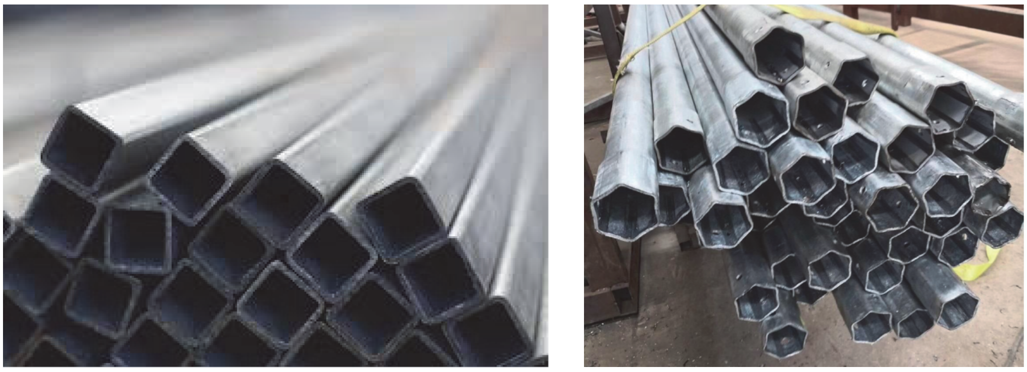

Fig. 1. Long prismatic thin-walled pipes with a cross section in the form of a square and a regular hexagon.

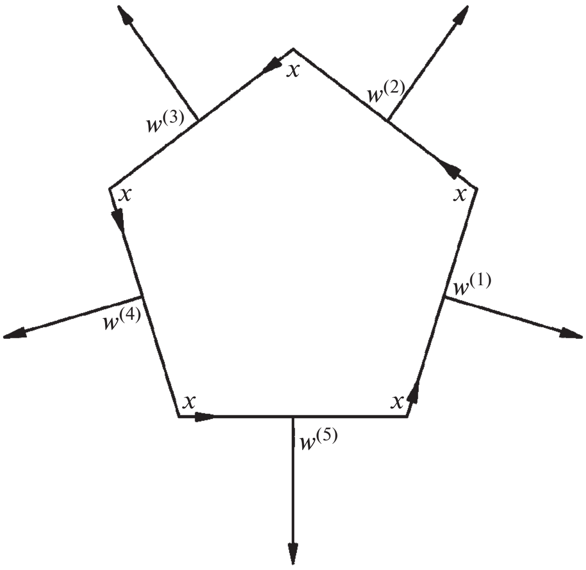

Fig. 2. Local coordinates in the shell cross section.

We use the linear Germain–Lagrange Equations to describe the small transverse deflection $w^{(i)}(x, y)$ of the ith plate

$$D\Delta\Delta w^{(i)} - \rho h w^{(i)}\omega^2 = 0, \quad i = \overline{1, n}, \tag{1}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, $D = \frac{Eh^3}{12(1-\nu^2)}$ is the cylindrical stiffness, and ω is the angular frequency of natural oscillations of the plate.

To formulate the problem, it is necessary to specify the boundary conditions at the lower and upper edges, as well as at the coupling lines between the plates. We assume that the lower and upper edges of the shell are hinged. To determine the conditions at the coupling lines between the plates, we assume, unless otherwise specified, that the deformations in the plane of each plate are negligible, the displacements and bending moments at the coupling points of the plates are equal, and the angles between adjacent plates remain equal to $\pi\frac{(n-2)}{n}$ under deformation.

Next, we use the dimensionless parameters of the natural frequency $\lambda = \omega^2 R^2 \frac{\rho}{E}$ and the length $L = \frac{l}{R}$, where R is the characteristic size of the shell. Since we will consider shells with a regular polygonal cross

section of equal perimeter, we will choose the radius of the shell with a circular cross section ($n = \infty$) as the characteristic size. Then the perimeter of the shell cross section is equal to $2\pi R$, and the length of the
1 side of the regular n-gon is $a = 2\pi R/n$.

The finite-element method (COMSOL) is used to numerically determine the frequencies and modes of oscillations. Whenever possible, a comparison is made with the oscillation frequencies obtained by analytical methods. In the analytical study of the natural oscillations of shells, we will limit ourselves to considering the case when the shell has an even number of faces. As was shown in [1], with a small (even) number of sides, the oscillation mode corresponding to the fundamental frequency consists of the first modes of oscillation of the plates forming the shell. In this case, the assumption is made that the angles between the faces of the shell are constant during deformation.

3. SHELL WITH A SQUARE CROSS SECTION

Let us first consider a thin cylindrical shell with a square cross section.

In work [1] it is shown that for shells of average length the lower part of the spectrum consists of "plate" frequencies, i.e., those for which the rectangular faces of the shell perform natural oscillations with frequencies and shapes close to the oscillations of hinged plates. The dimensionless parameter λ for such natural frequencies is found by the formula

$$\lambda = \pi^4 \left(\frac{m_1^2}{A^2} + \frac{m_2^2}{L^2} \right)^2 \frac{H^2}{12(1 - \nu^2)}, \tag{2}$$

where m_1 and m_2 are the wavenumbers in the x and y directions, respectively, $H = h/R$ is the dimensionless thickness of the plates, and $A = a/R = \pi/2$ is the dimensionless width of the plates. In particular, for the fundamental frequency ($m_1 = m_2 = 1$)

$$\lambda_0^p = \pi^4 \left(\frac{1}{A^2} + \frac{1}{L^2} \right)^2 \frac{H^2}{12(1 - \nu^2)}. \tag{3}$$

At the same time, the shell also has beam-type natural frequencies, i.e., those at which the shell cross section is not deformed, and the shell behaves like a long beam. In particular, a beam hinged at both ends has natural frequencies [4]

$$\omega^2 = \pi^4 m^4 \frac{EJ}{Spl^4}, \quad m = 1, 2, \ldots, \tag{4}$$

where J is the moment of inertia of the shell section relative to the axis passing through its center in its plane. For a shell of square cross section, the corresponding moment of inertia is equal to

$$J = \frac{2}{3} ah(a^2 + h^2),$$

and the cross-sectional area is $S = 4ah$. Then for the dimensionless parameter of the lowest "beam" natural frequency

$$\lambda_0^b = \pi^4 \frac{((\pi/2)^2 + H^2)}{6L^4}. \tag{5}$$

Figure 3 shows the dependence of the natural-frequency parameter of the first "plate" and "beam" modes on the shell length for $H = 0.01$ and $\nu = 0.33$. Solid lines correspond to the analytical results obtained using formulas (2) and (4), dots correspond to the numerical results obtained in the COMSOL finite-element package.

The solid green line is the values of the "beam" frequency according to formula (4), the blue dots with a dashed line are the numerical values of the "beam" frequency (COMSOL), the solid red line is the values of the first "plate" frequency according to formula (2), the purple dots with a dashed line are the numerical values of the first "plate" frequency (COMSOL), the solid yellow line is the values of the second "plate" frequency according to formula (2), and the black dots with a dashed line are the numerical values of the second "plate" frequency (COMSOL).

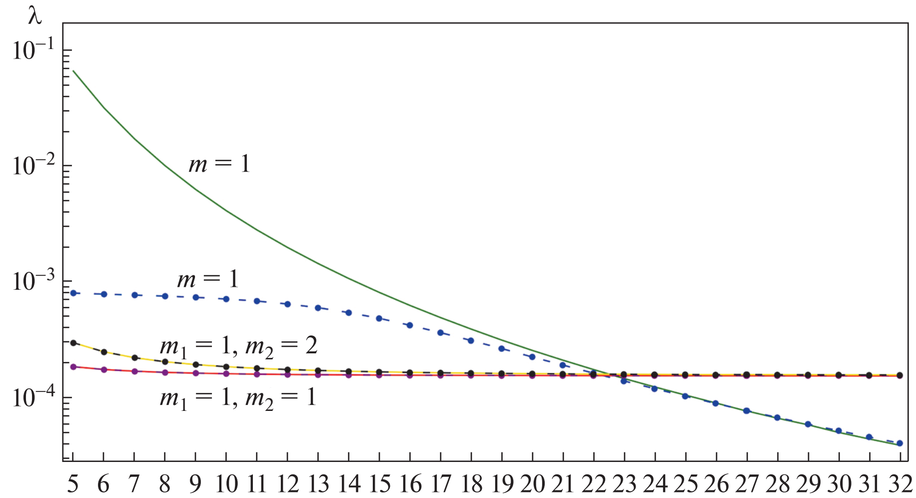

Fig. 3. Dependence of the lower "plate" and "beam" frequencies on the length of the square cross-sectional shell. Solid lines are analytical results, dotted lines are numerical results.

As the shell length increases, the lower plate frequencies decrease to a constant value $\lambda_\infty^p = \frac{\pi^4}{12(1-\nu^2)}H^2$, while the beam frequency quickly tends to 0. The value of L_0, at which the beam frequency becomes fundamental, is found from the condition

$$\lambda_0^p = \lambda_0^b,$$

whence for $H = 0.01$ and $\nu = 0.33$ we find $L_0 = 22.7$.

4. SHELL WITH A CIRCULAR CROSS SECTION

The vibrational modes and frequencies of a cylindrical shell with a large number of faces are close to the modes and frequencies of a cylindrical shell of circular cross section. For a hinged circular cylindrical thin shell, the dimensionless frequencies of the lower part of the spectrum are determined by the formulas [5]

$$\lambda^s(m) = \frac{\pi^4}{m^2(m^2+1)L^4} + \frac{m^2(m^2-1)^2}{(m^2+1)}\frac{H^2}{12(1-\nu^2)}, \tag{6}$$

where m is the wavenumber in the circular direction.

At $m = 1$, the form of the "shell" vibrations coincides with the form of the beam's natural vibrations. The corresponding frequencies can be obtained using the asymptotic formula (6)

$$\lambda_0^b = \frac{\pi^4}{2L^4} \tag{7}$$

and directly from formula (4), taking into account that for a circular cylindrical shell

$$J = \frac{\pi}{4}\left(\left(R+\frac{h}{2}\right)^4 - \left(R-\frac{h}{2}\right)^4\right) = \frac{\pi}{4}Rh(h^2+4R^2), \qquad S = 2\pi hR,$$

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$$\lambda_0^b = \frac{\pi^4(4+H^2)}{8L^4}. \tag{8}$$

When $H \to 0$, formulas (7) and (8) coincide.

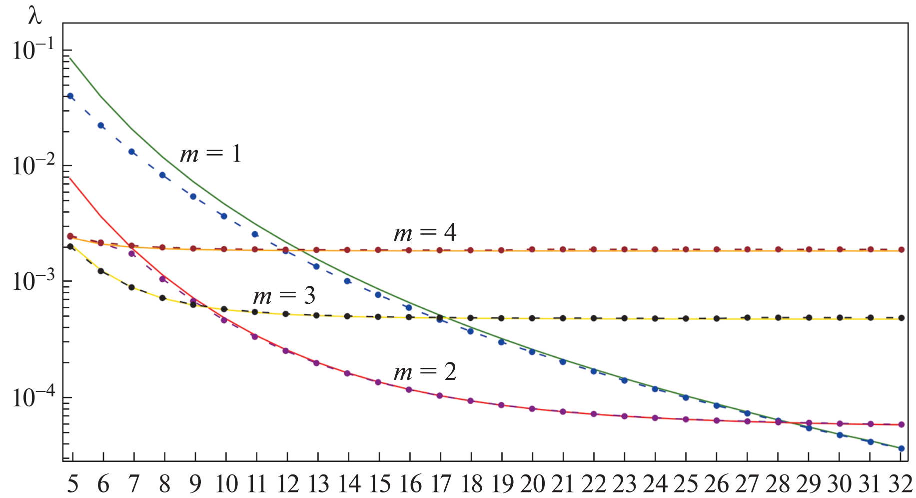

Fig. 4. Dependence of the lowest "shell" and "beam" frequencies on the length of the circular shell. Solid lines are analytical results, dotted lines are numerical results.

In Fig. 4, for the lowest "shell" and "beam" natural frequencies their dependence on the shell length is shown: the green line is the values of the "beam" frequency according to formula (4), the blue dots with a dashed line are the numerical values of the "beam" frequency (COMSOL), the solid red line is the values of frequencies with $m = 2$ according to the asymptotic formula (6), the purple dots with a dashed line are the values of frequencies with $m = 2$ (COMSOL), the solid yellow line is the values of frequencies with $m = 3$ according to the asymptotic formula (6), the black dots with a dashed line are the values of frequencies with $m = 3$ (COMSOL), the solid orange line is the values of frequencies with $m = 4$ according to the asymptotic formula (6), the brown dots with a dashed line are the values of frequencies with $m = 4$ (COMSOL).

With increasing L, the lowest "shell" frequency is achieved at ever smaller values of m. The value of L_0 at which the beam frequency becomes fundamental is found from the condition

$$\lambda^s(2) = \lambda_0^b,$$

from where for $H = 0.01$ and $\nu = 0.33$ we find $L_0 = 28.4$.

1 5. SHELL WITH A CROSS SECTION IN THE FORM OF A REGULAR n-GON

Obtaining analytical formulas for describing the frequencies of natural oscillations of a prismatic shell
1 with a cross section in the form of a regular n-gon is hardly possible. The fact is that for $n > 4$ the assumption that for forms with lower frequencies the angle between the faces is preserved during deformations turns out to be incorrect. For shells with a small number of faces, formula (3) gives an upper bound for the frequencies, which quickly loses accuracy with increasing n. The forms corresponding to the lower frequencies for shells with a large number of faces can be called quasi-shell-like, since with increasing n they increasingly resemble the lower natural forms of natural oscillations of a circular shell with m waves in the circumferential direction.

At the same time, formula (4) can still be used to estimate the frequencies of beam vibrations. The only problem is determining the moment of inertia of the section of such a shell relative to the axis passing through the center of the polygon.

Calculating the moment of inertia for a regular polygon is made a little easier by the fact that for a homogeneous body the intersection of the planes of symmetry is the principal axis of inertia. Consequently, for a flat figure the principal axis of inertia is the axis of symmetry. The principal axes of inertia are the axes of the ellipsoid of inertia (in the flat case, the ellipse of inertia). A regular polygon has axes of symmetry that form an angle different from a right angle. This means that the ellipse of inertia is a circle. Therefore, any line passing through its center is the principal axis of inertia, and the moments of inertia relative to these axes are equal.

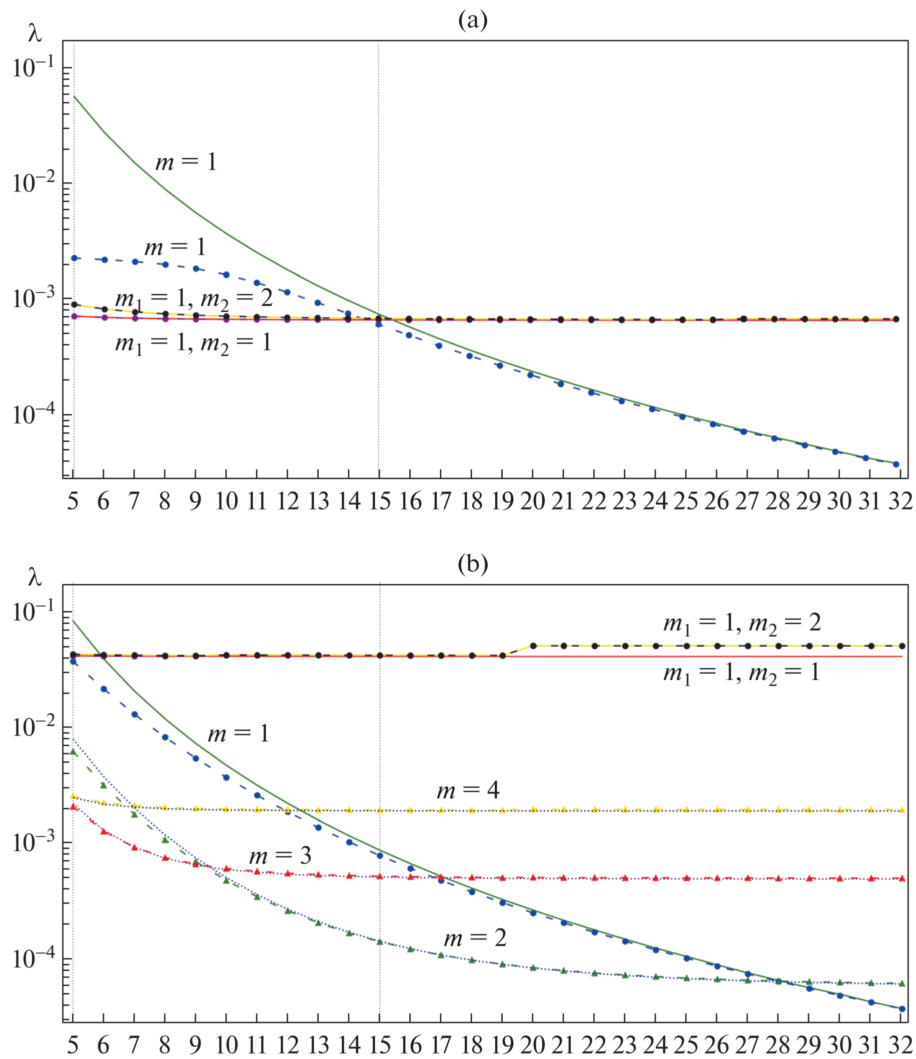

Fig. 5. Dependence of natural frequencies on the shell length for $n = 6$ (a) and for $n = 16$ (b).

The directions of the principal axes of inertia coincide with the directions of the eigenfunctions of the matrix J, so from an algebraic point of view the situation can be interpreted as follows: a symmetric matrix J of the second order for a regular polygon has two eigenfunctions that are not orthogonal. This can only be the case when its eigenvalue is a multiple, so the moments of inertia of a regular polygon relative to any axes passing through its center are equal.

For arbitrary n, the moment of inertia of a polygonal section for small values of h can be represented as

$$J = \pi h c(n) R^3 (1 + O((h/R)^2)).$$

The values of the coefficients $c(n)$ for regular polygons with perimeter 2π increase rapidly with increasing n from $c(4) = \pi^2/12 \approx 0.82$ for a shell with a square cross section to $c(\infty) = 1$ for a circular shell, and already at $c(6) = 0.91$.

Figure 5a shows the dependences of the natural frequency parameter for the “plate” and first “beam” forms for $n = 6$ and $n = 16$. The designations are similar to Fig. 3.

In Fig. 5b, the “shell” frequencies with m waves in the circumferential direction are additionally marked: the blue dotted line is the frequency values with $m = 2$ according to the asymptotic formula (6), the green triangles with a dashed line are the frequency values with $m = 2$ (COMSOL), the purple dotted line is the frequency values with $m = 3$ according to the asymptotic formula (6), the red triangles with a dashed line are the frequency values with $m = 3$ (COMSOL), the black dotted line is the frequency values

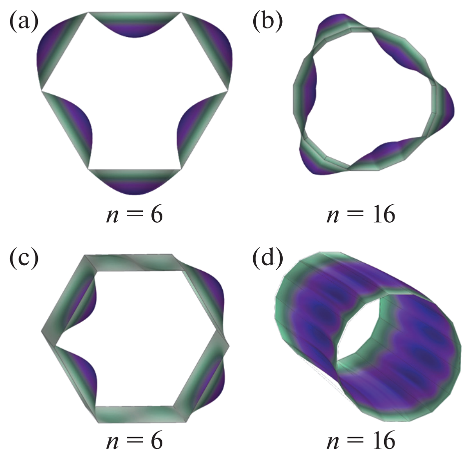

Fig. 6. Comparison of the vibrational modes of a shell of length $L = 5$: fundamental mode for $n = 6$ (a), fundamental mode for $n = 16$ (b), first beam mode for $n = 6$ (c), first beam mode for $n = 16$ (d).

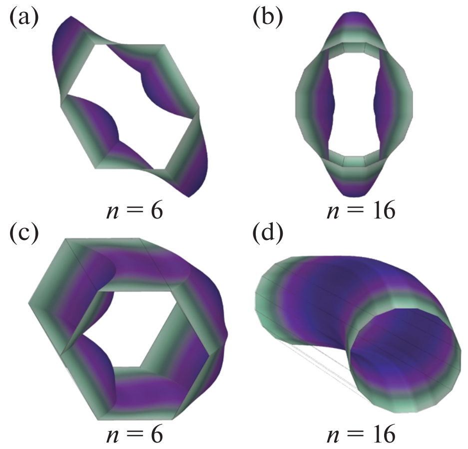

Fig. 7. Comparison of vibrational modes of a shell of length $L = 15$: fundamental mode for $n = 6$ (a), fundamental mode for $n = 16$ (b), first beam mode for $n = 6$ (c), first beam mode for $n = 16$ (d).

with $m = 4$ according to the asymptotic formula (6), and the yellow triangles with a dashed line are the frequency values with $m = 4$ (COMSOL).

For a fixed number of sides, with increasing length, the "plate" frequencies decrease, but with a numerical solution they can undergo a short-term small jump in growth. This can be due to both the features of the FEM solver and possible internal resonance due to frequency bunching.

For small n and a small shell length, the fundamental frequency corresponds to the plate mode of oscillations (Fig. 6a). With increasing n at a fixed length, the fundamental mode of oscillations is a combination of the shell and plate modes (Fig. 6b), gravitating with an increase in the number of sides to the shell mode of oscillations with m waves in the circumferential direction.

The first beam mode of vibrations at small n is realized not in pure form, but as a combination of beam and plate modes (Fig. 6c), the frequency corresponding to it is far from the lower part of the spectrum. With an increase in the number of shell faces, the beam mode of vibrations, in which one wave arises in the axial direction, and the cross section is not deformed, becomes more and more pronounced (Fig. 6d).

With increasing shell length, the behavior of the oscillation modes changes. The fundamental oscillation mode in this case tends to the shell mode with the corresponding number of waves in the circumferential direction m, and the stronger, the greater the number of sides of the cross section (Figs. 7a and 7b). In particular, $m = 2$ at $L = 15$ (see Fig. 5).

The first beam mode of vibrations at small *n* is realized not in pure form, but as a combination of beam and plate modes (Fig. 7c), the frequency corresponding to it is far from the lower part of the spectrum (see Fig. 5). For shells with an equal number of faces, with an increase in the length of the shell, the beam mode of vibrations becomes more and more pronounced (Figs. 6d and Fig. 7d).

6. CONCLUSIONS

With a small number of sides and a small length, the lower part of the spectrum of natural frequencies of a thin prismatic shell consists of "plate" frequencies. With increasing shell length, starting from a certain value L_0, the "beam" frequency becomes fundamental. With an increase in the number of sides, the oscillation mode approaches that of a shell of circular cross section. The oscillation modes have a mixed type of "plate," "shell," and "beam" oscillations and the assumption of preservation of the angles between the faces for the oscillation modes is fulfilled increasingly worse. With increasing length, the "beam" frequencies of the shell quickly decrease, approaching the natural frequencies of a beam of the same length.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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