Author Proof

Harmonic Numbers in Gambler's Ruin Problem

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Abstract. The gambler's ruin problem is studied. At each of n steps, the probability that the player wins at the next step depends on the win/lose ratio in previous steps. The player's payoff and the asymptotic formula for large game durations were determined. The numerical results of payoff simulation for different n values are reported. AQ1

Keywords: Random walk · Gambler's ruin problem · Ruin probability · Reflection principle

1 Introduction

This paper considers the following multistage model in discrete time. A random walk related to the ruin problem [1,2] is being monitored. In such problems, at each step, a particle moves one unit to the right on the integer number line when the player wins and one unit to the left if the player loses. Accordingly, the player's capital increases or decreases by a unit depending on whether they win or lose. The player's initial capital is fixed and when the random walk reaches this level, it is absorbed, which is considered as the moment of ruin of the player. In the classical model, the random walk is symmetrical, which corresponds to equal chances for a player to win and to lose. In this paper, we investigate a model in which the probability of winning increases with an increase in the total number of wins, and decreases with an increase in the number of losses. The player's goal is to augment his/her capital as much as possible without going ruin.

Models with symmetric random walks were considered in different ways depending on the player's goal. Shepp [3] studied a problem in which the goal was to maximize the value of the payoff per unit time. Tamaki [4] solved the problem of maximizing the probability of stopping on any of the last few maximum values.

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Problems related to the sequence of dependent random variables were considered in various urn schemes. Tamaki [5], Mazalov, Tamaki [6] considered variants of the problem of maximizing the probability of stopping at the largest value in an urn scheme, where the probability of transition to the next state depends on the trajectory at the previous steps. Shepp [3], Boyce [7], Ivashko [8] studied setups in which the goal was to maximize the value of the trajectory. Variants of balls-and-bins problem were considered by Tijms [9] and Ivashko [10]. Other extensions of the gambler's ruin problem to the case of multidimensional random walks [11] and the case of two players [12] have been studied.

In this paper, we consider the ruin problem associated with a random walk, where the probability of transition to the next state depends on the ratio of the number of wins and losses at the previous steps: the more wins, the greater the probability of success at the next step.

This paper is structured as follows. Section 2 gives the statement of gambler's ruin problem. Section 3 suggests an analytical solution of the problem for different cases and asymptotic behavior for large values of n. Finally, in Sect. 4, we present the findings and conclusions, and draft plans for the future.

2 Gambler's Ruin Problem

The paper considers the ruin problem of the following form. A time interval n is set at the beginning of the game. A random walk on the integer line starts from 0 and at each step i, $(i = 1, 2, ..., n)$ moves $+1$ to the right or -1 to the left. At the beginning of the game, the player can win or lose with equal probability of $\frac{1}{2}$. In the following, the transition probabilities are calculated based on the following assumptions. Suppose that at step i we are in the state (p, q), where p is the quantity $+1$ and q is the quantity -1, $i = p + q$. We will move to the state $(p + 1, q)$ with the probability $\frac{p+1}{p+q+2}$ and to the state $(p, q + 1)$ with the probability $\frac{q+1}{p+q+2}$.

Let $X_1, ..., X_n$ be a sequence of random variables that take the values $+1$ or -1 corresponding to the random walk considered above. Then, $S_i = \sum_{j=1}^{i} X_j$, $i = 1, 2, ..., n$ is the difference between $+1$ and -1 during i steps, i.e. the position of the particle at time i, $S_0 = 0$.

Moving over to coordinates on the plane (i, j): $i = p+q$ and $j = p-q$, we find that if $S_i = j$, then X_{i+1} takes the value $+1$ with a probability $\frac{1}{2}\left(1 + \frac{j}{i+2}\right)$ and the value -1 with a probability $\frac{1}{2}\left(1 - \frac{j}{i+2}\right)$, $j = \overline{-i, i}$, $i = \overline{0, n}$.

The value sequence $X_1, ..., X_n$ form a certain trajectory on the plane (see, e.g., Fig. 1).

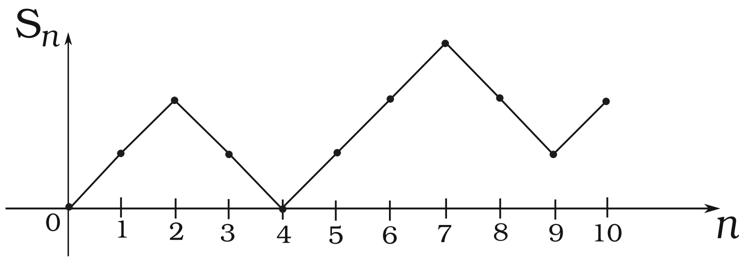

Fig. 1. Example of trajectory S_n for $n = 10$.

Let us describe the ruin problem associated with the specified random walk. We take the case of a cautious gambler who uses the following strategy. He/she leaves the game as soon as the number of losses exceeds the number of wins. Otherwise, he/she continues playing until the end instant n. The payoff when stopping at the time instant $\tau \leq n$ is denoted as S_τ (difference between the number of wins and the number of losses). Then, $S_\tau = -1$ if the random walk goes down to the level -1 or $S_\tau = S_n \geq 0$ if the game continues to the end time instant.

The player's payoff in this problem is

$$V_n = \overline{U}_n + U_n,$$

where

$$\overline{U}_n = \sum_{j \geq 0}^{n} j \cdot \mathbf{P}\{S_1 > -1, S_2 > -1, ..., S_{n-1} > -1, S_n = j\},$$

$$U_n = \sum_{j=1}^{n} (-1)\mathbf{P}\{S_1 > -1, ..., S_{j-1} > -1, S_j = -1\}$$
$$= (-1)\Big(1 - \mathbf{P}\{S_1 > -1, ..., S_{n-1} > -1, S_n > -1\}\Big).$$

Here, $\overline{U}_n$ is the payoff when stopping at the last step at a non-negative value (see Fig. 2), U_n is the payoff when stopping at -1 (see Fig. 3).

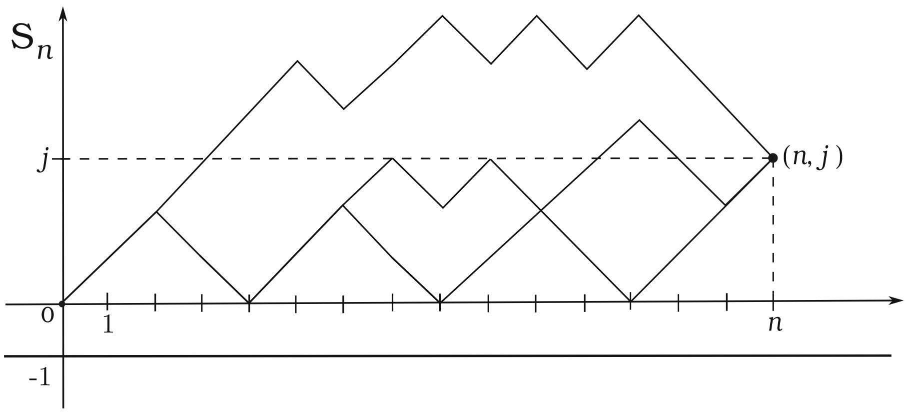

Fig. 2. Payoff when stopping at the last step at a non-negative value.

Remark 1. The probability P_n^r of the player's ruin with this strategy is

$$P_n^r = \mathbf{P}\big\{\inf_{1\le j\le n} S_j < 0\big\} = \sum_{j=1}^{n} \mathbf{P}\{S_1 > -1, ..., S_{j-1} > -1, S_j = -1\}$$
$$= 1 - \mathbf{P}\{S_1 > -1, ..., S_{n-1} > -1, S_n > -1\}.$$

To compute the payoff, we need to know the probability of an arbitrary trajectory getting from point $(0,0)$ to point (n,j). Let us find this probability, e.g., for the case where p successes were followed by $n-p$ failures.

$$P(n,j) = P((0,0);(n,j)) = \prod_{i=0}^{p-1} \frac{1}{2}\left(1 + \frac{i}{i+2}\right) \prod_{i=p}^{n-1} \frac{1}{2}\left(1 - \frac{2p-i}{i+2}\right),$$

where $p = \dfrac{n+j}{2}$ is the number of successes.

Simplifying the last expression, we get

$$P(n,j) = \frac{2}{n+j+2} \prod_{i=p}^{n-1} \frac{2i-n-j+2}{2(i+2)} = \frac{1}{p+1} \prod_{i=p}^{n-1} \frac{i-p+1}{i+2} = \frac{1}{\binom{n}{p}(n+1)}.$$

This proves to be valid for any trajectory.

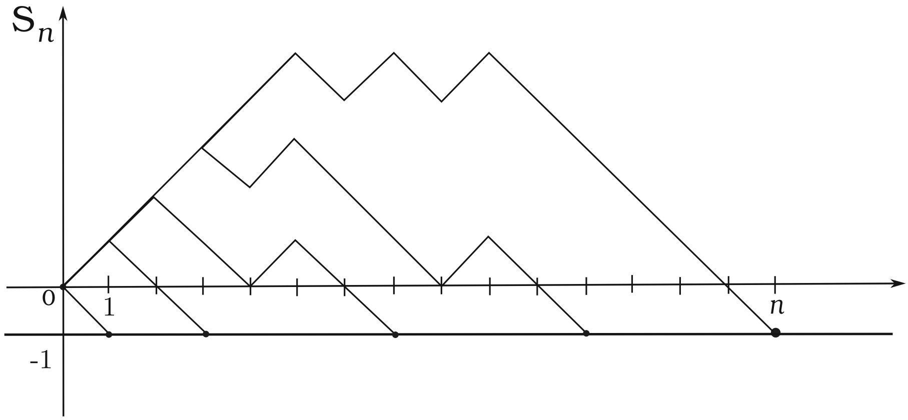

Fig. 3. Payoff when stopping at -1.

Lemma 1. *The probability of getting from point* $(0,0)$ *to point* (n,j)*, where* $n+j$ *is even and* $-n \le j \le n$*, via any trajectory does not depend on the travel path and equals*

$$P(n,j) = \frac{1}{\binom{n}{p}(n+1)}, \tag{1}$$

where $p = \dfrac{n+j}{2}$.

Proof. Let us prove the lemma by induction on n. For $n=1$ and $n=2$, formula 1 is validated directly. E.g., for $n=2$ the point $(2,0)$ can be reached by two paths via $(1,1)$ and $(1,-1)$ with equal probability $\frac{1}{6}$, which coincides with (1) since $n=2$, $p=1$, and $P(2,0) = \frac{1}{2\cdot 3} = \frac{1}{6}$.

Assuming this statement has been proved for any $n-1$, let us prove it for n. Let j be such that $n+j$ is an even number and $-n \le j \le n$. There are two ways for the trajectory to get to the point (n,j) from the initial point $(0,0)$: via the point $(n-1,j-1)$ and via the point $(n-1,j+1)$ (see Fig. 4). The probability of getting from the point $(n-1,j-1)$ to the point (n,j) is

$$P\big((n-1,j-1);(n,j)\big) = \frac{1}{2}\left(1 + \frac{j-1}{n+1}\right),$$

and the probability of getting from the point $(n-1,j+1)$ to the point (n,j) is

$$P\big((n-1,j+1);(n,j)\big) = \frac{1}{2}\left(1 - \frac{j+1}{n+1}\right).$$

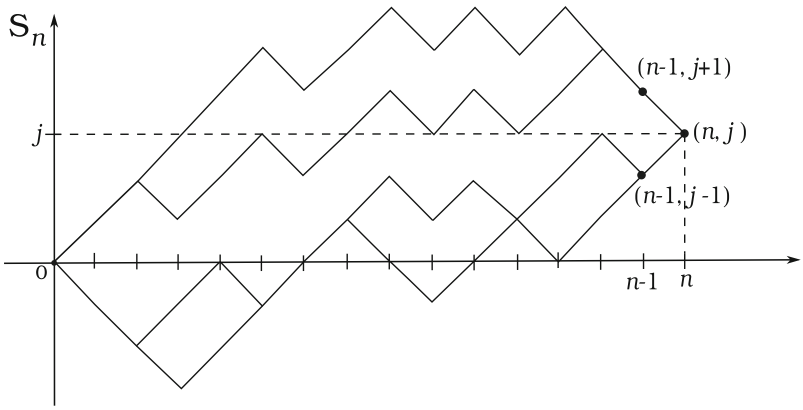

Fig. 4. Trajectories of getting from point $(0,0)$ to point (n,j).

Then, according to the induction proposition for $n-1$, the probability of the trajectory connecting the points $(0,0)$ and (n,j) and traveling via the point $(n-1,j-1)$ will be

$$P_1(n,j) = \frac{1}{2}\left(1 + \frac{j-1}{n+1}\right) \cdot \frac{1}{\binom{n-1}{\frac{n+j-2}{2}} n} = \frac{1}{\binom{n}{\frac{n+j}{2}}(n+1)}.$$

Similarly, the probability of the trajectory connecting the points $(0,0)$ and (n,j) and traveling via the point $(n-1,j+1)$ will be

$$P_2(n,j) = \frac{1}{2}\left(1 - \frac{j+1}{n+1}\right) \cdot \frac{1}{\binom{n-1}{\frac{n+j}{2}} n} = \frac{1}{\binom{n}{\frac{n+j}{2}}(n+1)}.$$

The two probabilities are equal and coincide with the expression (1). The Lemma is proved.

To find $\overline{U}_n$ and U_n, we use the following lemma based on the geometrical principle of trajectory reflection.

Lemma 2 (Feller [1]). *If* $a > 0$ *and* $b > 0$*, then the number of paths* $(s_1, s_2, ..., s_n)$ *such that* $s_1 > -b$*,* $s_2 > -b$*, ...,* $s_{n-1} > -b$*,* $s_n = a$ *equals* $N_{n,a} - N_{n,a+2b}$*, where* $N_{n,x} = \binom{n}{p}$*,* $p = \dfrac{n+x}{2}$.

Applying this lemma, we find that the number of trajectories from point $(0,0)$ to point (n,j) lying in the non-negative half-plane is

$$N_{n,j} - N_{n,j+2} = \binom{n}{p} - \binom{n}{p+1},$$

where $p = \dfrac{n+j}{2}$.

Then, for $j = p - q > 0$ (q is the number of losses) the following holds

$$\mathbf{P}\{S_1 > -1, S_2 > -1, ..., S_{n-1} > -1, S_n = j\} = \left(\binom{n}{p} - \binom{n}{p+1}\right) P(n,j)$$
$$= \binom{n}{p} \frac{p-q+1}{p+1} P(n,j) = 2\binom{n}{\frac{n+j}{2}} \frac{j+1}{n+j+2} P(n,j).$$

The payoff when stopping at the last step at a non-negative value is

$$\begin{aligned}
\overline{U}_n &= \sum_{j \geq 0}^{n} j \cdot \mathbf{P}\{S_1 > -1, S_2 > -1, ..., S_{n-1} > -1, S_n = j\} \\
&= \sum_{j > 0}^{n} j \cdot \mathbf{P}\{S_1 > -1, S_2 > -1, ..., S_{n-1} > -1, S_n = j\} \\
&= \sum_{p-q>0}^{n} (p-q) \cdot \binom{n}{p} \frac{p-q+1}{p+1} \cdot \frac{1}{\binom{n}{p}(n+1)} \\
&= \sum_{p-q>0}^{n} \frac{(p-q)(p-q+1)}{(p+1)(n+1)}. \qquad (2)
\end{aligned}$$

The payoff when stopping at the value -1 has the form

$$U_n = -\left(1 - \sum_{p-q>-1}^{n} \frac{p-q+1}{(p+1)(n+1)}\right). \qquad (3)$$

3 Payoff for Different Values of n

3.1 The Case of Even Values of n

For even $n = p+q = 2m$ ($j = p-q = 2s$, $p = m+s$, $s = -m, ..., m$) the formulas 2 and 3 take the form

$$\begin{aligned}
\overline{U}_n &= \sum_{p-q>0}^{n} \frac{(p-q)(p-q+1)}{(p+1)(n+1)} = \sum_{s=1}^{m} \frac{2s(2s+1)}{(m+s+1)(2m+1)} \\
&= 2\left(\frac{-m^2+m+1}{2m+1} + (m+1)\left(\sum_{s=1}^{2m} \frac{1}{s} + \sum_{s=1}^{m+1} \frac{1}{s}\right)\right),
\end{aligned}$$

$$\begin{aligned}
U_n &= -\left(1 - \sum_{p-q>-1}^{n} \frac{p-q+1}{(p+1)(n+1)}\right) = -\left(1 - \frac{1}{2m+1} \sum_{s=0}^{m} \frac{2s+1}{m+s+1}\right) \\
&= -\sum_{s=0}^{m-1} \left(\frac{1}{2s+1} - \frac{1}{2s+2}\right) = -\sum_{s=m+1}^{2m} \frac{1}{s} = -\sum_{s=1}^{2m} \frac{1}{s} + \sum_{s=1}^{m} \frac{1}{s}.
\end{aligned}$$

Then the payoff in this problem is

$$V_n = \overline{U}_n + U_n$$
$$= 2\left(\frac{-m^2+m+1}{2m+1} + (m+1)\left(\sum_{s=1}^{2m}\frac{1}{s} + \sum_{s=1}^{m+1}\frac{1}{s}\right)\right) - \sum_{s=1}^{2m}\frac{1}{s} + \sum_{s=1}^{m}\frac{1}{s}$$
$$= -\frac{2m(m+1)}{2m+1} + (2m+1)\left(\sum_{s=1}^{2m}\frac{1}{s} - \sum_{s=1}^{m}\frac{1}{s}\right).$$

3.2 The Case of Odd Values of n

For odd values of $n = 2m+1$ ($j = 2s+1$, $p = m+s+1$, $s = -m, ..., m$), the formulas 2 and 3 have the form

$$\overline{U}_n = \sum_{p-q>0}^{n} (p-q)\cdot\frac{p-q+1}{p+1}\frac{1}{n+1}$$
$$= \sum_{s=0}^{m}\frac{(2s+1)(s+1)}{(m+s+2)(m+1)} = -m-1+(2m+3)\sum_{s=m+2}^{2m+2}\frac{1}{s},$$
$$U_n = -\left(1 - \sum_{p-q>-1}^{n}\frac{p-q+1}{(p+1)(n+1)}\right) = -\left(1-\sum_{s=0}^{m}\frac{s+1}{(m+s+2)(m+1)}\right)$$
$$= -\sum_{s=m+2}^{2m+2}\frac{1}{s}.$$

Then the payoff in this problem is

$$V_n = \overline{U}_n + U_n = -m-1+(2m+3)\sum_{s=m+2}^{2m+2}\frac{1}{s} - \sum_{s=m+2}^{2m+2}\frac{1}{s}$$
$$= -m-1+(2m+2)\sum_{s=m+2}^{2m+2}\frac{1}{s} = -m+(2m+2)\sum_{s=m+2}^{2m+1}\frac{1}{s}.$$

Thus, the following theorem has been proved.

Theorem 1. *In the gambler's ruin problem, the payoff V_n has the form*

1) for even values of $n = 2m$

$$V_n = -\frac{2m(m+1)}{2m+1} + (2m+1)\left(H_{2m} - H_m\right),$$

2) for odd values of $n = 2m+1$

$$V_n = -m + (2m+2)\left(H_{2m+1} - H_{m+1}\right),$$

where $H_m = \sum_{s=1}^{m}\frac{1}{s}$ is a harmonic number.

The results presented in the theorem show that the payoff in this ruin problem contains harmonic numbers. Interestingly, harmonic numbers occur in optimal stopping problems. To wit, in the best choice problem in [13], the probability of choosing the best item from the set of items N using an optimal strategy k^* is equal to $\frac{k^*-1}{N}\Big(H_{N-1} - H_{k^*-2}\Big)$.

For large values of $n = 2m$ we have an approximation $H_m \approx \ln(m)$. Then,

$$\begin{aligned} V_n &= -\frac{2m(m+1)}{2m+1} + (2m+1)\Big(H_{2m} - H_m\Big) \\ &\approx -\frac{2m(m+1)}{2m+1} + (2m+1)\ln 2 \approx m(\ln 4 - 1) \approx 0.193n. \end{aligned}$$

Remark 2. For the probability of gambler's ruin we get an asymptotic estimate

$$P_n^r = H_{2m} - H_{m-1} \approx \ln 2 \approx 0.693.$$

Although the probability of gambler's ruin in the strategy under study is high, his/her average payoff grows without limit with a growing number of game rounds.

Remark 3. For a symmetric random walk, the probability of the gambler's ruin with the given strategy is approximately $1 - \frac{1}{\sqrt{\pi m}}$ for even values of $n = 2m$. For greater values of n, the probability of ruin tends to 1, contrary to the suggested random walk.

Table 1 gives the numerical results for payoff values V_n at different values of n.

Table 1. Player's payoff for different values of n

n	2	3	4	5	6	7	8	9	10	100	200	300
V_n	0.167	0.333	0.517	0.7	0.888	1.076	1.266	1.456	1.647	19.010	38.324	57.638

4 Conclusion

We investigated the problem of gambler's ruin during a given time interval n. It is assumed that the probability of the player winning at each next step depends on the ratio of wins and losses in the previous steps. The player continues playing until the number of losses exceeds the number of wins. The player's payoff in this problem was determined for different values of game duration. The payoff is related to harmonic numbers. An asymptotic formula was built for computing the player's payoff at large values of n.

Further studies may focus on other models related to the ruin problem based on the suggested random walk scheme. Here the player's goal is to increase his/her capital without going ruin. Or, in other words, it is required to stop the random walk with the maximum possible value. Note that the payoff function can have a different form, for example, it can be any arbitrary increasing function of the stopped value. In particular, it would be interesting to examine the ruin problem with two players where the game continues until one of the players is ruined.

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Chapter 19

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