2023 DAYS on DIFFRACTION (DD), pp. 1–6 

# Interfering circle combs and uniform hyperbolicity of cocycles

**Alexey V. Ivanov**
Saint-Petersburg State University, Universitetskaya nab. 7/9,
Saint-Petersburg, 199034, Russian Federation;
e-mail: a.v.ivanov@spbu.ru

**We study interference of two circle combs shifted one to another. Assuming the number of jags of the combs to be large, we estimate the Lebesgue measure of those values of the shift parameter which correspond to overlapping of jags with some given precision. We apply the obtained results to study hyperbolic properties of cocycles generated by a smooth family of skew products $F_{A,t} = (\sigma_\omega, A_t)$ over irrational rotation $\sigma_\omega(x) = x + \omega$ of a circle $\mathbb{T}^1$ with a real parameter $t$. It is supposed that the transformation $A_t : \mathbb{T}^1 \to SL(2,\mathbb{R})$ is of the form $A_t(x) = R(\varphi(x))Z(\lambda(x))$, where $R(\varphi)$ stands for a rotation in $\mathbb{R}^2$ over an angle $\varphi$ and $Z(\lambda) = \mathrm{diag}\{\lambda, \lambda^{-1}\}$ is a diagonal matrix. Under assumption that $\cos(x)$ possesses only two simple zeroes and $\lambda(x) \geq \lambda_0 \gg 1$, we obtain a low bound for the Lebesgue measure of values of the parameter $t$ which correspond to the uniform hyperbolicity for the cocycle.**

## 1 Introduction

In [1] V. F. Lazutkin found that two identical combs overlapping with a small shift may produce an interfering picture. Namely, let $f : [\alpha, \beta] \to \mathbb{R}$ be a smooth function such that

$$f'(\xi) > 0, \quad f''(\xi) > 0, \quad \forall\, \xi \in [\alpha, \beta]. \tag{1}$$

Take a large positive integer $N$ and consider a sequence $x_k = f\left(\frac{k}{N}\right)$ for $k : \frac{k}{N} \in [\alpha, \beta]$. Then we define a comb, *Cmb*, with jags at the points $\{x_k\}$ as

$$Cmb = \left\{(x, z) \in \mathbb{R}^2 : x = x_k,\ z \in [0, 1]\right\}.$$

Such a comb is shown in Fig. 1. For given $B > 0$, the shifted comb, $Cmb(B)$, is

$$Cmb(B) = \left\{(x, z) \in \mathbb{R}^2 : x = x_k + \frac{B}{N}, z \in [0, 1]\right\}.$$

If one superposes $Cmb(B)$ over $Cmb(0) = Cmb$, one can see the interfering picture as in Fig. 2. Note that changing the parameter $B$ may lead to a situation when two or more jags overlap. The problem of describing the values of $B$ that guarantee the disjointness of two combs was studied in [1].

To be more precise, define for $(k, s)$ such that $\frac{k}{N}$, $\frac{k+s}{N} \in [\alpha, \beta]$, $s > 0$ the following characteristics

$$\Delta_{k,s}(B) = \left| x_{k+s} - x_k - \frac{B}{N} \right|,$$
$$\Delta(B) = \min_{k,s\neq 0} \Delta_{k,s}(B). \tag{2}$$

Then, under some genericity condition, the following theorem holds ([1]).

**Theorem 1.** *For any $\varepsilon > 0$, there exist $N_0 \in \mathbb{N}$, $D > 0$ such that for any $N \geq N_0$ there exists an interval $J_N$ of the length $|J_N| = D/N$, such that*

$$\Delta(B) \geq \frac{\left(\frac{1}{2} - \varepsilon\right) \min f''}{N^2}, \quad \forall\, B \in J_N.$$

Thus, if conditions (1) are satisfied, one may separate two identical combs with precision of the order $O(N^{-2})$. In this paper we consider the opposite question for similar objects — circle combs. We study the problem of description of those values of $B$, which correspond to overlapping of jags with some precision.

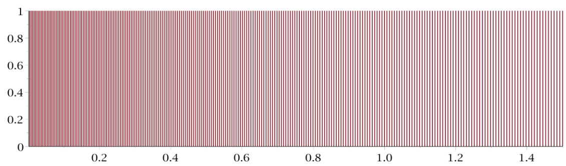


Figure 1: A comb generated by the function $f(t) = t + t^2/2$ with $\alpha = 0$, $\beta = 1$, $N = 250$.

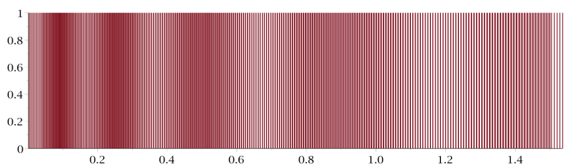


Figure 2: Interference of two identical combs with parameters as in Fig. 1 and $B = 9.1$.

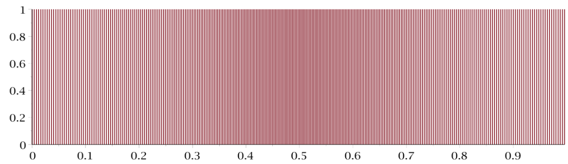


Figure 3: A circle comb generated by the function $f(t) = t + \frac{1}{30}\sin(2\pi t) + \frac{1}{10}\cos(2\pi N t)$ with $N = 300$.

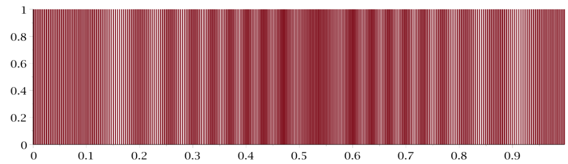


Figure 4: Interference of two circle combs with parameters as in Fig. 3 and $B = 1.2$.

## 2 Circle combs

Let $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ be the unit circle and $N$ be a large positive integer. We consider a smooth function $f : \mathbb{T}^1 \to \mathbb{T}^1$ and its lift $F : \mathbb{R} \to \mathbb{R}$ such that $\pi \circ F = f$ and

$$(A_1) \quad \operatorname{ind}(f) = 1,$$
$$(A_2) \quad F\left(\xi + \frac{1}{N}\right) > F(\xi), \ \ \forall \xi \in \mathbb{R},$$

where $\operatorname{ind}(f)$ stands for the index of the map $f$ and $\pi$ is the quotient map $\pi : \mathbb{R} \to \mathbb{T}^1$.

Then, for a given real $B \in [0, N)$, one may construct a sequence

$$x_k(B) = f\left(\left\{\frac{k+B}{N}\right\}\right), \quad k = 0, \dots, N-1,$$

where $\{x\}$ stands for the fractional part of $x$, and define a circle comb, $Cmb_c$, with jags at $\{x_k\}$ as

$$Cmb_c(B) = \big\{(x, z) \in \mathbb{T}^1 \times [0,1] : x = x_k, \\ z \in [0,1]\big\}.$$

An example of such circle comb is shown in Fig. 3.

As in the previous case the superposition of $Cmb_c(B)$ and $Cmb_c(0)$ leads to the interfering picture as shown in Fig. 4.

It has to be noted that definition of circle combs differs from definition given in [1]. Indeed, the second condition in (1) can never be satisfied on the circle. We also emphasize that the first condition in (1) is replaced by $(A_2)$.

Similarly to (2) we define for $(k, s)$ such that $k, k+s \in I_N = [0, N) \cap \mathbb{Z}$ the following charateristics:

$$\Delta_{k,s}(B) = \rho(x_{k+s}(B), x_k(0)),$$
$$\Delta(B) = \min_{k,s\neq 0} \Delta_{k,s}(B),$$

where $\rho$ denotes the standard distance on $\mathbb{T}^1$.

As mentioned above, we investigate the following problem: given $\delta > 0$, estimate the Lebesgue measure of a set $J_B$

$$J_B = \{B \in [0, N) : \Delta(B) < \delta\}.$$

First, we note that due to relation

$$x_k(B) = x_p(\{B\}), \quad p = k + [B] \mod N,$$

where $[B]$ stands for the integer part of $B$, one has

$$Cmb_c(B) = Cmb_c(\{B\}).$$

This enables us to make a reduction and consider $B \in [0, 1)$.

Besides, the function $F$ can be represented as

$$F(x) = x + G(x),$$

where $G : \mathbb{R} \to \mathbb{R}$ is a 1-periodic function. We denote its projection by $g = \pi \circ G : \mathbb{T}^1 \to \mathbb{T}^1$. It satisfies $\operatorname{ind}(g) = 0$.

Introduce

$$\tilde{\Delta}_{ks}(B) = F\left(\frac{k+s+B}{N}\right) - F\left(\frac{k}{N}\right).$$

Then $\tilde{\Delta}_{ks}(B)$ can be rewritten as

$$\begin{aligned}\tilde{\Delta}_{ks}(B) &= \frac{B}{N} + G\left(\frac{k+s+B}{N}\right) - G\left(\frac{k+s}{N}\right) \\ &\quad + \frac{s}{N} + G\left(\frac{k+s}{N}\right) - G\left(\frac{k}{N}\right) \\ &= F\left(\frac{k+s+B}{N}\right) - F\left(\frac{k+s}{N}\right) \\ &\quad + F\left(\frac{k+s}{N}\right) - F\left(\frac{k}{N}\right)\end{aligned}$$

and has the following properties:

1. $\tilde{\Delta}_{k,0}(0) = 0, \ \ \forall k \in I_N,$
2. $\tilde{\Delta}_{k,1}(0) > 0, \ \ \forall k \in I_N,$
3. $\tilde{\Delta}_{ks}(B) = \tilde{\Delta}_{k+s,0}(B) + \sum_{j=k}^{k+s-1} \tilde{\Delta}_{j,1}(0).$

For any $k, s \in I_N$, we consider an equation

$$\tilde{\Delta}_{k+s,0}(B) = -\sum_{j=k}^{k+s-1} \tilde{\Delta}_{j,1}(0) \mod 1 \tag{3}$$

with respect to $B \in [0,1)$ and introduce a set

$$J_{k,s} = \{B \in (0,1) : B \text{ is a solution of (3)}\}.$$

One may remark that under two following conditions the set $J_{k,s}$ is empty:

$$1.\ |G'(\xi)| < 1 \implies J_{k,s} = \emptyset,$$
$$2.\ |g(\xi)| < \frac{1}{2N} \implies J_{k,s} = \emptyset.$$

We will suppose that that the sets $J_{k,s}$ are either empty or consists of finite number, $M_{k,s}$, of points:

$$J_{k,s} = \bigcup_{j=1}^{M_{k,s}} \{B_{k,s,j}\}, \quad M_{k,s} > 0. \tag{4}$$

We also assume that the following non-degeneracy condition is fulfilled

$$G'(B) \neq -1, \ \forall\, B \in J_{k,s}. \tag{5}$$

Then we arrive at

**Theorem 2.** *Under conditions* (4), (5), *the following holds true:*
*1. If*

$$J_{k,s} \cap J_{p,l} = \emptyset, \ \forall\, (k,s) \neq (p,l),$$

*then for sufficiently small* $\delta > 0$ *there exists a positive constant* $C$ *such that the Lebesgue measure* $\operatorname{leb}(J_B)$ *admits an estimate*

$$\operatorname{leb}(J_B) \geq CN\delta \sum_{k=0}^{N-1} \sum_{s=0}^{N-1} M_{k,s}. \tag{6}$$

*2. If* $G$ *is* $\frac{1}{N}$*-periodic function, then for sufficiently small* $\delta > 0$ *there exists a positive constant* $C$ *such that the Lebesgue measure* $\operatorname{leb}(J_B)$ *admits an estimate*

$$\operatorname{leb}(J_B) \geq CN\delta \sum_{s=0}^{N-1} M_{k,s}. \tag{7}$$

Note that the first statement is a direct consequence of the implicit function theorem. Whereas the second one is a consequence of the $\frac{1}{N}$-periodicity, since in this case

$$M_{k,s} = M_{p,s}, \ \forall\, k,p,s \in I_N.$$

Finally, one may obtain that if $B/N \in [a,b] \subset [0,1)$, then estimates (6), (7) take the following form

$$\operatorname{leb}(J_B) \geq C(b-a)N\delta \sum_{k=0}^{N-1} \sum_{s=0}^{N-1} M_{k,s}\big(1 + O(N^{-1})\big),$$
$$\operatorname{leb}(J_B) \geq C(b-a)N\delta \sum_{s=0}^{N-1} M_{k,s}\big(1 + O(N^{-1})\big),$$

respectively. Moreover, if the variance, $\operatorname{Var}(g)$, of the function $g$ is bounded from below, then $M_{k,s}$ admits an estimate

$$M_{k,s} \geq 2\left[\frac{N \operatorname{Var}(g)}{s+1}\right]. \tag{8}$$

## 3 $SL(2,\mathbb{R})$-cocycles over irrational rotation

In [2] we studied hyperbolic properties of linear cocycles generated by a family of skew-products over irrational rotation

$$F_{A,t} : \mathbb{T}^1 \times \mathbb{R}^2 \to \mathbb{T}^1 \times \mathbb{R}^2 \tag{9}$$

defined by

$$(x,v) \mapsto (\sigma_\omega(x), A_t(x)v), \quad (x,v) \in \mathbb{T}^1 \times \mathbb{R}^2,$$

where $\sigma_\omega(x) = x + \omega$ is a rotation of the circle $\mathbb{T}^1$ with an irrational rotation number $\omega$, $A_t \in C(\mathbb{T}^1, SL(2,\mathbb{R}))$ and $t \in [a,b]$ is a real parameter.

One may associate with a skew-product (9) a difference equation

$$\psi(y+\omega) = \mathcal{A}(y)\psi(y), \quad y \in \mathbb{R},$$

where $\mathcal{A} = A \circ \pi$ is a 1-periodic matrix-valued function, $\pi : \mathbb{R} \to \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ is the quotient map and $\psi = (\psi_1, \psi_2)^{\mathrm{tr}}$ is an unknown vector-function. Such difference equations have many applications: spectral theory of the Schrödinger operator [3–6], stability theory [7], diffraction in a wedge-shaped domains [8] and others.

Besides, skew-products (9) are of great interest due their intermediate position between one- and two-dimensional cascades since they keep many features of multidimensional systems, but on the other hand the governing dynamics is much easier [9–12].

A skew product (9) generates a cocycle $M(x,n)$ by

$$M(x,n) = A(\sigma_\omega^{n-1}(x)) \ldots A(x), \ n > 0;$$
$$M(x,n) = \left[A(\sigma_\omega^{-n}(x)) \ldots A(\sigma_\omega^{-1}(x))\right]^{-1}, \ n < 0;$$
$$M(x,0) = I.$$

We remind definition of the uniform hyperbolicity.

**Definition 1.** *A cocycle $M$ is said to be uniformly hyperbolic (UH) if there exist continuous maps $E^{u,s} : \mathbb{T}^1 \to Gr(2,1)$ and positive constants $C$, $\Lambda$ such that the subspaces $E^{u,s}(x)$ are invariant with respect to the map (9), i.e. $E^{u,s}(\sigma_\omega(x)) = A(x)E^{u,s}(x)$, and $\forall x \in \mathbb{T}^1$, $n \geq 0$*

$$\left\|M(x,-n)|_{E^u(x)}\right\| \leq C\mathrm{e}^{-\Lambda n},$$
$$\left\|M(x,n)|_{E^s(x)}\right\| \leq C\mathrm{e}^{-\Lambda n}.$$

Here $Gr(2,1)$ stands for the set of 1-dimensional subspaces of $\mathbb{R}^2$. Due to the Oseledets theorem, such invariant subspaces exist for a.e. $x \in \mathbb{T}^1$, but, in general, the maps $E^{u,s}$ are only measurable.

This definition is equivalent (see e.g. [11]) to existence of positive constants $C$ and $\Lambda_0$ such that $\forall x \in \mathbb{T}^1$ and $n \geq 0$

$$\|M(x,n)\| \geq C\mathrm{e}^{\Lambda_0 n}. \quad (10)$$

In the present paper, we consider a family of skew-products (9) satisfying the following assumptions. We suppose that transformation $A_t$ can be represented as

$$A_t(x) = R(\varphi(x)) \cdot Z(\lambda(x)),$$

where

$$R(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}, \quad Z(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

with some smooth functions $\varphi : \mathbb{T}^1 \to 2\pi\mathbb{T}^1$, $\lambda : \mathbb{T}^1 \to \mathbb{R}$, such that

$(H_1)$ $\quad \left\{x \in \mathbb{T}^1 : \cos(\varphi(x)) = 0\right\} = \bigcup_{j=0}^{1} \{c_j\};$

$(H_2)$ $\quad C_1\varepsilon^{-1} \leq |\varphi'(x)| \leq C_2\varepsilon^{-1}, \ \forall x \in U_\varepsilon(c_j);$

$(H_3)$ $\quad \left|\cos(\varphi(x))\right| \geq C_3, \ \forall x \in \mathbb{T}^1 \setminus \bigcup_{j=0}^{1} U_\varepsilon(c_j);$

$(H_4)$ $\quad \mathrm{ind}(\varphi) = 0;$

$(H_5)$ $\quad \lambda(x) \geq \lambda_0 \gg 1, \ \forall x \in \mathbb{T}^1;$

$(H_6)$ $\quad \left|\dfrac{\mathrm{d}\rho(c_0(t), c_1(t))}{\mathrm{d}t}\right| > C_4, \ \forall t \in [a,b].$

Here $C_k$ denotes a positive constant, $\varepsilon \ll 1$, $U_\varepsilon(x)$ is the $\varepsilon$-neighbourhood of a point $x \in \mathbb{T}^1$.

In the present paper, we continue studying the problem of constructive description of the set

$$\mathcal{T}_h = \{t \in [a,b] : M(x,n) \text{ is UH}\}.$$

It has to be noted that despite a cocycle $M$, which corresponds to (9), is a product of matrices $A_k$ with such sufficiently large norm $\|A_k\| \geq \lambda_0$, the product $\prod_{k=1}^{n} A_k$ may not admit estimate (10). The reason for this fact is the presence of the critical set, $\mathcal{C}_0$, defined as

$$\mathcal{C}_0 = \{c_0\} \cup \{c_1\}.$$

It was emphasized in [2] that hyperbolic properties of $M(x,n)$ strongly depend on the dynamics of the critical set itself. To describe the dynamics of the critical set, we introduced in [13] notions of collision and time of collision.

**Definition 2.** *For a given $\delta > 0$ define $\tau_{j,j'}$ to be the minimum of integer $k > 0$ such that*

$$\sigma_\omega^k(c_j) \cap U_\delta(c_{j'}) \neq \emptyset.$$

*We say that the points $c_j$ and $c_{j'}$ collide with accuracy $\delta$ at the time $\tau_{j,j'}$ and call such event a collision and $\tau_{j,j'}$ the time of collision. A collision is called primary if $j = j'$ and secondary otherwise.*

There is essential difference in behaviour of the primary and secondary collisions with respect to the parameter $t$. First, we note that the times of primary collisions $\tau_{j,j}$ do not depend on $j$ and we may denote them by $\tau_0$. It is a characteristic of the rotation number $\omega$ and the parameter $\delta$ only. On the other hand, the assumption $(H_6)$ implies that relative positions of the points $c_j$ vary with respect to $t$ and, hence, the times of secondary collisions depend on the parameter $t$ in a non-trivial way.

Since the type of collisions is descibed by the relative position of the points $c_0$ and $c_1$, we may always consider the case when only $c_1$ depends on the parameter $t$, but position of $c_0$ is constant. Moreover, due to assumption $H_6$, one may assume that

$$t = \rho(c_0, c_1).$$

In [2] we showed that the dominance of secondary collisions may lead to the uniform hyperbolicity of the cocycle (in contrary to the dominance of primary collisisons [13]). To formulate the result, we assume that for a fixed positive $\delta \ll 1$ the secondary collision occurs before the secondary one, i.e.

$$\tau_{0,1}(\delta) = n, \quad \tau_0(\delta) > n. \quad (11)$$

Moreover, we suppose that there exists $t_0 \in [a,b]$, such that

$$\sigma_\omega^{-n}(c_1(t_0)) = c_0. \quad (12)$$

Then the following theorem holds [2].

**Theorem 3.** *Let hypotheses* $(H_1)$–$(H_6)$ *and conditions* (11), (12) *be satisfied. If additionally*

$$\tau_0 > C_\lambda n, \quad C_\lambda = 1 + \frac{2\log\lambda_{\max}}{\log(C_3\lambda_{\min})},$$

*then there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$ *the cocycle* $M(x, n)$ *is uniformly hyperbolic for all* $t \in U_h(t_0)$, *where*

$$h = O(\varepsilon\lambda_0^{-n}).$$

Here $\lambda_{\min}$, $\lambda_{\max}$ stands for the minimum and maximum values of the function $\lambda$, respectively.

## 4 Circle combs generated by cocycles

In this section we investigate how a cocycle associated with the skew-product (9) may generate circle combs and consider their interference.

Each irrational number $\omega \in (0, 1)$ has a unique representation as a continued fraction

$$\omega = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\cdots}}}, \quad a_n \in \mathbb{N} \ \forall n \in \mathbb{N}$$

or, using the list form,

$$\omega = [a_1, a_2, \ldots, a_n, \ldots].$$

Then the rational number $p_n/q_n = [a_1, a_2, \ldots, a_n]$ is called the $n$-th convergent to $\omega$ and gives the minimum for $|q\omega - p|$ over all rationals $p/q$ with $q \le q_n$. The $n$-th convergents satisfy an estimate [14]

$$\frac{1}{2q_nq_{n+1}} < \left|\omega - \frac{p_n}{q_n}\right| < \frac{1}{q_nq_{n+1}}.$$

Since $\omega$ is assumed to be irrational, a trajectory of any point $x \in \mathbb{T}^1$ under the rotation $\sigma_\omega$ is dense in $\mathbb{T}^1$. We take $n_0 \in \mathbb{N}$ and choose sufficiently small positive $\delta$ such that

$$\frac{1}{q_{n+1}} < \delta < \frac{1}{2q_n}. \tag{13}$$

Then

$$\tau_0(\delta) = q_{n_0}$$

and we denote $N = q_{n_0} - 1$.

Using the singular value decomposition, one may represent the cocycle $M(x, N)$ as

$$M(x, N) = R(\Phi_N(x))Z(\mu_N(x))R(\chi_N(x))$$

and consider the following equation

$$\cos(\Phi_N(x)) = 0.$$

This equation defines the critical set of the $N$-th order, $\mathcal{C}_N$ (see [13])

$$\mathcal{C}_N = \{x \in \mathbb{T}^1 : \cos(\Phi_N(x)) = 0\}.$$

Then we arrive at the following lemma.

**Lemma 1.** *The $N$-th order critical set has the following representation*

$$\mathcal{C}_N = \bigcup_{k=0}^{N-1}\{x_k^{(0)}\} \cup \bigcup_{k=0}^{N-1}\{x_k^{(1)}(t)\},$$

*where*

$$x_k^{(0)} = c_0 + \frac{k}{N} + g\left(c_0 + \frac{k}{N}\right) + O\left(\frac{1}{q_{n_0+1}}\right),$$

$$x_k^{(1)} = c_0 + t + \frac{k}{N} + g\left(c_0 + t + \frac{k}{N}\right) + O\left(\frac{1}{q_{n_0+1}}\right),$$

*and* $g : \mathbb{T}^1 \to \mathbb{T}^1$ *is a smooth function such that* $\operatorname{ind}(g) = 0$ *and its lift, $G$, is a* $\frac{1}{N}$*-periodic function.*

We remark here that $x_k^{(1)}(0) = x_k^{(0)}$.

One may construct two circle combs with jags at the points $x_k^{(0)}$ and $x_k^{(1)}(t)$ generated by the cocycle $M(x, N)$. We denote them by $Cmb_c(0)$, $Cmb_c(B)$ with $t = B/N$, respectively, and consider their interference.

It can be proved that the variance of the function $g$ from Lemma 1 satisfies

$$\operatorname{Var}(g) \ge C_5 \operatorname{Var}(\varphi)\varepsilon\lambda_0^{-2}. \tag{14}$$

Thus, $\operatorname{Var}(g)$ is small due to assumptions $(H_2)$, $(H_5)$. However, the estimate (14) is uniform with respect to $N$.

We apply Theorems 2 and 3, Lemma 1 and estimates (8), (14) to get the following statement.

**Theorem 4.** *Let hypotheses* $(H_1)$–$(H_2)$ *be satisfied. Suppose that for a given* $n_0 \in \mathbb{N}$ *and* $\delta$ *satisfying* (13) *the conditions of Lemma 1 are fulfilled. Then the Lebesgue measure of a set*

$$\mathcal{T}_h^{(n_0)} = \{t \in [a, b] : M(x, N) \text{ is UH}, \ N = q_{n_0} - 1\}$$

*admits an estimate*

$$\operatorname{leb}\left(\mathcal{T}_h^{(n_0)}\right) \ge C(b-a)N\varepsilon^2\lambda_0^{-2} \times \sum_{k=k_1}^{k_2}\sum_{s=0}^{N-1}\lambda_0^{-(k+1)}\frac{\operatorname{Var}(\varphi)}{s+1},$$

*where*

$$k_1 = \max\left\{\left[\frac{\ln\varepsilon\delta^{-1}}{\ln\lambda_0}\right], 0\right\}, \quad k_2 = \left[\frac{N}{C_\lambda}\right].$$

Acknowledgements

The research was supported by RFBR grant (project № 20-01-00451/22).

References

[1] Lazutkin, V. F., 1997, Interfering combs and multiple horseshoe, *Reg. & Chaotic Dyn.*, Vol. **2**(2), pp. 3–13.

[2] Ivanov, A. V., 2023, On $SL(2,\mathbb{R})$-cocycles over irrational rotations with secondary collisions, *Reg. & Chaotic Dyn.*, Vol. **28**(2), pp. 207–226.

[3] Avila, A., 2010, Almost reducibility and absolute continuity I, *arXiv*: 1006.0704.

[4] Avila, A., Bochi, J., Damanik, D., 2012, Opening gaps in the spectrum of strictly ergodic Schrödinger operators, *J. Eur. Math. Soc.*, Vol. **14**, pp. 61–106.

[5] Avila, A., Bochi, J., 2002, A formula with some applications to the theory of Lyapunov exponents, *Isr. J. Math.*, Vol. **131**, pp. 125–137.

[6] Buslaev, V. S., Fedotov, A. A., 1994, The monodromization and Harper equation, *Séminaire Équations aux Dérivées Partielles*, Vol. **XXI**, pp. 1–21, École Polytechnique, Centre de Mathématiques, Palaiseau.

[7] Ivanov, A. V., 2017, Connecting orbits near the adiabatic limit of Lagrangian systems with turning points, *Reg. & Chaotic Dyn.*, Vol. **22**(5), pp. 479–501.

[8] Lyalinov, M. A., Zhu, N. Y., 2003, A solution procedure for second-order difference equations and its application to electromagnetic-wave diffraction in a wedge-shaped region, *Proc. Roy. Soc. Lond. A*, Vol. **459**(2040), pp. 3159–3180.

[9] Herman, M., 1983, Une methode pour minorer les exposants de Lyapunov et quelques exemples montrant le charactere local d'un theoreme d'Arnold et de Moser sur le tore en dimension 2, *Commun. Math. Helv.*, Vol. **58**, pp. 453–502.

[10] Young, L.-S., 1997, Lyapunov exponents for some quasi-periodic cocycles, *Ergod. Th. & Dynam. Sys.*, Vol. **17**, pp. 483–504.

[11] Avila, A., Bochi, J., 2007, A uniform dichotomy for generic SL(2,$\mathbb{R}$)-cocycles over a minimal base, *Bull. Soc. Math. France*, Vol. **135**, pp. 407–417.

[12] Benedicks, M., Carleson, L., 1991, The dynamics of the Hénon map, *Ann. Math.*, Vol. **133**, pp. 73–169.

[13] Ivanov, A. V., 2021, On singularly perturbed linear cocycles over irrational rotations, *Reg. & Chaotic Dyn.*, Vol. **26**(3), pp. 205–221.

[14] Khinchin, A. Ya., 1964, *Continued Fractions*, Chicago University Press, Chicago, London.