Obtaining a parametric equation for the road trajectory which is optimal in terms of construction costs

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Abstract. There are different approaches to define trajectory that is optimal from the point of view of construction costs. We study the problem of obtaining a parametric equation of the cost-optimal trajectory for the road connecting two points on an initially given terrain. By means of mathematical modelling approach we construct the integral cost functional, which arguments are parametric functions describing the trajectory. Therefore, we get the problem of the calculus of variations, the solution of which defines the most cost-effective way. We derive the optimality condition which has a form of a system of integro-differential equations. We solve the equations from the resulting system using the Galerkin method. The optimal solution is presented as linear combination of the first n functions of a system of twice continuously differentiable compactly supported functions on a given interval. The paper also presents the results of numerical experiments for various surfaces on which the road is laid.

Keywords: Calculus of variations, Mathematical modelling, Integro-differential equation, Shooting method, Linearization, Optimal trajectory.

1 Introduction

In this paper, we present a method to determine the most cost-effective path. Such tasks are very important, as they have many applications in various fields [1-3]. Researchers often use popular methods, which sometimes cannot provide the expected quality of the solution [4-21]. In contrast, our method is based on the variational principle and avoids these shortcomings. In this study, we apply the results obtained in our previous works [22, 23], in which the problem was formulated in variational form and optimality conditions were described. These conditions represent an integral-differential equation, the solution of which is a difficult task. Here we have solved this equation using the Galerkin method and given numerical examples.

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2 Problem statement

Consider two points: O and A, which define the starting and ending point of the route that we need to find. The task is to connect these two expensive points, which will be the most optimal in terms of construction costs. We hold the following main meetings:

- Construction materials are supplied from the starting point
- Delivery of materials to the construction site can only be carried out along the already constructed part of the road.
- The production technology is the same everywhere.
- The total cost of roads includes the cost of shipping construction materials and the cost of most construction.
- Change in elevation in the construction area.

Let α the cost of delivery per unit road length of the volume of materials required for the construction of its single section. Since construction conditions vary from point to point due to various factors, the cost β of constructing a single section of road at each point on the route is different. Introduce a Cartesian coordinate system with the origin O. Here (l, y_l) are the coordinates of the end point A, and $y: \mathbb{R} \to \mathbb{R}$ is any twice continuously differentiable function which satisfies boundary conditions

$$y(0) = 0, \quad y(l) = y_l. \tag{1}$$

We assume that owning such a curve is acceptable. The cost of the road, a certain discount y(x), can be expressed as follows:

$$J(y) = \int_0^l \alpha\sqrt{1+y'^2(x)} \int_0^x \sqrt{1+y'^2(\xi)} \quad d\xi dx + \int_0^l \beta(x,y)\sqrt{1+y'^2(x)}\, dx, \tag{2}$$

where $\beta: \mathbb{R}^2 \to \mathbb{R}$ is a given function with continuous second order partial derivatives.

Lemma 1. Let $f(x)$ be an arbitrary function from $C[0, l]$. Then the following equation holds

$$\int_0^l f(x) \int_0^x f(\xi)\, d\xi\, dx = \frac{1}{2}\left(\int_0^l f(x)\, dx\right)^2.$$

The proof is presented in [22].

By means of Lemma 1 we can rewrite functional (2) as

$$J(y) = \frac{\alpha}{2}\left(\int_0^l \sqrt{1+y'^2(x)} dx\right)^2 + \int_0^l \beta(x,y)\sqrt{1+y'^2(x)}\, dx.$$

Theorem 1. *For the admissible curve $y_*(x) \in C^2[0,l]$ to be a minimizer of the cost functional J the following conditions must be satisfied:*

$$\frac{y_{*}''(x)}{1+y_{*}'^{2}(x)}\left(\alpha\int_0^l \sqrt{1+y_{*}'^{2}(x)}dx \quad +\beta(x,y_*(x))\right) +y_{*}'(x)\frac{\partial\beta(x,y_*(x))}{\partial x}-\frac{\partial\beta(x,y_*(x))}{\partial y}=0. \tag{3}$$

The theorem is proved in [22, 23].

To solve equation (3) on a uniform grid, you can calculate the values of y at the nodes of the variables and use them to construct an interpolation polynomial (see [26]). A similar approach for this problem is used in [22, 23, 27]. Applying this idea, we obtain a nonlinear algebraic system based on the results of functions at network nodes. The method becomes extremely unstable at high levels of the interpolation polynomial. Galerkin method is free of these shortcomings.

The main goal of the current research is to derive necessary condition for the minimum for a cost-optimal trajectory in a parametric form. Parametric form allows us to work with a wider set of trajectories than explicit one since parametrically represented curve can have more than one intersection point with a line parallel to axe Oy while this not true for explicitly given curve. We derive necessary condition for optimality in parametric form and use Galerkin method in order to solve the obtained integro-differential equations.

Consider a parametric definition of the curve $r(t)=\big(x(t),y(t)\big)$ such that $x(0)=y(0)=0$ and $x(1)=l,\ y(1)=y_l$. The cost of the curve is the value of the following functional

$$J(y)=\frac{\alpha}{2}\left(\int_0^1\sqrt{x'^2+y'^2}dt\right)^2+\int_0^1\beta(x,y)\sqrt{x'^2+y'^2}\,dt.$$

Theorem 2. For the admissible curve $r_*(t)=(x_*(t),y_*(t))\in C^2[0,l]$ to be a minimizer of the cost functional J it is necessary that

$$\begin{cases}\frac{x_*''(t)y_*'(t)-y_*''(t)x_*'(t)}{y_*'^2(t)+x_*'^2(t)}\left(\alpha\int_0^1\sqrt{x_*'^2(t)+y_*'^2(t)}dt+\beta\big(x_*(t),y_*(t)\big)\right) \\ \qquad +x_*'(t)\frac{\partial\beta(x_*(t),y_*(t))}{\partial y}-y_*'(t)\frac{\partial\beta\big(x_*(t),y_*(t)\big)}{\partial x}=0 \\ \frac{y_*''(t)x_*'(t)-x_*''(t)y_*'(t)}{y_*'^2(t)+x_*'^2(t)}\left(\alpha\int_0^1\sqrt{x_*'^2(t)+y_*'^2(t)}dt+\beta\big(x_*(t),y_*(t)\big)\right) \\ \qquad +y_*'(t)\frac{\partial\beta\big(x_*(t),y_*(t)\big)}{\partial x}-x_*'(t)\frac{\partial\beta(x_*(t),y_*(t))}{\partial y}=0\end{cases}. \tag{4}$$

Proof. For the sake of convenience introduce the notion

$$F(x',y') = \sqrt{x'^2 + y'^2}. \qquad (5)$$

Then

$$J(y) = \frac{\alpha}{2}\left(\int_0^1 F(y'(x))dt\right)^2 + \int_0^1 \beta(x,y)F(y'(x))\,dt. \qquad (6)$$

Let $\delta_1(t), \delta_2(t)$ be a continuously differentiable functions with compact support [0,1] and $\varepsilon_1, \varepsilon_2$ be a scalar. The variation of the functional is found as (see [24])

$$\delta J(x_*, y_*) = \begin{pmatrix} \frac{d}{d\varepsilon_1} J(x_* + \varepsilon_1\delta_1, y_* + \varepsilon_2\delta_2) \\ \frac{d}{d\varepsilon_2} J(x_* + \varepsilon_1\delta_1, y_* + \varepsilon_2\delta_2) \end{pmatrix} |_{\varepsilon_1,\varepsilon_2=0} = 0. \qquad (7)$$

We have

$$\frac{d}{d\varepsilon_1}\left[\begin{matrix}\frac{\alpha}{2}\left(\int_0^1 F\big(x'_* + \varepsilon_1\delta_1', \quad y'_* + \varepsilon_2\delta_2'\big)dt\right)^2 + \int_0^1 \beta(x_* + \varepsilon_1\delta_1, y_* + \varepsilon_2\delta_2) * \\ * F\big(x'_* + \varepsilon_1\delta_1', \quad y'_* + \varepsilon_2\delta_2'\big)dt\end{matrix}\right]|_{\varepsilon_1,\varepsilon_2=0}$$

$$= \alpha\int_0^l \frac{\partial F}{\partial x'}\delta_1' dt \int_0^1 F(x'_*, y'_*)\,dt + \int_0^1 \frac{\partial \beta}{\partial x_*} F(x'_*, y'_*)\delta_1 dt + \int_0^1 \beta\frac{\partial F}{\partial x'_*}\delta_1' dt.$$

Using the formula for integration by parts, we consider separately the expressions in the summands which are in the right side of this equality.

$$\int_0^1 \frac{\partial F}{\partial x'}\delta_1' dt = \frac{\partial F}{\partial x'}\delta_1|_0^1 - \int_0^1 \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)\delta_1 dt = -\int_0^1 \frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)\delta_1 dt,$$

$$\int_0^1 \beta\frac{\partial F}{\partial x'}\delta_1' dt = \beta\frac{\partial F}{\partial x'}\delta_1|_0^1 - \int_0^1 \frac{d}{dt}\left(\beta\frac{\partial F}{\partial x'}\right)\delta_1 dt = -\int_0^1 \frac{d}{dt}\left(\beta\frac{\partial F}{\partial x'}\right)\delta_1 dt.$$

Thus, we can write the first equation from necessary minimum condition in the form

$$\frac{d}{d\varepsilon_1}J(x_*, y_*) = \int_0^1 \left(-\alpha\frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)\int_0^1 F(x'_*, y'_*)\,dt + \frac{\partial\beta}{\partial x_*}F(x'_*, y'_*) - \frac{d}{dt}\left(\beta\frac{\partial F}{\partial x'}\right)\right)\delta_1 dt = 0.$$

The function under the integral and being the multiplier of δ_1 belongs to $C^1[0,1]$. Since $C^1[0,1] \subset \mathcal{L}_2[0,1]$, and a set of continuously differentiable functions with compact support [0,1] is dense $\mathcal{L}_2[0,1]$ (see [25]), we obtain

$$-\alpha\frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)\int_0^1 F\big(x'_*, y'_*\big)dt + \frac{\partial\beta}{\partial x}F\big(x'_*, y'_*\big) - \frac{d}{dt}\left(\beta\frac{\partial F}{\partial x'}\right) = 0. \qquad (8)$$

Consider the individual terms of the left side of the equation

$$\frac{\partial F}{\partial x'} = \frac{x'}{\sqrt{x'^2 + y'^2}},$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right) = \frac{x''\sqrt{x'^2 + y'^2} - \dfrac{x''(x')^2 + y''x'y'}{\sqrt{x'^2 + y'^2}}}{x'^2 + y'^2} = \frac{x''y'^2 - y''x'y'}{(x'^2 + y'^2)^{\frac{3}{2}}}$$

$$\frac{d}{dt}\left(\beta\frac{\partial F}{\partial x'}\right) = \frac{\partial\beta}{\partial x}x'\frac{\partial F}{\partial x'} + \frac{\partial\beta}{\partial y}y'\frac{\partial F}{\partial x'} + \beta\frac{d}{dt}\left(\frac{\partial F}{\partial x'}\right)$$

$$= \frac{\partial\beta}{\partial x}x'\frac{x'}{\sqrt{x'^2 + y'^2}} + \frac{\partial\beta}{\partial y}y'\frac{y'}{\sqrt{x'^2 + y'^2}} + \beta\frac{x''y'^2 - y''x'y'}{(x'^2 + y'^2)^{\frac{3}{2}}}.$$

Substituting this into the equation (8)

$$-\alpha\frac{x''y'^2 - y''x'y'}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}\int_0^1 F(x,y)dt + \frac{\partial\beta}{\partial x}\sqrt{x'^2 + y'^2} - \frac{\partial\beta}{\partial x}\frac{x'^2}{\sqrt{x'^2 + y'^2}}$$

$$-\frac{\partial\beta}{\partial y}\frac{x'y'}{\sqrt{x'^2 + y'^2}} - \beta\frac{x''y'^2 - y''x'y'}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}} = 0.$$

Make transformations

$$-\alpha\frac{x''y'^2 - y''x'y'}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}\int_0^1 F(x,y)dt - \frac{\dfrac{\partial\beta}{\partial x}x'^2 + \dfrac{\partial\beta}{\partial x}y' - \dfrac{\partial\beta}{\partial x}x'^2 - \dfrac{\partial\beta}{\partial y}x'y'}{\sqrt{x'^2 + y'^2}}$$

$$-\beta\frac{x''y'^2 - y''x'y'}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}} = 0,$$

$$-\frac{y'}{\sqrt{x'^2 + y'^2}}\left[\alpha\frac{x''y'^2 - y''x'y'}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}\int_0^1 F(x,y)dt - \beta\frac{x''y'^2 - y''x'y'}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}} + \frac{\partial\beta}{\partial y}x' - \frac{\partial\beta}{\partial x}y'\right] = 0.$$

Finally get

$$\frac{x''y'^2 - y''x'y'}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}\left[\alpha\int_0^1 F(x,y)dt - \beta(x,y)\right] + \frac{\partial\beta}{\partial y}x' - \frac{\partial\beta}{\partial x}y' = 0.$$

When applying the same actions to the second for, we obtain the second equation of the system.

Thus, we arrive at the minimum condition

$$\begin{cases} \dfrac{x_*''(t)y_*'(t) - y_*''(t)x_*'(t)}{y_*'^2(t) + x_*'^2(t)}\left(\alpha \displaystyle\int_0^1 \sqrt{x_*'^2(t) + y_*'^2(t)}dt + \beta\big(x_*(t), y_*(t)\big)\right) \\ \qquad + x_*'(t)\dfrac{\partial \beta(x_*(t), y_*(t))}{\partial y} - y_*'(t)\dfrac{\partial \beta\big(x_*(t), y_*(t)\big)}{\partial x} = 0 \\ \dfrac{y_*''(t)x_*'(t) - x_*''(t)y_*'(t)}{y_*'^2(t) + x_*'^2(t)}\left(\alpha \displaystyle\int_0^1 \sqrt{x_*'^2(t) + y_*'^2(t)}dt + \beta\big(x_*(t), y_*(t)\big)\right) \\ \qquad + y_*'(t)\dfrac{\partial \beta\big(x_*(t), y_*(t)\big)}{\partial x} - x_*'(t)\dfrac{\partial \beta\big(x_*(t), y_*(t)\big)}{\partial y} = 0. \end{cases}$$

In the present work we solve (3) and (4) via Galerkin method which belongs to the class of approximate methods.

3 Galerkin metod

Let us initially briefly describe the main idea of Galerkin method [28] for the problem (3). Consider the operator

$$L(y) = \frac{y''}{1 + y'^2}\left(\alpha \int_0^l \sqrt{1 + y'^2}dx + \beta(x, y)\right) + y'\frac{\partial \beta(x,y)}{\partial x} - \frac{\partial \beta(x, y)}{\partial y}.$$

The necessary condition of the minimum (3) then can be rewritten as an equation

$$L(y) = 0. \tag{9}$$

We find a solution of the equation in the form

$$y = \tfrac{y_l}{l}x + \textstyle\sum_{k=1}^{\infty} a_k \phi_k(x), \tag{10}$$

where $\{\phi_k(x) | k = 1,2, \dots\}$ is a system of basis functions in the space $C_0^2[0, l]$ of twice continuously differentiable functions with compact support $[0, l]$ (i.e. satisfying boundary conditions $y(0) = y(l) = 0$). It is clear that function (10) satisfies the boundary conditions (1).

We can use for example system of functions

$$\phi_k(x) = \sin\tfrac{k\pi x}{l}, \quad k = 1,2, \dots. \tag{11}$$

or

$$\phi_k(x) = (l - x)x^k, \quad k = 1,2, \dots \tag{12}$$

This fact immediately follows from the Weierstrass approximation theorems.

Note that $L(y)$ is continuous for any admissible y. We can consider problem (9) in $\mathcal{L}_2[0,l]$. Then it is obvious that the function y_* fulfills equation (9) if and only if $L(y_*)$ is orthogonal to all functions of system $\{\phi_k(x)|k=1,2,\dots\}$. However, if we work solely with the sum of the first n terms of the series (10), we can satisfy only n orthogonality conditions, i.e.

$$\int_0^l L(y_*(x))\phi_k(x)dx = \int_0^l L\left(\frac{y_l}{l}x + \sum_{k=1}^{n} a_k\phi_k(x)\right)\phi_k(x)dx = 0, \quad k=1,\dots,n.$$

We act similarly with system (4).

$$\mathrm{M}\big(\mathrm{x}_*(\mathrm{t}),\mathrm{y}_*(\mathrm{t})\big)=\Big(\frac{\mathrm{x}_*''(\mathrm{t})\mathrm{y}_*'(\mathrm{t})-\mathrm{y}_*''(\mathrm{t})\mathrm{x}_*'(\mathrm{t})}{\mathrm{y}_*'^2(\mathrm{t})+\mathrm{x}_*'^2(\mathrm{t})}\left(\alpha\int_0^{\mathrm{l}}\sqrt{\mathrm{x}_*'^2(\mathrm{t})+\mathrm{y}_*'^2(\mathrm{t})}\mathrm{dt}+\beta\big(\mathrm{x}_*(\mathrm{t}),\mathrm{y}_*(\mathrm{t})\big)\right)+\mathrm{x}_*'(\mathrm{t})\frac{\partial\beta(\mathrm{x}_*(\mathrm{t}),\mathrm{y}_*(\mathrm{t}))}{\partial\mathrm{y}}-\mathrm{y}_*'(\mathrm{t})\frac{\partial\beta\big(\mathrm{x}_*(\mathrm{t}),\mathrm{y}_*(\mathrm{t})\big)}{\partial\mathrm{x}}$$

$$\mathrm{N}\big(\mathrm{x}_*(\mathrm{t}),\mathrm{y}_*(\mathrm{t})\big)=\frac{\mathrm{y}_*''(\mathrm{t})\mathrm{x}_*'(\mathrm{t})-\mathrm{x}_*''(\mathrm{t})\mathrm{y}_*'(\mathrm{t})}{\mathrm{y}_*'^2(\mathrm{t})+\mathrm{x}_*'^2(\mathrm{t})}\left(\alpha\int_0^{\mathrm{l}}\sqrt{\mathrm{x}_*'^2(\mathrm{t})+\mathrm{y}_*'^2(\mathrm{t})}\mathrm{dt}+\beta\big(\mathrm{x}_*(\mathrm{t}),\mathrm{y}_*(\mathrm{t})\big)\right)+\mathrm{y}_*'(\mathrm{t})\frac{\partial\beta\big(\mathrm{x}_*(\mathrm{t}),\mathrm{y}_*(\mathrm{t})\big)}{\partial\mathrm{x}}-\mathrm{x}_*'(\mathrm{t})\frac{\partial\beta(\mathrm{x}_*(\mathrm{t}),\mathrm{y}_*(\mathrm{t}))}{\partial\mathrm{y}}$$

We find a solution of the equations (4) in the form

$$x(t) = lt + \sum_{k=1}^{\infty} a_k\phi_k(t),$$

$$y(t) = y_l t + \sum_{k=1}^{\infty} b_k\phi_k(t).$$

Then

$$\int_0^1 M\big(x_*(t),y_*(t)\big)\phi_k(t)dt = \int_0^1 M\big(lt+\sum_{k=1}^{\infty} a_k\phi_k(t), y_l t+\sum_{k=1}^{\infty} b_k\phi_k(t)\big)\phi_k(t)dt = 0,$$
k=1,...,n.

$$\int_0^1 N\big(x_*(t),y_*(t)\big)\phi_k(t)dt = \int_0^1 N\big(lt+\sum_{k=1}^{\infty} a_k\phi_k(t), y_l t+\sum_{k=1}^{\infty} b_k\phi_k(t)\big)\phi_k(t)dt = 0,$$

k=1,…,m.

More details regarding the method can be found in [28].

4 The results of numerical experiments

Let's look at some examples in which we prefer that the ground surface on which the road will be built is also described by a moderate β. This assumption allows us to obtain a visual graphical representation and interpretation of the results obtained.

Example 1. Let $\alpha = 0.1$, $l = 1$, $y_l = 1$ and $\beta: \mathbb{R}^2 \to \mathbb{R}$

$$\beta(x, y) = 1 + \sin 5x \cdot \sin y.$$

First, we will use system (11) and search the solution in the form

$$y(x) = \frac{y_l}{l} x + \sum_{k=1}^{5} a_k \sin\frac{\pi k}{l} x. \tag{13}$$

Via Galerkin method we obtain (black curve in Fig. 2)

$$a_1 = -0.31489, \quad a_2 = 0.07442, \quad a_3 = -0.03199, \quad a_4 = 0.01256, \quad a_5 = -0.00424.$$

If we substitute (10) to functional J itself and minimize it with respect to a_k, $k = 1, \dots, 5$ we come to Ritz method. It produces the following solution

$$a_1 = -0.31397, \quad a_2 = 0.07367, \quad a_3 = -0.03138, \quad a_4 = 0.01212, \quad a_5 = -0.00396,$$

for which the cost also equals 1.279. As we see in Fig. 1 these two solutions are practically identical.

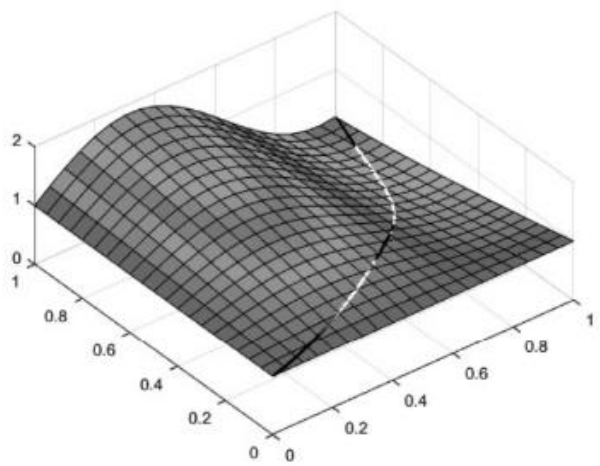

Fig. 1. Illustration of the obtained curve of the form (13) in Example 1.

Now let us find the solution of the same problem in parametric form via conditions of Theorem 2.

$$x(t) = t + \sum_{k=1}^{3} a_k \sin(\pi k t),$$

$$y(t) = t + \sum_{k=1}^{3} b_k \sin(\pi k t).$$

Using the Galerkin method

$$a_1 = 0.14642926, \quad a_2 = 0.00245703, \quad a_3 = 0.01382993,$$

$$b_1 = -0.17766913, \quad b_2 = 0.0075474, \quad b_3 = 0.00617156.$$

This solution is demonstrated in Fig. 2 and the cost of solution (i.e., the value of functional J) on the solution produced is 1.28.

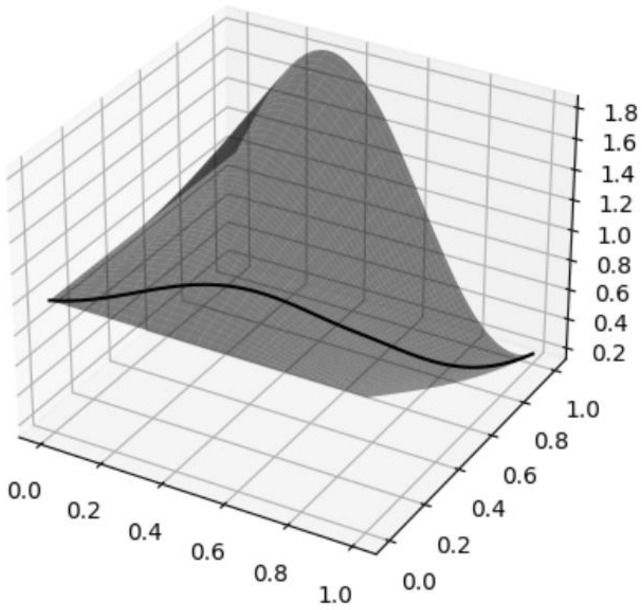

Fig. 2. Illustration of the obtained curve in the parametric form in Example 1.

Example 2. Let $\alpha = 0.5$, $l = 1$, $y_l = 1$ and $\beta: \mathbb{R}^2 \to \mathbb{R}$

$$\beta(x, y) = 5 + 2\cos 2x \cdot \sin y.$$

Fig. 3 illustrates the approximate solution in the form

$$y(x) = \frac{y_l}{l} x + \sum_{k=1}^{5} a_k x^k (1 - x) \qquad (14)$$

obtained via Galerkin method, where

$$a_1 = -0.13852, \quad a_2 = 0.02332, \quad a_3 = -0.01159, \quad a_4 = 0.00629,$$
$$a_5 = -0.03046.$$

The cost is 7.796.

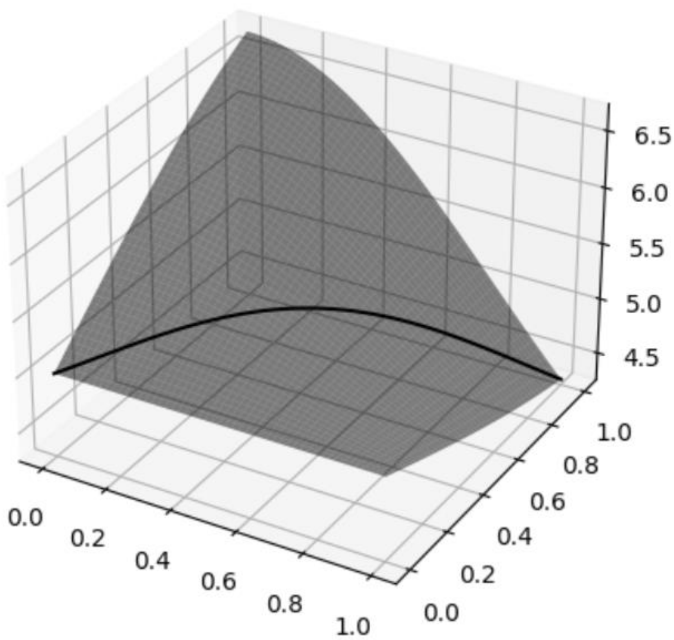

Fig. 3. Illustration of the obtained curve in form (14) in Example 1.

Now let us find the solution of the same problem in parametric form via conditions of Theorem 2.

$$x(t) = t + \sum_{k=1}^{3} a_k \sin(\pi k t),$$

$$y(t) = t + \sum_{k=1}^{3} b_k \sin(\pi k t).$$

Using the Galerkin method

$$a_1 = 0.04190654, \quad a_2 = 0.01219011, \quad a_3 = -0.00278807,$$

$$b_1 = -0.06898219, \quad b_2 = 0.01886075, \quad b_3 = -0.01038629.$$

This solution is demonstrated in Fig. 4 and the cost of solution (i.e., the value of functional J) on the solution produced is 7.797.

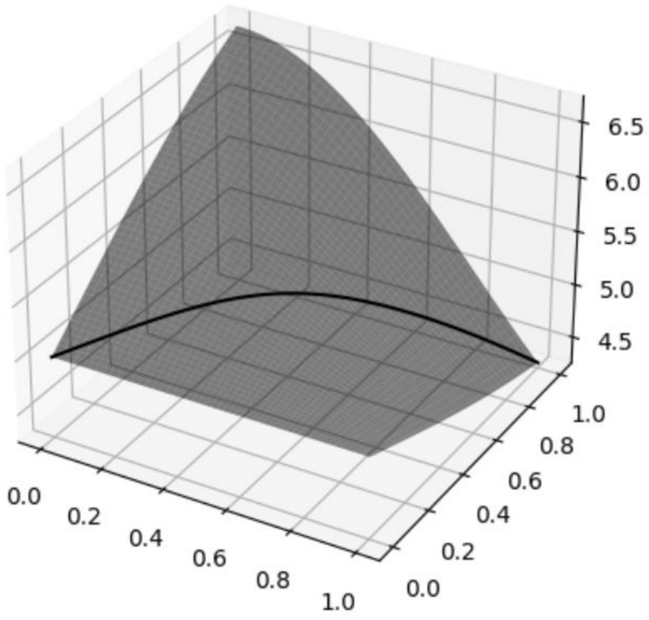

Fig. 4. Illustration of the obtained curve in the parametric form in Example 2.

5 Conclusion

We considered the problem of finding the cost optimal path in the parametric form. The proposed approach is based on calculus of variations. Unlike widely used heuristics, our method guarantees the quality of the obtained result. We obtain the optimal solution via the developed approach and therefore can be sure that we do not waste resources while building roads and other transport networks such as waterways, pipelines and so on.

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