Saint-Petersburg State University

Conditions of local identifiability of discrete infinite-dimensional parameter

Shalgin Vladimir Sergeevich,
postgraduate student

Scientific advisor: Professor,
DSc in Physics and Mathematics
Pilyugin Sergey Yuryevich

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The parameter identification problem

→ A real object or process that needs to be investigated

→ a suitable class of models describing an object or a process

→ the necessity to determine relevant parameter values of a model.

The purpose of parameter identification

is to establish the condition of *principal possibility* of determining unknown parameters based on the results of an experiment (observations).

The property of local (parametric) identifiability

is the possibility of estimating an unknown parameter in a *neighborhood of initially given value of a parameter* (that is usually nominal value of it).

- Often, differential equations (DEs) or dynamical systems (DSs) containing parameters play the role of models.

Motivation and issues

- The problem of local identifiability has been intensively studied in the case of a *finite-dimensional parameter*.
- However, the problem in the case of an *infinite-dimensional parameter* has been studied much less.

Issues

- Establishing *sufficient conditions* for the property of local identifiability for various types of DEs and DSs with infinite-dimensional parameters.
- Establishing the *genericity and prevalence* of the obtained conditions, whether almost all systems and parameters will satisfy these conditions in terms of topology or measure.

1 Bodunov N. A. *An introduction to the theory of local parameter identifiability.* (2006)
— A monograph covering various formulations of the local identification problem, but with a finite-dimensional parameter.

2 Bodunov N. A., Volfson G. I. *Local identifiability of systems with a variable parameter.* (2009)
— The authors obtained a sufficient condition for the local identifiability of the parameter-function for a system of DEs based on observation at one point.

3 Pilyugin S. Y., Shalgin V. S. *Conditions for local parameter identifiability for systems of differential equations with an infinite-dimensional parameter.* (2023)
— The same problem is considered but observation at a finite number of time moments.

4, 5 Bodunov N. A., Kolbina S. A., Pilyugin S. Y. *Locally parameter identifiable systems are generic.* (2012) And *Prevalence of locally parameter identifiable systems.* (2015)
— The authors showed the genericity (topological) and the prevalence (measured) of locally identifiable "input-output" systems.

1. General condition of local identifiability

- We consider a discrete time dynamical system
$$x_{k+1} = f_k(x_k, p_k), \quad k \geq 0, \tag{1}$$
depending on an infinite-dimensional parameter $P = (p_k)_{k=0}^{\infty}$, where $x_k \in \mathbb{R}^n$, $p_k \in \mathbb{R}^m$.
- We assume that the functions f_k are continuously differentiable in a domain D of the space $\mathbb{R}^{n+m}$.
- Parameter $P = (p_k)_{k=0}^{\infty}$ belongs to a family $\mathcal{P}$ with norm $||P|| = \sup_{k\geq 0} |p_k|$.
- We fix x_0 and denote by $x(P) = (x_k(P))$ the trajectory of the system (1) with initial value x_0 in which parameter $P \in \mathcal{P}$ is fixed.
- Observations at all time moments $k \geq 0$.
- Here we work with the following definition of local parameter identifiability.

Definition (1)

System (1) is locally parameter identifiable on the trajectory of the point x_0 at parameter $P^0 \in \mathcal{P}$ if there exists a $\delta > 0$ such that if $P \in \mathcal{P}$ and $0 < ||P - P^0|| < \delta$, then $x_k(P) \neq x_k(P^0)$ for some $k > 0$.

1. General condition of local identifiability

Theorem (1)

Assume that any sequence $P^l \in \mathcal{P}$ such that $||P^l - P^0|| > 0$ and $||P^l - P^0|| \to 0$, $l \to \infty$, contains a subsequence P^{l_q} with the following property: the sequence

$$\frac{P^{l_q} - P^0}{||P^{l_q} - P^0||}$$

converges to a sequence $R = (r_k)$ such that its element r_j with some $j > 0$ does not belong to the kernel of the matrix

$$\frac{\partial f_j}{\partial p}\left(x_j\left(P^0\right), p_j^0\right).$$

Then the system (1) *is locally parameter identifiable on the trajectory of the point x_0 at parameter $P^0 \in \mathcal{P}$ in the sense of Definition 1.*

- This type of condition imposed on parameters is called *normalized separability* in the literature.
- Such a condition of local identifiability is called *conditional local identifiability* in the literature, since the parameter must belong to a certain class of parameters.

1. General condition of local identifiability

- The proof of the Theorem (1) relies on the linear representation of $\Delta x_k = x_k(P) - x_k(P^0)$.
- Denote

$$\Delta P = (\Delta p_k) = (p_k - p_k^0),$$

$$A_k = \frac{\partial f_k}{\partial x}\left(x_k\left(P^0\right), p_k^0\right), \ \ k \geq 1,$$

$$B_k = \frac{\partial f_k}{\partial p}\left(x_k\left(P^0\right), p_k^0\right), \ \ k \geq 0.$$

- Then for all $k \geq 0$

$$\Delta x_{k+1} = A_k \ldots A_1 B_0 \Delta p_0 + \ldots + A_k B_{k-1} \Delta p_{k-1} + B_k \Delta p_k + G_k(\Delta P),$$

where

$$\frac{|G_k(\Delta P)|}{||\Delta P||} \to 0, \quad ||\Delta P|| \to 0.$$

- The very proof is conducted by assuming the contrary: that the system is not locally identifiable at the parameter P^0.

2. Observation at an arbitrary countable set of time moments

- We again consider the dynamical system (1)

$$x_{k+1} = f_k(x_k, p_k), \quad k \geq 0,$$

fix an initial value x_0 and a parameter $P^0 = (p_k^0)$.

- Observations at any countable set of time moments $0 = t_0 < t_1 < t_2 \dots$
- For any integers τ, θ with $0 \leq \tau < \theta$ and parameter P denote

$$|\Delta P|_{\tau,\theta} = \max_{\tau \leq k < \theta} |p_k - p_k^0|.$$

- For τ, θ we have the following representation

$$\Delta x_\theta(P) - \Delta x_\theta(P^0) = A_{\theta-1} \dots A_{\tau+1} B_\tau \Delta p_\tau + \dots + B_{\theta-1} \Delta p_{\theta-1} + G_{\tau,\theta}(\Delta P),$$

where

$$\frac{|G_{\tau,\theta}(\Delta P)|_{\tau,\theta}}{|\Delta P|_{\tau,\theta}} \to 0, \quad |\Delta P|_{\tau,\theta} \to 0.$$

2. Observation at an arbitrary countable set of time moments

Theorem (2)

Let an increasing sequence of integers $0 = t_0 < t_1 < \ldots$ *and a parameter* P *have the following property:*

$$x_k(P) = x_k(P^0), \quad k = t_0, t_1, \ldots .$$

We select the family of parameters $\mathcal{P}$ *by the following two conditions.*

1) There exists a positive number a *such that for any pair* $(\tau, \theta) = (t_k, t_{k+1})$, $k = 0, 1, \ldots,$ *the inequality*

$$|A_{\theta-1} \ldots A_{\tau+1} B_\tau \Delta p_\tau + \ldots + B_{\theta-1} \Delta p_{\theta-1}| > a |\Delta P|_{\tau,\theta}$$

holds if $P \in \mathcal{P}$ *and* $|\Delta P|_{\tau,\theta} > 0$.
2) There exists a $\delta > 0$ *such that*

$$|G_{\tau,\theta}(\Delta P)|_{\tau,\theta} \le a |\Delta P|_{\tau,\theta}$$

for any pair $(\tau, \theta) = (t_k, t_{k+1})$ *and any* $P \in \mathcal{P}$ *with* $|\Delta P|_{\tau,\theta} < \delta$.

Then any $P \in \mathcal{P}$ *with* $||P - P^0|| < \delta$ *coincides with* P^0.

3. Local identifiability in a neighborhood of a hyperbolic set

Definition (2)

Let f be a C^1 diffeomorphism of $\mathbb{R}^n$. A compact set Λ is said to be *hyperbolic* if there exist constants $C \geq 1$ and $\lambda \in (0,1)$, and for any $x \in \Lambda$, there exist two projections $S(x)$ and $U(x)$ of the space $\mathbb{R}^n$ with the following properties:

$$\mathbb{R}^n = S(x)\mathbb{R}^n \oplus U(x)\mathbb{R}^n,$$

$$Df^k(x)T(x) = T(f^k(x))Df^k(x), \quad k \in \mathbb{Z},\ T = S, U;$$

$$|Df^m(x)S(x)v| \leq C\lambda^m |S(x)v|, \quad m \geq 0, v \in \mathbb{R}^n;$$

$$|Df^{-m}(x)U(x)v| \leq C\lambda^m |U(x)v|, \quad m \geq 0, v \in \mathbb{R}^n.$$

$$||S(x)||, ||U(x)|| \leq C.$$

Definition (3)

A diffeomorphism f is said to have *the Lipschitz shadowing property* in $U \subset \mathbb{R}^n$ with constants L, d_0 if for any sequence of points $y_k \in U$, $k \in \mathbb{Z}$, such that $|f(y_k) - y_{k+1}| \leq d \leq d_0$ (d-*pseudotrjectory of* f) there exists a point $x \in U$ (*the shadowing point*) such that $|y_k - f^k(x)| \leq Ld$ for all k.

3. Local identifiability in a neighborhood of a hyperbolic set

- Let f be a C^1 diffeomorphism of $\mathbb{R}^n$ and let Λ be a hyperbolic set of f.
- It is known that there exists a neighborhood U of Λ in which f has the Lipschitz shadowing property with some constants L, d_0.
- We may assume that for any d_0-pseudotrajectory, its shadowing point belongs to Λ. We also may assume that the shadowing point is unique.
- We observe sequence $Y = \{y_k\}$ of points in U which are trajectories of the discrete time dynamical system

$$y_{k+1} = f(y_k) + p_k, \quad k \in \mathbb{Z}, \tag{2}$$

  depending on a sequence $P = (p_k \in \mathbb{R}^n)_{k \in \mathbb{Z}}$ of parameters.
- Our purpose is to find sufficient conditions under which the zero parameter $P = (p_k = 0)$ is locally identifiable in certain sense, i.e. the observed sequence Y is the exact trajectory $X = \{x_k = f^k(x)\}$ of f.

3. Local identifiability in a neighborhood of a hyperbolic set

- It is assumed that parameters satisfy the inequality

$$||P|| = \sup_{k\in\mathbb{Z}} |p_k| \leq d_0.$$

so any trajectory $Y = \{y_k\}$ of system $y_{k+1} = f(y_k)+p_k$ is a $||P||$-pseudotrajectory of f in U and hence there exists a unique trajectory $X = \{x_k = f^k(x)\} \subset \Lambda$ of f such that if

$$y_k = x_k + r_k,$$

then

$$|r_k| \leq L||P||, \quad k \in \mathbb{Z}.$$

- Using equalities $x_{k+1} + r_{k+1} = y_{k+1} = f(y_k) + p_k$ we can represent

$$r_{k+1} = Df(x_k)r_k + p_k + g_k(r_k), \quad k \in \mathbb{Z}. \quad (3)$$

where $g_k(0) = 0$ and Lipschitz constants of g_k are uniformly small for small arguments.

3. Local identifiability in a neighborhood of a hyperbolic set

- If the number $||P||$ is small enough, then equations (3) have a unique bounded solution

$$\begin{aligned} r_k = & \sum_{i=-\infty}^{k} Df^{k-i}(x_i)S(x_i)(p_{i-1} + g_{i-1}(r_{i-1})) \\ & - \sum_{i=k+1}^{\infty} Df^{-(i-k)}(x_i)U(x_i)(p_{i-1} + g_{i-1}(r_{i-1})), \quad k \in \mathbb{Z}. \end{aligned} \tag{4}$$

- Denote $r_k^s = S(x_k)r_k$ and $r_k^u = U(x_k)r_k$ it follows from (4) and from properties of the projections S and U that

$$r_k^s = \sum_{i=-\infty}^{k} Df^{k-i}(x_i)S(x_i)(p_{i-1} + g_{i-1}(r_{i-1})),$$

$$r_k^u = - \sum_{i=k+1}^{\infty} Df^{-(i-k)}(x_i)U(x_i)(p_{i-1} + g_{i-1}(r_{i-1})).$$

3. Local identifiability in a neighborhood of a hyperbolic set

- Fix a number $M > 0$ and denote by $\mathcal{P}(X, M)$ the class of parameters $P = (p_k)_{k\in\mathbb{Z}}$ satisfying the inequality

$$\liminf_{k\to+\infty} \left| \sum_{i=k+1}^{\infty} Df^{-(i-k)}(x_i)U(x_i)p_{i-1} \right| \geq M||P||.$$

- Note that writing $\mathcal{P}(X, M)$ we emphasize the fact that this class depends on the trajectory X.

Theorem (3)

Fix $\nu \in (0, 1)$. For any $M > 0$, the zero sequence $P = (p_k = 0)$ is ***conditionally locally identifiable*** *in the class $\mathcal{P}(X, M)$ in the following sense: there exists $D = D(M) > 0$ such that if $P \in \mathcal{P}(X, M)$, $||P|| \leq D$, and the following relation holds:*

$$\sup \{k : |r_k^u| \leq \nu M||P||\} = \infty,$$

then $P = (p_k = 0)$.

Conclusion

Three different formulations of the problem of local identifiability of the parameter-sequence have been considered and sufficient conditions for its local identifiability have been found:

- One-sided dynamical system $x_{k+1} = f_k(x_k, p_k)$ when observations at all time moments.
- The same dynamical system when observations at arbitrary countable set of time moments.
- The local identifiability of the zero parameter-sequence in the neighborhood of a hyperbolic set of a linearly perturbed diffeomorphism: $y_{k+1} = f(y_k) + p_k$.

Thank you!

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