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General control problems

Convergence of trajectories and stability of fixed points in a modified Hegselmann – Krause model

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Abstract. In this paper, we study a modified Hegselmann – Krause model of opinion dynamics based on the bounded confidence principle. This model is formulated as a discontinuous and nonlinear dynamical system. At any time moment of the process of opinion formation, the operator of forming the next opinion of an agent is two-step; first, one takes the average of opinions of agents sharing similar opinions plus his/her own; in the second step, a regularization procedure is performed. A new regularization procedure is applied. We find conditions under which every trajectory tends to a fixed point of the system and study stability properties of fixed points.

Keywords: Opinion dynamics, bounded confidence principle, Hegselmann – Krause model, convergence of trajectories, stability of fixed points

1. Introduction. At present, opinion studies are a well-developed field of research (see, for example, the monographs [18, 23] and the recent surveys [2, 26]). The main goal of opinion dynamics is to analyze evolution of public opinion in social systems.

Various (mostly linear) models of opinion dynamics have been studied since 1950s ([10, 11]). The linearity of a model allows one to apply more or less standard methods of linear dynamical systems.

One of the first nonlinear models was suggested in [13, 16], where the notion of "bounded confidence" had been introduced. This notion formalizes the fact that, in the course of formation of public opinion, a member of the society is mostly influenced by individuals sharing a similar opinion.

The first opinion model based on the notion of bounded confidence, introduced by Hegselmann and Krause, was later called the Hegselmann – Krause (HK) model; this model and its generalizations have been intensively studied by various authors, see, for example, [9, 20, 17, 5, 19, 8, 7, 14, 24, 25, 12]. Mostly, the results were based on computer simulations, and it was noted that "rigorous analytical results are difficult to obtain [13].

To introduce the HK model, let us consider the dynamics of opinions in a society of voters who have to choose between two options, -1 and 1. Assume that the society is formed by N individuals (usually called "agents"), and let $v_k^n \in [-1, 1]$, $k \in \{1, \dots, N\}$, be the opinion of individual k at time moment n.

Fix a positive $\varepsilon < 1$ (the level of bounded confidence in the society) and consider for $k \in \{1, \dots, N\}$ the set of indices

$$J(v_k^n) = \{l \in \{1, \dots, N\} \ : |v_l^n - v_k^n| \le \varepsilon\}.$$

This is the set of indices of agents whose opinions influence agent k at time moment n. Denote by $I(v_k^n)$ the cardinality of the set $J(v_k^n)$ (note that $I(v_k^n) > 0$).

In the original HK model (see [15]), the dynamics is governed (in our notation) by the equalities

$$v_k^{n+1} = \frac{1}{I(v_k^n)} \sum_{l \in J(v_k^n)} v_l^n, \quad k \in \{1, \dots, N\}, n \ge 0. \tag{1}$$

In the paper [21], the following modification of the HK model had been studied. Fix a number $h > 0$ and consider the dynamics governed by the equalities

$$v_k^{n+1} = v_k^n + \frac{h}{I(v_k^n)} \sum_{l \in J(v_k^n)} v_l^n, \quad k \in \{1, \dots, N\}, n \ge 0. \tag{2}$$

In the paper [6], it is explained that system (2) appears as an application of the one-step Euler method to the differential equation

$$\frac{d}{dt}v_k^t = \frac{1}{I(v_k^t)} \sum_{l \in J(v_k^t)} v_l^t, \quad k \in \{1, ..., N\}, t \geq 0$$

which is naturally a variant of the "bounded confidence" model with continuous time.

System (2) is the main object of study in the present paper.

Let us explain the main differences between systems (2) and (1).

First, if $J(v_k^n) = \{k\}$ (i.e., agent k does not have agents with ε-close opinions at time n), then $v_k^{n+1} = v_k^n$ in model (1), i.e., his/her opinion does not change. In contrast, $v_k^{n+1} \neq v_k^n$ in model (2). A sociological meaning of this phenomenon is explained in [21].

The second very important difference is that the dynamics of models (2) and (1) are essentially different.

It is easily seen that since $|v_k^n| \leq 1$ in model (1), similar inequalities hold for v_k^{n+1}, i.e., formulas (1) define a dynamical system on $[-1, 1]^N$.

At the same time, this is not so for model (2); hence, one has to apply some regularization procedure to get a properly defined dynamical system on $[-1, 1]^N$.

Two such regularization procedures have been applied in the papers [21] and [1].

For an array $V^n = (v_1^n, \dots, v_N^n)$, introduce an auxiliary array $W(V^n) = (w_1^n, \dots, w_N^n)$ by setting

$$w_k^n = v_k^n + \frac{h}{I(v_k^n)} \sum_{l \in J(v_k^n)} v_l^n. \tag{3}$$

Clearly, $|w_k^n| \leq 1 + h$.

In the paper [21], a "cutting" procedure had been applied.

Set

$$u_k^n = v_k^n \text{ if } |w_k^n| \leq 1,$$
$$u_k^n = 1 \text{ if } w_k^n > 1,$$
$$u_k^n = -1 \text{ if } w_k^n < -1,$$

and

$$\Phi(V^n) = (u_1^n, \dots, u_N^n).$$

Then $\Phi([0,1]^N) \subset [0,1]^N$, and we get a properly defined dynamical system on $[-1,1]^N$.

In the paper [1], a different procedure had been applied. Take a nonzero array V^n, define an auxiliary array $W(V^n)$ by formulas (3) (clearly, $W(V^n)$ is also nonzero), and set

$$u_k^n = \frac{w_k^n}{\max_{l=1,\dots,N} |w_l^n|}$$

and

$$\Phi(V^n) = (u_1^n, \dots, u_N^n).$$

Clearly, the system Φ is also a properly defined dynamical system on $[-1,1]^N$.

In this paper, we consider one more natural regularization procedure.

For a nonzero array $V^n = (v_1^n, \dots, v_N^n)$ and the corresponding array $W(V^n)$ given by (3), define

$$\mu(W(V^n)) = \max(1, \max_{l=1,\dots,N} |w_l^n|)$$

and set

$$\Phi(V^n) = (u_1^n, \dots, u_N^n), \tag{4}$$

where

$$u_k^n = \frac{w_k^n}{\mu(W(V^n))}.$$

As in the previous cases, formula (4) defines a dynamical system Φ on $[-1,1]^N$ which we study in this paper.

One of the main properties of systems modeling opinion dynamics is the convergence of their trajectories as time tends to infinity. For the above dynamical systems, known sufficient conditions of such a convergence are essentially different. In the paper [21], this condition has the form $\varepsilon \le 1/2$, while in the paper [1] it has the form $\varepsilon(N-1) < 1$.

In this paper, we show that if

$$\varepsilon < 1, \tag{5}$$

then all the trajectories of system Φ defined by formula (4) converge as time tends to infinity.

In addition, we show that any trajectory of Φ tends to a fixed point and study stability properties of fixed points.

Everywhere in the paper, it is assumed that condition (5) is satisfied.

The structure of the paper is as follows. In Section 2, we describe basic properties of the system and define two classes of initial points: standard and singular. Section 3 is devoted to trajectories of singular points. In Section 4, we show that trajectories of standard points converge; in Section 5, it is shown that all trajectories tend to fixed points of Φ. In Sections 6, 7, we study stability properties of fixed points. It is shown that singular fixed points are Lyapunov unstable, the system Φ has two asymptotically stable fixed points $P_+ = (1, \dots, 1)$ and $P_- = (-1, \dots, -1)$, and, finally, that any standard fixed point different from P_+ and P_- is Lyapunov stable but not asymptotically stable.

2. Basic properties of the system. Denote by Ψ the set of points $V = (v_1, \dots, v_N)$ such that $v_1 \le v_2 \le \dots \le v_N$.

Proposition 1 *If $V \in \Psi$, then $\Phi(V) \in \Psi$.*

The proof of this Proposition repeats the proof of Proposition 3.1 of [PC].

Since the set of elements of $\Phi(V)$ does not depend on the order of the values $v_1, \dots, v_N$, in what follows, we only consider trajectories of points $V \in \Psi$.

Proposition 2 *If $|v_k^m - v_{k+1}^m| > \varepsilon$, then $|v_k^{m+1} - v_{k+1}^{m+1}| \ge |v_k^m - v_{k+1}^m|$.*

Proof. Since $|v_k^m - v_{k+1}^m| > \varepsilon$, $J(v_k^m) \subset \{1, \dots, k\}$ and $J(v_{k+1}^m) \subset \{k+1, \dots, N\}$. Hence, the following inequalities are valid:

$$v_k^m \ge \frac{\sum\limits_{i \in J(v_k^m)} v_i^m}{I(v_k^m)}$$

and

$$v_{k+1}^m \le \frac{\sum\limits_{j \in J(v_{k+1}^m)} v_j^m}{I(v_{k+1}^m)}.$$

We apply these inequalities to estimate the difference $v_{k+1}^{m+1} - v_k^{m+1}$:

$$v_{k+1}^{m+1} - v_k^{m+1} = \frac{1}{\mu(W(V^m))} \left(v_{k+1}^m + h \cdot \frac{\sum\limits_{j \in J(v_{k+1}^m)} v_j^m}{I(v_{k+1}^m)} - v_k^m - h \cdot \frac{\sum\limits_{i \in J(v_k^m)} v_i^m}{I(v_k^m)} \right) \ge$$

$$\geq \frac{1}{\mu(W(V^m))}(v_{k+1}^m(1+h) - v_k^m(1+h)) = \frac{1+h}{\mu(W(V^m))}(v_{k+1}^m - v_k^m) \geq v_{k+1}^m - v_k^m,$$

as required. □

Corollary 1 *If $v_{k+1}^m - v_k^m > \varepsilon$, then*

$$v_{k+1}^m - v_k^m \leq v_{k+1}^{m+1} - v_k^{m+1} \leq v_{k+1}^{m+2} - v_k^{m+2} \leq \dots .$$

Definition 1 *We call a point (array V^0)* ***standard*** *if the inequalities $\mu(W(V^k)) > 1$ hold for all k large enough. The points that do not satisfy this condition are called* ***singular****.*

In what follows, we prove that all singular points are fixed points of Φ, and they are Lyapunov unstable. First we prove the following simple Proposition.

Proposition 3 *If $\mu(W(V^{k_0})) > 1$, then $\mu(W(V^k)) > 1$ for all $k > k_0$.*

Proof. We prove the Proposition using induction on k.

Base. The statement is obviously valid for $k = k_0$.

Induction step. Let us show that if $\mu(W(V^k)) > 1$, then $\mu(W(V^{k+1})) > 1$ as well. By definition, the first inequality means that $\max_{t=1,\dots,N} |w_t(V^k)| > 1$. As was noted above,

$$w_1(V^k) \leq w_2(V^k) \leq \dots \leq w_N(V^k).$$

Hence, either $|w_1(V^k)| = \mu(W(V^k))$ or $|w_N(V^k)| = \mu(W(V^k))$. This implies that $|v_j^k| = 1$ either for $j = 1$ or for $j = N$.

Consider for definiteness the second case and assume that $v_N^k = 1$ (the remaining possible cases are considered similarly).

Note that

$$w_N(V^{k+1}) = v_N^{k+1} + \frac{h}{I(v_N^{k+1})} \sum_{t \in J(v_N^{k+1})} v_t^{k+1}.$$

We claim that the sum $S := \sum_{t \in J(v_N^{k+1})} v_t^{k+1}$ is nonzero and has the same sign as v_N^k. The statement of our Proposition is a corollary of this claim since in this case,

$$\mu(W(V^{k+1})) \geq w_N(V^{k+1}) = 1 + \frac{S}{I(v_N^{k+1})} > 1.$$

To prove the claim, note that if $t \in J(v_N^{k+1})$, then v_t^{k+1} has the same sign as v_N^{k+1} since $v_N^{k+1} - v_t^{k+1} \leq \varepsilon < 1$ and $v_N^{k+1} = 1$. In addition, the sum S is nonzero since it contains the nonzero term v_j^{k+1}. The claim is proved. □

Remark. In the proof of Proposition 3, the inequality $\varepsilon < 1$ has been explicitly applied.

Corollary 2 *If V^0 is a singular point, then $\mu(W(V^n)) = 1$ for all n.*

Corollary 3 *If V^0 is a standard point, then either $v_1^n = -1$ or $v_N^n = 1$ for all n large enough.*

3. Trajectories of singular points. In this section, we classify all singular points and describe their trajectories.

Proposition 4 *If $v_1 \leq -\varepsilon$ or $v_N \geq \varepsilon$, then the point $V = (v_1, \dots, v_N)$ is standard.*

Proof. Assume that, under our assumption, the equality $\mu(W(V^n)) = 1$ holds for all n. Consider the case where $v_N \geq \varepsilon$. In this case, $v_l \geq 0$ for any $l \in J(v_N)$, and it follows that

$$v_N^1 = v_N + \frac{h}{I(v_N)} \sum_{l \in J(v_N)} v_l \geq v_N + \frac{h}{N} v_N = v_N(1 + \frac{h}{N}).$$

Now we apply induction to show that $v_N^{k+1} \geq v_N(1 + \frac{h}{N})^k$ for $k > 1$. Indeed, if this inequality holds for some k, then $v_N^{k+1} \geq v_N \geq \varepsilon$ and $v_l^{k+1} \geq 0$ for any $l \in J(v_N^{k+1})$. Since $\mu(W(V^{k+1})) = 1$, we can estimate v_N^{k+2} as follows:

$$v_N^{k+2} = v_N^{k+1} + \frac{h}{I(v_N^{k+1})} \sum_{l \in J(v_N^{k+1})} v_l^{k+1} \geq v_N^{k+1} + \frac{h}{I(v_N^{k+1})} v_N^{k+1} \geq$$

$$\geq \left(1 + \frac{h}{N}\right) v_N^{k+1} \geq \left(1 + \frac{h}{N}\right)^{k+1} v_N.$$

Hence, $v_N^{k+1} \geq v_N(1 + \frac{h}{N})^k$ for all k, which contradicts the inequalities $v_N^{k+1} \leq 1$. The case $v_1 \leq -\varepsilon$ is considered similarly.

Hence, $\mu(W(V^n)) > 1$ for some n. Consequently, point V is standard by Corollary 2. □

Proposition 5 *If $v_N - v_1 \le \varepsilon$ and, in addition, the sum $v_1 + \dots + v_N$ is nonzero, then the point $V = (v_1, \dots, v_N)$ is standard.*

Proof. To get a contradiction, assume that $\mu(W(V^n)) = 1$ for all n.

Denote by s the sum $\frac{1}{N}\sum_{l=1}^{N} v_l$. Assume that $s > 0$ (the case $s < 0$ is symmetric). We prove by induction that $v_N^n - v_1^n = v_N - v_1$ and $v_d^n \ge v_d + nhs$, $d = 1, \dots, N$. The base $n = 0$ is obvious.

Assuming that for $n = k$ our claim is valid, let us prove it for $n = k + 1$.

Due to our assumption,

$$v_d^{k+1} = v_d^k + \frac{h}{I(v_d^k)} \sum_{l \in J(v_d^k)} v_l^k = v_d^k + \frac{h}{N} \sum_{l=1}^{N} v_l^k.$$

Substitute $d = 1$ and $d = N$ in the above formula to conclude that $v_N^{k+1} - v_1^{k+1} = v_N^k - v_1^k = v_N - v_1$.

In addition, our assumption implies that

$$v_d^{k+1} = v_d^k + \frac{h}{N} \sum_{l=1}^{N} v_l^k \ge v_d + khs + \frac{h}{N} \sum_{l=1}^{N} (v_l + khs) \ge$$

$$\ge v_d + khs + \frac{h}{N} \sum_{l=1}^{N} v_l = v_d + (k+1)hs, \quad d = 1, \dots, N.$$

Hence, our claim is valid for all n. But $v_d^N \le 1$, while $v_d + nhs > 1$ for large n which is a contradiction. It follows that $\mu(W(V^n)) > 1$ for some n which means (see Corollary 2) that the point V is standard. □

Remark. It is easily seen that if $v_N - v_1 \le \varepsilon$ and $v_1 + \dots + v_N = 0$, then $(v_1, \dots, v_N)$ is a singular fixed point.

Proposition 6 *If $\varepsilon < v_N - v_1 < 2\varepsilon$, then $V = (v_1, \dots, v_N)$ is a standard point.*

Proof. To get a contradiction, assume that $\mu(W(V^n)) = 1$ for all n.

Denote by x the value $v_N - v_1 - \varepsilon$. Clearly, $x > 0$.

Apply induction to prove that

$$v_N^n - v_1^n \ge v_N - v_1 + \frac{hnx}{N}, \quad n \ge 0.$$

The base $n = 0$ is obvious. Assume that the induction assumptions is valid for some n; we prove it for $n + 1$.

The equalities $\mu(W(V^n)) = 1$ imply that

$$v_1^{n+1} = v_1^n + h\frac{v_1^n + \cdots + v_m^n}{m}$$

and

$$v_N^{n+1} = v_N^n + h\frac{v_N^n + v_{N-1}^n + \cdots + v_{N-k+1}^n}{k},$$

where $k = I(v_N^n)$ and $m = I(v_1^n)$. It follows from the inequalities $v_N^n - v_1^n > \varepsilon + x$ and $v_m^n - v_1^n \leq \varepsilon$ that $v_N^n - v_m^n > x$.

In the case where $N - k + 1 \geq m$,

$$v_N^{n+1} \geq v_N^n + h\frac{v_N^n + (k-1)v_m^n}{k} \geq v_N^n + h\frac{x + kv_m^n}{k} \geq v_N^n + \frac{hx}{N} + hv_m^n$$

and

$$v_1^{n+1} \leq v_1^n + h\frac{mv_m^n}{m} \leq v_1^n + hv_m^n.$$

The induction assumption implies that

$$v_N^{n+1} - v_1^{n+1} \geq (v_N^n + \frac{hx}{N} + hv_m^n) - (v_1^n + hv_m^n) \geq v_N - v_1 + \frac{hnx}{N} + \frac{hx}{N} =$$

$$= v_N - v_1 + \frac{h(n+1)x}{N}.$$

In the case where $N - k + 1 < m$, denote by y the arithmetic mean of the values $v_{N-k+1}^n, \ldots, v_m^n$. In other words,

$$v_{N-k+1}^n + \cdots + v_m^n = y(m + k - N).$$

Then

$$v_1^n \leq \cdots \leq v_{N-k+1}^n \leq y \leq v_m^n \leq \cdots \leq v_N.$$

In addition, it follows from the above reasoning that $v_N^n - y \geq v_N^n - v_m^n \geq x$.

We apply these two inequalities to estimate v_N^{n+1} and v_1^{n+1} in terms of y:

$$v_N^{n+1} = v_N^n + h\frac{v_N^n + \cdots + v_m^n + y(m + k - N)}{k} \geq$$

$$\geq v_N^n + h\frac{v_N^n + y(k-1)}{k} \geq v_N^n + h\frac{x + ky}{k} \geq v_N^n + hy + \frac{hx}{N}$$

and

$$v_1^{n+1} = v_1^n + h\frac{v_1^n + \cdots + v_m^n}{m} = v_1^n + h\frac{v_1^n + \cdots + v_{N-k}^n + y(m+k-N)}{m} \leq$$

$$\leq v_1^n + h\frac{ym}{m} = v_1^n + hy.$$

Now the induction hypothesis implies that

$$v_N^{n+1} - v_1^{n+1} \geq (v_N^n + hy + \frac{hx}{N}) - (v_1^n + hy) = v_N^n - v_1^n + \frac{hx}{N} \geq v_N - v_1 + \frac{h(n+1)x}{N},$$

which completes the induction step.

It follows that

$$v_N^n - v_1^n \geq v_N - v_1 + \frac{hnx}{N} > 2\varepsilon$$

for n large enough.

But if the inequality $v_N^t - v_1^t \geq 2\varepsilon$ holds for some t, then v_1^t and v_N^t cannot simultaneously belong to the interval $(-\varepsilon, \varepsilon)$; hence, V^t satisfies the assumptions of Proposition 4, which means that V is a standard point. □

Thus, we can characterize all the singular points.

Corollary 4 *If $V = (v_1, \ldots, v_N)$ is a singular point, then V is a fixed point of Φ, $v_N - v_1 \leq \varepsilon$, and $v_1 + \cdots + v_N$ =0.*

4. Converegence of trajectories of standard points. Clearly, we have the following representaion for points of trajectories of our dynamical system:

$$V^{r+1} = \frac{E + hA_r}{\mu(W(V^r))} \cdots \frac{E + hA_0}{\mu(W(V^0))} V^0,$$

where $\{A_r\}$ is a sequence of stochastic $N \times N$ matrices such that the l-th row of a matrix A_r contains precisely $I = I(v_l^{r-1})$ nonzero entries and has the form

$$\left(0, 0, .., 0, \frac{1}{I}, \ldots, \frac{1}{I}, 0, 0, \ldots, 0\right)$$

in which the nonzero entries a_{il} have indices $i \in J(v_l^{r-1})$.

Denote

$$G_k := \frac{E + hA_k}{1 + h};$$

it is easily seen that G_k is a stochastic matrix.

We can represent the vector V^{r+1} in terms of these matrices as follows:

$$V^{r+1} = \frac{(1+h)^r}{\mu(W(V^0)) \cdots \mu(W(V^r))} G_r G_{r-1} \cdots G_0 V^0. \tag{6}$$

To prove the convergence of such products for standard initial points V^0, we refer to the following Theorems 1 and 2.

Theorem 1 *(Lorenz, [3]). Let $\{G_k\}$ be a sequence of stochastic matrices satisfying the following conditions:*

(a) the diagonal entries of all he matrices G_k are positive;

(b) $(G_k)_{ij} = 0$ if and only if $(G_k)_{ji} = 0$;

(c) there exists a $\delta > 0$ such that any nonzero entry of any matrix G_k is larger than δ.

Then the infinite product $G_k G_{k-1} \cdots G_0$ converges as $k \to \infty$.

Let us show that the matrices G_k satisfy the assumptions of Theorem 1.

• The diagonal entries of the matrices E and A_k are positive; hence, assumption (a) is satisfied;

• the matrices E and A_k satisfy assumption (b), hence, G_k satisfies this assumption as a linear combination of E and A_k.

• any nonzero entry of a matrix A_k is not less than $\frac{1}{N}$; hence, any nonzero entry of the matrix G_k is not less than $\frac{h}{(1+h)N}$. Hence, G_k satisfies condition (c).

Thus, it follows from Theorem 1 that the product $G_r G_{r-1} \cdots G_0$ converges. Next, we show that this product does not converge to the zero matrix. First, we need the following two definitions.

Definition 2 *A sequence $\{G_t\}$ of $n \times n$ stochastic matrices is called* ***balanced*** *if there exists a number $\alpha \in (0,1)$ such that for any matrix G in this sequence and any subset $S \subset \{1, 2, \ldots, n\}$, the following inequality holds:*

$$\sum_{i \in S, j \in \overline{S}} G_{ij} \geq \alpha \cdot \sum_{i \in S, j \in \overline{S}} G_{ji},$$

where $\overline{S}$ is the complement to S in $\{1, \ldots, n\}$.

Definition 3 *We say that a sequence of matrices has* ***feedback property*** *if there exists a number* $\gamma > 0$ *such that all diagonal entries of all matrices in the sequence are not less than* γ.

The feedback property is introduced in ([4], Section 4.2) for chains of random matrices. Namely, strong feedback property, feedback property, and weak feedback property are considered (see [4], page 37). For chains of deterministic matrices, the strong feedback property coincides with the feedback property, while they differ from the weak feedback property (see [4], Example 5.2, pages 38-39).

Definition 4 *(see [4], Definition 4.5.) We say that a chain of random matrices* $\{W(k)\}$ *belongs to the class* $\mathcal{P}^*$ *if there exists a vector-valued random process* $\pi(k+1) = W(k)\pi(k)$ *such that* $\pi(k) > p^*$ *almost surely for all* $k \geq 0$ *and some number* $p^* > 0$.

Remark. As in [4], we write $\pi(k) > p^*$ if any component of the vector $\pi(k)$ is larger than p^* (see [4], page 8).

Theorem 2 *(Touri, Nedić, [4]). Let* $\{G_k\}$ *be a balanced sequence of stochastic matrices having the feedback property. Then* $\{G_k\}$ *belongs to the class* $\mathcal{P}^*$.

Let us check that that the sequence of matrices $\{G_k\}$ satisfies the conditions of the above theorem.

First we show that it is balanced. It was mentioned above that if c_{ij} are the entries of a matrix G_k, then $c_{ij} \neq 0$ if and only if $c_{ji} \neq 0$. In addition, if c_{ij} is a nonzero entry, then $1 \geq c_{ij} \geq \frac{h}{N(1+h)}$. It follows that for any $i, j \in \{1, \ldots, N\}$, the inequality $c_{ij} \geq \frac{h}{N(1+h)} c_{ji}$ holds.

Thus, if we fix an arbitrary set of indices $S \subset \{1, 2, \ldots, N\}$, then the following inequality is valid:

$$\sum_{i \in S,\ j \in \overline{S}} c_{ij} \geq \frac{h}{N(1+h)} \sum_{i \in S,\ j \in \overline{S}} c_{ji}.$$

The feedback property is obvious since the diagonal entries of any of the matrices G_k are not less than $\frac{1}{1+h}$.

Thus, the sequence $\{G_k\}$ belongs to the class $\mathcal{P}^*$, i.e., there exists a vector Z^0 such that the sequence $G_r G_{r-1} \cdots G_0 Z^0$ does not converge to zero. Hence, the product $G_r G_{r-1} \cdots G_0$ does not converge to the zero matrix as $r \to \infty$.

Since $\mu(W(V^r)) \le 1 + h$, the sequence

$$k_r := \frac{(1+h)^r}{\mu(W(V^0)) \cdots \mu(W(V^r))}$$

is nondecreasing.

The left-hand sides of equalities (6) are bounded; the convergence of the products $G_r G_{r-1} \cdots G_0$ implies the convergence of the sequence k_r.

Note that, in addition, if V^0 is a standard point, then

$$\mu(W(V^r)) \to 1 + h, \quad r \to \infty. \tag{7}$$

Thus, we have shown that the trajectory of any standard point converges; below we show that it converges to a fixed point of Φ.

5. Convergence of trajectories to fixed points. As was shown above, any singular point V is a fixed point of Φ; hence, its trajectory obviously converges to V.

Let us show that the trajectory of any standard point also converges to a fixed point.

Definition 5 *For a sequence v_n^k, consider the value*

$$S(v_k^n) = \frac{1}{I(v_k^n)} \cdot \sum_{j \in J(v_k^n)} v_j^n.$$

Proposition 7 *If V is a standard point, then $\lim\limits_{n\to\infty} S(v_k^n) = \lim\limits_{n\to\infty} v_k^n$.*

Proof. Fix an arbitrary $n \in \mathbb{N}$ and represent the difference $v_k^{n+1} - v_k^n$ as follows:

$$v_k^{n+1} - v_k^n = \frac{v_k^n + hS(v_k^n)}{\mu(W(V^n))} - v_k^n =$$

$$= \frac{v_k^n(1+h)}{\mu(W(V^n))} + \frac{-hv_k^n + h \cdot S(v_k^n)}{\mu(W(V^n))} - v_k^n =$$

$$= v_k^n \left(\frac{1+h}{\mu(W(V^n))} - 1 \right) + \frac{h}{\mu(W(V^n))} (S(v_k^n) - v_k^n).$$

As was shown, the trajectory of any standard point converges; hence, the limit of the left-hand side as $n \to \infty$ in the first line above is 0:

$$0 = \lim_{n\to\infty} \left[v_k^n \left(\frac{1+h}{\mu(W(V^n))} - 1 \right) + \frac{h}{\mu(W(V^n))} (S(v_k^n) - v_k^n) \right].$$

In addition, for a standard point, equality (7) holds; hence,

$$\lim_{n\to\infty} v_k^n \left(\frac{1+h}{\mu(W(V^n))} - 1 \right) = 0,$$

and it follows that

$$\lim_{n\to\infty} \frac{h}{\mu(W(V^n))} (S(v_k^n) - v_k^n) = 0$$

and

$$\lim_{n\to\infty} S(v_k^n) = \lim_{n\to\infty} v_k^n.$$

□

Proposition 8 *If V is a standard point, then there exists a $\tau \in \mathbb{N}$ such that $J(V_k^\tau) = J(V_k^t)$ for any $t > \tau$ and $k \in \{1, \ldots, N\}$.*

Proof. Denote $\lim_{n\to\infty} V^n = V^* = (v_1^*, ..., v_N^*)$.

Find the maximal index k in $\{1, \ldots, N\}$ for which there exists an $l \in \{1, \ldots, N\}$ such that the set $\{t \mid l \notin J(v_k^t),\ l \in J(v_k^{t+1})\}$ is infinite. If such a pair of indices (k, l) does not exist, then the statement of our Proposition is valid.

The condition $l \notin J(v_k^t),\ l \in J(v_k^{t+1})$ is equivalent to the condition

$$|v_k^t - v_l^t| > \varepsilon \geq |v_k^{t+1} - v_l^{t+1}|.$$

Note that $l \notin J(V_k^t),\ l \in J(v_k^{t+1})$ if and only if $k \notin J(V_l^t),\ k \in J(v_l^{t+1})$. Thus, the set $\{t \mid k \notin J(v_l^t),\ k \in J(v_l^{t+1})\}$ is infinite as well. Since we selected the maximal such k, $k > l$.

Hence, if $m \in \{k, k+1, \ldots, N\}$, then the set $\{t \mid m \notin J(v_k^t),\ m \in J(v_k^{t+1})\}$ is finite. Thus, if a number τ is large enough, then

$$J(v_k^\tau) \cap \{k, k+1, \ldots, N\} = J(v_k^{\tau+1}) \cap \{k, k+1, \ldots, N\} =$$

$$= J(v_k^{\tau+2}) \cap \{k, k+1, \ldots, N\} = \ldots .$$

Select an arbitrary $t > \tau$ in the set $\{t \mid l \notin J(v_k^t),\ l \in J(v_k^{t+1})\}$. The sets $J(v_k^t)$ and $J(v_k^{t+1})$ consist of consecutive indices, $J(v_k^t) \cap \{k, k+1, \ldots, N\} = J(v_k^{t+1}) \cap \{k, k+1, \ldots, N\}$, and $l \notin J(v_k^t),\ l \in J(v_k^{t+1})$ for some $l < k$. Hence, $J(v_k^t) = \{l_2, \ldots, r\}$ and $J(v_k^{t+1}) = \{l_1, \ldots, r\}$ for some $l_1, l_2, r \in \{1, \ldots, N\}$ such that $l_1 \le l < l_2 \le k \le r$.

According to our notation,

$$S(v_k^t) = \frac{v_{l_2}^t + \cdots + v_r^t}{r - l_2 + 1}$$

and

$$S(v_k^{t+1}) = \frac{v_{l_1}^{t+1} + \cdots + v_r^{t+1}}{r - l_1 + 1}.$$

Due to the corresponding convergencies, we may assume that τ is so large that $|v_m^t - v_m^*| < \varepsilon/(12N)$ and $|S(v_m^t) - v_m^*| < \varepsilon/(12N)$ for all $t \ge \tau$ and $m = 1, \ldots, N$. In particular,

$$|v_m^t - v_m^{t+1}| \le |v_m^t - v_m^*| + |v_m^* - v_m^{t+1}| < \frac{\varepsilon}{6N}.$$

This inequality implies that

$$S(v_k^t) = \frac{v_{l_2}^t + \ldots + v_r^t}{r - l_2 + 1} \ge \frac{(v_{l_2}^{t+1} - \frac{\varepsilon}{6N}) + \ldots + (v_r^{t+1} - \frac{\varepsilon}{6N})}{r - l_2 + 1} =$$

$$= \frac{v_{l_2}^{t+1} + \ldots + v_r^{t+1}}{r - l_2 + 1} - \frac{\varepsilon}{6N}.$$

Hence,

$$S(v_k^t) + \frac{\varepsilon}{6N} \ge \frac{v_{l_2}^{t+1} + \cdots + v_r^{t+1}}{r - l_2 + 1}.$$

Apply this inequality to estimate $S(v_k^{t+1})$ from above:

$$S(v_k^{t+1}) = \frac{v_{l_1}^{t+1} + \cdots + v_r^{t+1}}{r - l_1 + 1} = \frac{v_{l_1}^{t+1} + \cdots v_{l_2-1}^{t+1} + (v_{l_2}^{t+1} + \cdots + v_r^{t+1})}{r - l_2 + 1} =$$

$$= \frac{v_{l_1}^{t+1} + \cdots + v_{l_2-1}^{t+1} + (r - l_2 + 1)\frac{v_{l_2}^{t+1} + \cdots + v_r^{t+1}}{r - l_2 + 1}}{r - l_1 + 1} =$$

$$= \frac{v_{l_1}^{t+1} + \cdots + v_{l_2-1}^{t+1}}{r - l_1 + 1} + \frac{r - l_2 + 1}{r - l_1 + 1} \cdot \frac{v_{l_2}^{t+1} + \cdots + v_r^{t+1}}{r - l_2 + 1} \le$$

$$\leq \frac{v_{l_1}^{t+1} + \cdots + v_{l_2-1}^{t+1}}{r - l_1 + 1} + \frac{r - l_2 + 1}{r - l_1 + 1} \cdot \left(S(v_k^t) + \frac{\varepsilon}{6N} \right).$$

Let us continue the chain of inequalities taking into account that $\varepsilon/(6N) \geq |v_m^{t+1} - v_m^t|$:

$$\frac{v_{l_1}^{t+1} + \cdots + v_{l_2-1}^{t+1}}{r - l_1 + 1} + \frac{r - l_2 + 1}{r - l_1 + 1} \cdot \left(S(v_k^t) + \frac{\varepsilon}{6N} \right)$$

$$\leq \frac{(v_{l_1}^t + \frac{\varepsilon}{6N}) + \cdots + (v_{l_2-1}^t + \frac{\varepsilon}{6N})}{r - l_1 + 1} + \frac{r - l_2 + 1}{r - l_1 + 1} \cdot \left(S(v_k^t) + \frac{\varepsilon}{6N} \right) =$$

$$= \frac{v_{l_1}^t + \cdots + v_{l_2-1}^t}{r - l_1 + 1} + \frac{l_2 - l_1}{r - l_1 + 1} \cdot \frac{\varepsilon}{6N} + \frac{r - l_2 + 1}{r - l_1 + 1} \cdot \left(S(v_k^t) + \frac{\varepsilon}{6N} \right).$$

Since $v_{l_1}^t \leq \cdots \leq v_{l_2-1}^t \leq v_k^t$ and $l_2 - 1 \notin J(v_k^t)$ (i.e., $|v_{l_2-1}^t - v_k^t| > \varepsilon$), we conclude that $v_{l_1}^t \leq \cdots \leq v_{l_2-1}^t < v_k - \varepsilon$.

Hence,

$$S(v_k^{t+1}) \leq \frac{v_{l_1}^t + ... + v_{l_2-1}^t}{r - l_1 + 1} + \frac{l_2 - l_1}{r - l_1 + 1} \cdot \frac{\varepsilon}{6N} + \frac{r - l_2 + 1}{r - l_1 + 1} \cdot \left(S(v_k^t) + \frac{\varepsilon}{6N} \right) \leq$$

$$\leq \frac{(l_2 - l_1)(v_k^t - \varepsilon)}{r - l_1 + 1} + \frac{l_2 - l_1}{r - l_1 + 1} \cdot \frac{\varepsilon}{6N} + \frac{r - l_2 + 1}{r - l_1 + 1} \cdot \left(S(v_k^t) + \frac{\varepsilon}{6N} \right) \leq$$

$$\leq \frac{l_2 - l_1}{r - l_1 + 1} v_k^t - \frac{l_2 - l_1}{r - l_1 + 1} \varepsilon + \frac{l_2 - l_1}{r - l_1 + 1} \cdot \frac{\varepsilon}{6N} + \frac{r - l_2 + 1}{r - l_1 + 1} \cdot \left(S(v_k^t) + \frac{\varepsilon}{6N} \right).$$

The corresponding convergencies imply that

$$|v_m^t - S(v_m^t)| \leq |v_m^t - v_m^*| + |v_m^* - S(v_m^t)| < \frac{\varepsilon}{6N} \quad \text{for } m = 1, ..., N;$$

hence, $v_k^t \leq S(v_k^t) + \frac{\varepsilon}{6N}$. Apply this inequality to rewrite the estimate of $S(v_k^{t+1})$:

$$S(v_k^{t+1}) \leq \frac{l_2 - l_1}{r - l_1 + 1}(S(v_k^t) + \frac{\varepsilon}{6N}) - \frac{l_2 - l_1}{r - l_1 + 1}\varepsilon + \frac{l_2 - l_1}{r - l_1 + 1} \cdot \frac{\varepsilon}{6N} +$$

$$+ \frac{r - l_2 + 1}{r - l_1 + 1} \cdot \left(S(v_k^t) + \frac{\varepsilon}{6N} \right) = S(v_k^t) \left[\frac{l_2 - l_1}{r - l_1 + 1} + \frac{r - l_2 + 1}{r - l_1 + 1} \right] +$$

$$+ \varepsilon \left[\frac{l_2 - l_1}{r - l_1 + 1} \cdot \frac{1}{6N} - \frac{l_2 - l_1}{r - l_1 + 1} + \frac{l_2 - l_1}{r - l_1 + 1} \cdot \frac{1}{6N} + \frac{r - l_2 + 1}{r - l_1 + 1} \cdot \frac{1}{6N} \right] =$$

$$= S(v_k^t) + \varepsilon \left[\frac{1}{6N} - \frac{l_2 - l_1}{r - l_1 + 1} + \frac{l_2 - l_1}{r - l_1 + 1} \cdot \frac{1}{6N} \right] =$$

$$= S(v_k^t) + \varepsilon \left[\frac{1}{6N} - \frac{l_2 - l_1}{r - l_1 + 1} \cdot \frac{6N - 1}{6N} \right].$$

The inequalities $1 \le l_1 < l_2 \le r \le N$ imply that $l_2 - l_1 \ge 1$, $r - l_1 + 1 \le N$, and $(l_2 - l_1)/(r - l_1 + 1) \ge 1/N$. Substitute this into the above inequality:

$$S(v_k^{t+1}) \le S(v_k^t) + \varepsilon \left[\frac{1}{6N} - \frac{1}{N} \cdot \frac{6N - 1}{6N} \right] \le$$

$$\le S(v_k^t) + \varepsilon \left[\frac{1}{6N} - \frac{1}{N} \cdot \frac{5}{6} \right] = S(v_k^t) - \frac{2}{3N}\varepsilon,$$

i.e., $|S(v_k^{t+1}) - S(v_k^t)| \ge 2\varepsilon/(3N)$ for an infinite set of t. But this contradicts the relation $\lim\limits_{t\to\infty} S(v_k^t) = v_k^*$, which completes the proof. □

Now we prove the main result of this section.

Theorem 3 *Trajectory of any standard point tends to a standard fixed point.*

Proof. Let $\lim\limits_{n\to\infty} V^n = V^* = (v_1^*, \ldots, v_N^*)$. Without loss of generality, we may assume that $v_1^* = -1$ (see Corollary 3). By the previous Proposition, for a fixed $k \in \{1, \ldots, N\}$, the sets $J(v_k^t)$ coincide starting from some t. For brevity, denote $J_k = J(v_k^t) = J(v_k^{t+1}) = \ldots$, and the cardinality of this set will be denoted by I_k.

Let us show that if indices $k, l \in \{1, \ldots, N\}$ satisfy the inequality $v_l \in [v_k^*, v_k^* + \varepsilon]$, then $v_k^* = v_l^*$. We prove this using induction on k.

Base. $k = 1$. By Proposition 7, $\lim\limits_{n\to\infty} \frac{1}{I_1} \sum\limits_{j\in J_1} v_j^n = \lim\limits_{n\to\infty} S(v_1^n) = -1$. Passing to the limit, we conclude that

$$\frac{\sum\limits_{j\in J_1} v_j^*}{I_1} = -1.$$

Since $v_j^* \ge -1$ for all $j \in J_1$ and the arithmetic mean of these numbers equals -1, $v_j^* = -1$ for all $j \in J_1$.

The equalities $v_q^* = -1$ hold for all $q \in J_1$; thus, a similar reasoning shows that

$$\frac{\sum\limits_{j\in J_q} v_j^*}{I_q} = -1$$

and $v_j^* = -1$ for all $j \in J_q$. In particular, this means that

$$J_q = J_1 = \{s \in \{1, \dots, N\} \mid v_s^* = -1\}$$

for any $q \in J_1$. Further we assume that q is the largest index in J_1.

Since $q+1 \notin J_1$ and $J_1 = J_q$, there exists a large enough $\tau \in \mathbb{N}$ and a number $\alpha > 0$ such that $|v_q^\tau - v_{q+1}^\tau| > \varepsilon + \alpha$. By Corollary 1, $|v_q^t - v_{q+1}^t| > \varepsilon + \alpha$ for all $t > \tau$. Hence, $|v_q^* - v_{q+1}^*| \geq \varepsilon + \alpha > \varepsilon$. Since $v_q^* = -1$, $-1 + \varepsilon < v_{q+1}^* \leq \dots \leq v_N^*$. This proves the induction base.

Induction step. Assume that our statement is proved for k and prove it for $k+1$. If $v_{k+1}^* = v_k^*$, then the statement is obviously true. Otherwise, $v_{k+1}^* > v_k^* + \varepsilon$ and $k \notin J_{k+1}$.

By Proposition 7, $\lim\limits_{n\to\infty} \frac{1}{I_{k+1}} \sum\limits_{j \in J_{k+1}} v_j^n = \lim\limits_{n\to\infty} S(v_{k+1}^n) = v_{k+1}^*$. Passing to the limit, we get the equality

$$\frac{\sum\limits_{j \in J_{k+1}} v_j^*}{I_{k+1}} = v_{k+1}^*.$$

The arithmetic mean of numbers in the set J^{k+1} equals v_{k+1}^*, and $v_j^* \geq v_{k+1}^*$ for all $j \in J_{k+1}$ since $k \notin J_{k+1}$. Hence, $v_j^* = v_{k+1}^*$ for all $j \in J_{k+1}$.

Since $v_q^* = v_{k+1}^*$ for all $q \in J_{k+1}$, a similar reasoning shows that

$$\frac{\sum\limits_{j \in J_q} v_j^*}{I_q} = v_{k+1}^*$$

and $v_j^* = v_{k+1}^*$ for all $j \in J_q$. In particular, this means $J_q = J_{k+1} = \{s \in \{1, \dots, N\} \mid v_s^* = v_{k+1}^*\}$ for any $q \in J_{k+1}$. Further we assume that q is the largest index in J_{k+1}.

Since $q+1 \notin J_q$, there exists a large enough $\tau \in \mathbb{N}$ and a number $\alpha > 0$ such that $|v_q^\tau - v_{q+1}^\tau| > \varepsilon + \alpha$. By Corollary 1, $|v_q^t - v_{q+1}^t| > \varepsilon + \alpha$ for all $t > \tau$. Thus, $|v_q^* - v_{q+1}^*| \geq \varepsilon + \alpha > \varepsilon$. Since $v_q^* = v_{k+1}^*$, $v_{k+1}^* + \varepsilon < v_{q+1}^* \leq \dots \leq v_N^*$, which completes the induction step.

Thus, for any indices $i, j \in \{1, ..., N\}$ either $v_i^* = v_j^*$ or $|v_i^* - v_j^*| > \varepsilon$. This precisely means that $S(v_k^*) = v_k^*$ for any $k \in \{1, \dots, N\}$ and $w_k(V^*) = v_k^* + hS(v_k^*) = (1+h)v_k^*$. It is easily seen that $\mu(W(V^*)) = |w_1(V^*)| = 1 + h$. Hence,

$$\frac{w_k(v^*)}{\mu(W(V^*))} = \frac{(1+h)v_k^*}{1+h} = v_k^*,$$

which means that V^* is a fixed point. $\square$

Corollary 5 *$V = (v_1, \dots, v_N)$ is a standard fixed point if and only if for any pair of indices $i, j \in \{1, \dots, N\}$ either $v_i = v_j$ or $|v_i - v_j| > \varepsilon$.*

Proof. If $\lim_{n\to\infty} V^n = V^* = (v_1^*, \dots, v_N^*)$, then, according to the proof of Theorem 3, for any pair of indices $i, j \in \{1, \dots, N\}$ either $v_i^* = v_j^*$ or $|v_i^* - v_j^*| > \varepsilon$. Now suppose V is a standard fixed point, $V^* = V$. Hence, for any pair of indices $i, j \in \{1, \dots, N\}$ either $v_i = v_j$ or $|v_i - v_j| > \varepsilon$. □

6. Stability of fixed points.

Proposition 9 *Singular fixed points are Lyapunov unstable.*

Proof. If $V = (v_1, \dots, v_N)$ is a singular fixed point, then $|v_1 - v_N| \le \varepsilon$ and $v_1 + \dots + v_N = 0$. Fix a small $\delta > 0$ and consider the point $U = (v_1 + \delta, v_2, \dots, v_N)$.

Since $v_1 + \dots + v_N + \delta > 0$, the point U is standard. The trajectory of the point U converges to a fixed point $P = (p_1, \dots, p_N)$, and $\mu(W(P)) = \lim_{k\to\infty} \mu(W(V^k)) = 1 + h$. This means that P is a standard point. By Corollary 3, $p_1 = -1$ or $p_n = 1$. It follows from the classification of singular points (see Corollary 4) that $-\varepsilon \le v_1 \le \dots \le v_n < \varepsilon$. Since δ is arbitrary, this means that the fixed point V is unstable. □

Before classifying the standard points' stability, let us introduce one more object (see [1]).

Definition 6 *We call a subset $\{k, \dots, m\} \subset \{1, \dots, N\}$, where $k \le m$, a **band** for a point $U = (u_1, \dots, u_N)$ if $|u_i - u_j| < \varepsilon$ for $i, j \in \{k, \dots, m\}$, $|u_k - u_{k-1}| > \varepsilon$ (if $k > 0$), and $|u_m - u_{m+1}| > \varepsilon$ (if $m < N$). The value $|u_m - u_k|$ is called the **width** of the band $\{k, \dots, m\}$.*

Proposition 10 *Standard fixed points are Lyapunov stable.*

Proof. Consider a standard fixed point $V = (v_1, \dots, v_N)$ and an arbitrary $\varepsilon_0 > 0$. Without loss of generality, we may assume that $\varepsilon_0 < \varepsilon$, $v_N = 1$.

Find a positive $\delta < \varepsilon_0/(8N)$ such that for any point $U = (u_1, \dots, u_N)$ with $|u_i - v_i| < \delta$ for $i = 1, \dots, N$ there does not exist a pair of indices p, q such that $\varepsilon_0/(4N) \le |u_p - u_q| \le \varepsilon$. In other words, for any $p, q \in \{1, \dots, N\}$ either $|u_p - u_q| < \varepsilon_0/(4N)$ or $|u_p - u_q| > \varepsilon$. The existence of such a δ follows from Corollary 5.

Take an arbitrary point U^0 such that $|u_i^0 - v_i| < \delta$ for all $i = 1, \dots, N$. By definition, the set of indices $\{1, \dots, N\}$ can be represented as a disjoint union of bands whose widths are not more than $\varepsilon_0/(2N)$. Consider one of these bands $\{k, \dots, m\}$. It is easily seen that

$$|v_k - v_m| \le |v_k - u_k^0| + |u_k^0 - u_m^0| + |u_m^0 - v_m| < \varepsilon.$$

By Corollary 5, $v_k = ... = v_m$. Therefore,

$$|u_k^0 - u_m^0| \le |u_m^0 - v_m| + |v_m - v_k| + |v_k - u_k^0| < 2\delta.$$

By Corollary 1, $|u_k^n - u_{k-1}^n| > \varepsilon$ (if $k > 0$) and $|u_m^n - u_{m+1}^n| > \varepsilon$ (if $m < N$) for any natural n. Here, as above, we denote $\Phi^n(U^0)$ by $U^n = (u_1^n, \dots, u_N^n)$.

Hence, for any integer $n \ge 0$ and any index $i \in \{k, \dots, m\}$, the inclusion $J(u_i^n) \subset \{k, \dots, m\}$ holds. In addition, the inequalities $u_k^n \le \dots \le u_m^n$ imply the estimates

$$S(u_k^n) = \frac{\sum\limits_{j \in J(u_k^n)} u_j^n}{I(u_k^n)} \ge u_k^n$$

and

$$S(u_m^n) = \frac{\sum\limits_{j \in J(u_m^n)} u_j^n}{I(u_m^n)} \le u_m^n.$$

Thus,

$$|u_m^{n+1} - u_k^{n+1}| = \left| \frac{u_m^n + h \cdot \frac{\sum\limits_{j \in J(u_m^n)} u_j^n}{I(u_m^n)}}{\mu(W(U^n))} - \frac{u_k^n + h \cdot \frac{\sum\limits_{j \in J(u_k^n)} u_j^n}{I(u_k^n)}}{\mu(W(U^n))} \right| \le$$

$$\le \left| \frac{u_m^n(1+h)}{\mu(W(U^n))} - \frac{u_k^n(1+h)}{\mu(W(U^n))} \right| \le |u_m^n - u_k^n| \cdot \frac{1+h}{\mu(W(U^n))}$$

and

$$|u_m^{n+1} - u_k^{n+1}| \le |u_m^n - u_k^n| \cdot \frac{1+h}{\mu(W(U^n))} \le |u_m^{n-1} - u_k^{n-1}| \cdot \frac{(1+h)^2}{\mu(W(U^n))\mu(W(U^{n-1}))} \le$$

$$\le ... \le \frac{(1+h)^{n+1}}{\mu(W(U^n))\mu(W(U^{n-1}))...\mu(W(U^0))} |u_k^0 - u_m^0|.$$

Consider the band $\{L, ..., N\}$ for U^0. Applying $S(u_L^n) \geq u_L^n$, we conclude that

$$u_L^{n+1} = \frac{u_L^n + hS(u_L^n)}{\mu(W(U^n))} \geq \frac{u_L^n(1+h)}{\mu(W(U^n))} \geq u_L^n,$$

and

$$u_L^{n+1} \geq \frac{1+h}{\mu(W(U^n))} u_L^n \geq \frac{1+h}{\mu(W(U^n))} \cdot \frac{1+h}{\mu(W(U^{n-1}))} \cdot u_L^{n-1} \geq ... \geq$$

$$\geq \frac{(1+h)^{n+1}}{\mu(W(U^n))\mu(W(U^{n-1}))...\mu(W(U^0))} u_L^0.$$

As we already know, $u_L^{n+1} \leq 1$, $v_L = ... = v_N = 1$ and $|u_L^0 - v_L| < \delta$. Therefore

$$1 \geq \frac{(1+h)^{n+1}}{\mu(W(U^n))\mu(W(U^{n-1}))...\mu(W(U^0))} u_L^0,$$

and

$$\frac{1}{1-\delta} \geq \frac{(1+h)^{n+1}}{\mu(W(U^n))\mu(W(U^{n-1}))...\mu(W(U^0))}.$$

This means that

$$|u_m^{n+1} - u_k^{n+1}| \leq \frac{(1+h)^{n+1}}{\mu(W(U^n))\mu(W(U^{n-1}))...\mu(W(U^0))} \cdot |u_k^0 - u_m^0| \leq$$

$$\leq \frac{1}{1-\delta} \cdot |u_k^0 - u_m^0| \leq \frac{2\delta}{1-\delta}.$$

Hence, if $\delta < \varepsilon/(8N)$, then $\{k, ..., m\}$ is a band for all U^n, $n \in \mathbb{N}$, and its width does not exceed 3δ at any time moment. Consequently, $J(u_i^n) = \{k, ..., m\}$ for any integer $n \geq 0$ and any index $i \in \{k, ..., m\}$. Denote $T_n := S(u_k^n) = S(u_{k+1}^n) = ... = S(u_m^n)$. Now let us write a recurrence relation for T_n's:

$$T_{n+1} = \frac{u_k^{n+1} + ... + u_m^{n+1}}{m-k+1} = \frac{(u_k^n + hT_n) + ... + (u_m^n + hT_n)}{\mu(W(U^n)) \cdot (m-k+1)} = \frac{1+h}{\mu(W(U^n))} T_n.$$

Therefore,

$$T_{n+1} = \frac{(1+h)^{n+1}}{\mu(W(U^n))\mu(W(U^{n-1}))...\mu(W(U^0))} T_0.$$

This implies that

$$T_0 \leq T_{n+1} \leq \frac{1}{1-\delta} \cdot T_0 = T_0 + \frac{\delta}{1-\delta} \cdot T_0.$$

Since $u_k^0, ..., u_m^0 \in (v_k - \delta, v_k + \delta)$, then $T_0 \in (v_k - \delta, v_k + \delta)$ and

$$|T_n - v_k| \le \max(T_0 - v_k, T_0 + \frac{\delta}{1-\delta} T_0 - v_k) \le \delta + \frac{\delta}{1-\delta} \cdot |T_0| < \frac{2\delta}{1-\delta} < 3\delta.$$

Due to the fact that $u_k^n \le T_n \le u_m^n$ and $|u_m^n - u_k^n| < 3\delta$, we see that

$$|u_j^n - v_j| = |u_j^n - v_k| \le |u_j^n - T_n| + |T_n - v_k| < 6\delta < \varepsilon$$

for arbitrary n and $j \in \{k, ..., m\}$.

As was said above, the set $\{1, \dots, N\}$ decomposes into a disjoint union of bands for any point U^0 in the δ-neighborhood of V.

In addition, we have shown that for any time moment n and for any index i belonging to any band, the inequality $|u_i^n - v_i| < \varepsilon$ is satisfied. This means that the point V is Lyapunov stable. □

7. Asymptotic stability.

Proposition 11 *The points $P_- = (-1, \dots, -1)$ and $P_+ = (1, \dots, 1)$ are asymptotically stable.*

Proof. The proof of this fact is based on the same idea as the proof of item 1 of Theorem 5.1 in [1].

Let us prove the statement for the point P_+. Fix a positive $\delta < \varepsilon$. Consider an arbitrary point U^0 such that $u_i^0 \in [1-\delta, 1], i = 1, \dots, N$. Let n be an arbitrary time moment.

Since $u_1^n \le \dots \le u_N^n$,

$$\frac{\sum\limits_{j \in J(u_1^n)} u_j^n}{I(u_1^n)} \ge u_1^n$$

and

$$u_1^{n+1} = \frac{u_1^n + h \cdot \frac{\sum\limits_{j \in J(u_1^n)} u_j^n}{I(u_1^n)}}{\mu(W(V^n))} \ge \frac{u_1^n + h \cdot \frac{\sum\limits_{j \in J(u_1^n)} u_j^n}{I(u_1^n)}}{1+h} \ge u_1^n.$$

The point U^0 is standard, and $u_1^0 \le u_1^1 \le u_1^2 \le \cdots$. The trajectory of the point U^0 converges to a standard fixed poit $P = (p_1, \dots, p_N)$. Since $\lim\limits_{n\to\infty} u_1^n = p_1$ and the sequence $\{u_1^n\}_{n=1}^\infty$ is monotonically increasing, $p_1 \ge u_1^0 \ge 1 - \delta$.

By Corollary 3, $p_N = 1$. This means that $p_1 - p_N \le \delta < \varepsilon$, and $p_1 = \dots = p_N$ by Corollary 5. We have shown that $P = P_+$.

A similar reasoning can be applied in the case of the point P_-. □

Proposition 12 *Standard fixed points different from P_- and P_+ are not asymptotically stable.*

Proof. The proof of this fact is based on the same idea as the proof of item 2 of Theorem 5.1 in [1].

Assume that there exists an asymptotically stable standard fixed point $U = (u_1, \dots, u_N)$ different from P_- and P_+. By Corollary 3, either $u_1 = -1$ or $u_N = 1$. Without loss of generality, assume that $u_1 = -1$. Note that in this case $u_N \neq -1$ (since U and P_- are different) and $u_N > -1 + \varepsilon$ by Corollary 5.

For the point U, there exists a decomposition of $\{1, \dots, N\}$ into bands, and, as shown above, 1 and N belong to different bands. Consider the band $\{k, \dots, N\}$. By definition, $u_k - u_{k-1} > \varepsilon$. By Corollary 5, $u_k = \dots = u_N$. Take an arbitrary positive $\delta < u_k - u_{k-1} - \varepsilon$ and the point $U' = (u_1, \dots, u_{k-1}, u_k - \delta, u_{k+1} - \delta, \cdots, u_N - \delta)$.

Note that U' is a fixed point since $(u_k - \delta) - u_{k-1} > \varepsilon$ and $u_k - \delta = u_{k+1} - \delta = \dots = u_N - \delta$. Thus, any neighborhood of the point U contains a fixed point, which means that the point U is not asymptotically stable. □

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