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ON MORPHISMS KILLING WEIGHTS AND STABLE HUREWICZ-TYPE THEOREMS

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Abstract For a weight structure w on a triangulated category $\underline{C}$ we prove that the corresponding *weight complex* functor and some other (*weight-exact*) functors are 'conservative up to weight-degenerate objects'; this improves earlier conservativity formulations. In the case $w = w^{sph}$ (the *spherical* weight structure on SH), we deduce the following converse to the stable Hurewicz theorem: $H_i^{sing}(M) = \{0\}$ for all $i < 0$ if and only if $M \in SH$ is an extension of a connective spectrum by an acyclic one. We also prove an equivariant version of this statement.

The main idea is to study M that has *no weights* $m, \ldots, n$ ('in the middle'). For $w = w^{sph}$, this is the case if there exists a distinguished triangle $LM \to M \to RM$, where RM is an n-connected spectrum and LM is an $m-1$-*skeleton* (of M) in the sense of Margolis's definition; this happens whenever $H_i^{sing}(M) = \{0\}$ for $m \le i \le n$ and $H_{m-1}^{sing}(M)$ is a free abelian group. We also consider morphisms that *kill weights* $m, \ldots, n$; those 'send n-w-skeleta into $m-1$-w-skeleta'.

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Contents

Introduction

Let us recall that, for any object M of a triangulated category $\underline{C}$ and any integer n, a weight structure w on $\underline{C}$ gives (essentially by definition) an n-weight decomposition triangle $LM \to M \to RM \to LM[1]$, where LM is of weights at most n and RM is of weights at least $n+1$;[1] however, this triangle is not canonical. In particular, for the *spherical* weight structure w^{sph} on the stable homotopy category SH (see §4.2 of [7], Theorem 4.2.4 and Theorem 4.2.5(1) below) one can take LM to be an arbitrary choice of an *n-skeleton* for the spectrum M in the sense of §6.3 of [20]; thus, LM is not determined by M uniquely. However, it was noticed by J. Wildeshaus that if one can choose an n-weight decomposition such that LM is of weights at most $m-1$ for some $m \le n$, then this stronger assumption makes the decomposition canonical. In this case, M is said to be *without weights* $m,\dots,n$. In this paper, we prove that M satisfies this condition if and only if its *weight complex* $t(M)$ is homotopy equivalent to a complex with zero terms in degrees $-n,\dots,-m$;[2] recall here that t is a 'weakly exact' functor from $\underline{C}$ into a certain quotient $K_{\mathfrak{w}}(\underline{Hw})$ of the homotopy category of complexes in the *heart* $\underline{Hw}$ of w. It easily follows that one can find out whether M is without weights $m,\dots,n$ by applying functors that are *pure* (in the sense of Definition 2.4.1 below) to M. Moreover, one can put $m=-\infty$ or $n=+\infty$ in these statements to obtain that t is 'conservative up to *weight-degenerate* objects'. One may say that t is conservative up to objects of infinitely small and infinitely large weights. This is a significant improvement over previously known bounded conservativity results. One may say that objects of $\underline{C}$ may be 'detected' by means of objects of a much simpler category $K_{\mathfrak{w}}(\underline{Hw})$. We apply our conservativity of weight complexes result to calculate certain intersections of purely generated subcategories (this result was applied in [5]) and to prove that certain weight-exact functors are conservative. The latter statement generalizes Theorems 2.5 and 2.8 of [29]; in particular, we treat not necessarily bounded weight structures.

Moreover, we apply our general results to equivariant stable homotopy categories and *spherical weight structures* on them (as introduced in §4 of [7]). The aforementioned conservativity of weight complexes results yield a certain converse to the equivariant stable Hurewicz theorem. In particular, in the case of a trivial group the weight complex functor

[1] That is, $LM \in \underline{C}_{w\le n} = \underline{C}_{w\le 0}[n]$ and $RM \in \underline{C}_{w\ge n+1} = \underline{C}_{w\ge 0}[n+1]$.

[2] Here, one has to assume that $\underline{C}$ is *weight-Karoubian*, that is, that $\underline{Hw}$ is idempotent complete; see Theorem 2.3.1(4). However, this is a reasonable assumption since it is fulfilled whenever $\underline{C}$ is idempotent complete itself.

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essentially calculates singular homology; thus, we obtain that the singular homology of a spectrum $E \in \operatorname{Obj} SH$ vanishes in negative degrees if and only if E is an extension of a connective spectrum by an acyclic one. This statement appears to be completely new since in all the previously existing formulations only the case where E is *bounded below* was considered; see Theorem 2.1(i) of [18], Proposition 7.1.2(f) of [15] and Theorem 6.9 of [20]. Moreover, the vanishing of $H_i^{sing}(E)$ for two subsequent values of i gives a canonical 'decomposition' of E into a distinguished triangle. This result is new as well. Our main definitions and statements are nicely illustrated by Theorems 4.2.4 and 4.2.5. In particular, we prove that w^{sph}-Postnikov towers are the *cellular* ones in the sense of [20] (thus completing the proof of [7, Theorem 4.2.1]). Our central statements can also be applied to Tate motives; see Remark 4.1.8(1) below.

The main tool for obtaining these results is the new interesting notion of *morphisms killing weights* $m,\dots,n$; for a morphism $g: M \to N$ this means that g is 'compatible with' some morphism $w_{\le n}M \to w_{\le m-1}N$. This definition is equivalent to several other ones. In particular, if $m = n$, then one can easily reformulate this condition in terms of $t(g)$. Thus, an SH-morphism g kills weight m if and only if $H^m_{sing}(g,\Gamma) = 0$ for every abelian group Γ. More generally, an SH-morphism g kills weights $m,\dots,n$ whenever it 'sends n-skeleta into $m-1$-skeleta'; see Proposition 2.1.1(4) and Theorem 4.2.5.

Let us now describe the contents of the paper.

§1 contains some preliminaries, mostly on weight structures.

In §2, we define morphisms killing weights $m,\dots,n$ and objects without these weights and study these notions in detail. In particular, we relate killing weights to weight complexes and pure functors. This gives a new conservativity of the weight complex functor result.

In §3, we extend some of the results of the previous section to the case where $\underline{Hw}$ is not Karoubian (thus, $\underline{Hw}$-idempotents do not yield direct summands in $\underline{C}$), so we formulate Theorem 3.1.3 that is central for this paper. We also give applications to intersections of *purely* generated subcategories and prove that certain weight-exact functors are 'conservative up to weight-degenerate objects'.

In §4, we study *purely compactly generated categories*. We also consider equivariant stable homotopy examples to obtain certain converse stable Hurewicz theorems (see Theorems 4.2.2 and 4.2.4) and several related statements (including the aforementioned Theorem 4.2.5).

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An alternative version of this paper can be found at [6]. Another closely related text is the new [12]; see Remark 2.3.2(2) below.

1. Weight structures: reminder

In §1.1, we introduce some notation and conventions. In §1.2, we recall some basics on weight structures. The only new statement of this section is the technical (yet important) Lemma 1.2.7(2).

In §1.3, we recall some properties of weight complex functors and of the weak homotopy equivalence relation for morphisms between complexes.

1.1. Some (categorical) notation

- Let C be a category and $X,Y \in \operatorname{Obj} C$. Then we will write $C(X,Y)$ for the set of morphisms from X to Y in C.
- We say that X is a *retract* of Y if id_X can be factored through Y.
- A subcategory $\mathcal{P}$ of an additive category C is said to be *retraction-closed* in C if it contains all retracts of its objects in C.
- For any $(C,\mathcal{P})$ as above the full subcategory $\operatorname{Kar}_C(\mathcal{P})$ of C whose objects are all retracts of finite direct sums of objects $\mathcal{P}$ in C will be called the *retraction-closure* of $\mathcal{P}$ in C. Note that this subcategory is obviously additive and retraction-closed in C.
- Below $\underline{A}$ will always denote some abelian category; $\underline{B}$ is an additive category.
- We will write $\operatorname{Kar}(\underline{B})$ (no lower index) for the *idempotent completion* $\underline{B}$; see Definition 1.2 of [1]. Recall that its objects are the pairs (A,p) for $A \in \operatorname{Obj} \underline{B}$, $p \in \underline{B}(A,A)$, $p^2 = p$; the correspondence $A \mapsto (A, \mathrm{id}_A)$ (for $A \in \operatorname{Obj} \underline{B}$) fully embeds $\underline{B}$ into $\operatorname{Kar}(\underline{B})$; see Proposition 1.3 and Remark 1.4 of loc. cit. Moreover, $\operatorname{Kar}(\underline{B})$ is *Karoubian*, that is, every idempotent morphism gives a direct sum decomposition in $\operatorname{Kar}(\underline{B})$, and $\operatorname{Kar}(\underline{B})$ is triangulated if $\underline{B}$ is; see Theorem 1.5 of loc. cit.
- The symbol $\underline{C}$ below will always denote some triangulated category. Usually it is endowed with a weight structure w. The symbols $\underline{C}'$ and $\underline{D}$ will also be used for triangulated categories only.
- For any $A,B,C \in \operatorname{Obj} \underline{C}$ we say that C is an *extension* of B by A if there exists a distinguished triangle $A \xrightarrow{f} C \to B \to A[1]$.

  Moreover, we will write $B = \operatorname{Cone}(f)$. Recall here that different choices of cones are connected by nonunique isomorphisms.
- For $X,Y \in \operatorname{Obj} \underline{C}$ we write $X \perp Y$ if $\underline{C}(X,Y) = \{0\}$. If D and E are classes of objects or subcategories of $\underline{C}$, then we will write $D \perp E$ if $X \perp Y$ for all $X \in D$, $Y \in E$. Moreover, we write $D^{\perp}$ for the class

$$\{Y \in \operatorname{Obj} \underline{C} : \ X \perp Y \ \forall X \in D\};$$

  dually, ${}^{\perp}D$ is the class $\{Y \in \operatorname{Obj} \underline{C} : \ Y \perp X \ \forall X \in D\}$.
- We write $C(\underline{B})$ for the category of cohomological complexes in $\underline{B}$, and the corresponding homotopy category will be denoted by $K(\underline{B})$. We will write $M = (M^i)$ if M^i are the terms of the complex M.
- We say that an additive covariant (resp. contravariant) functor H from $\underline{C}$ into $\underline{A}$ is *homological* (resp. *cohomological*) if it converts distinguished triangles into long exact sequences. We will write H_i (resp. H^i) for the composition $H \circ [-i]$.

1.2. Weight structures: basics

Definition 1.2.1. We say that subclasses $\underline{C}_{w \le 0}$ and $\underline{C}_{w \ge 0} \subset \operatorname{Obj} \underline{C}$ give a *weight structure* w or $(\underline{C},w)$ on a triangulated category $\underline{C}$ and that $\underline{C}$ is *weighted* if the following conditions are fulfilled.

(i) $\underline{C}_{w \ge 0}$ and $\underline{C}_{w \le 0}$ are retraction-closed in $\underline{C}$, that is, they contain all $\underline{C}$-retracts of their objects).

(ii) $\underline{C}_{w \le 0} \subset \underline{C}_{w \le 0}[1]$ and $\underline{C}_{w \ge 0}[1] \subset \underline{C}_{w \ge 0}$.

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(iii) **Orthogonality:** $\underline{C}_{w\le 0} \perp \underline{C}_{w\ge 0}[1]$.
(iv) For any $M \in \operatorname{Obj}\underline{C}$ there exists a distinguished triangle

$$LM \to M \to RM \to LM[1]$$

such that $LM \in \underline{C}_{w\le 0}$ and $RM \in \underline{C}_{w\ge 0}[1]$.

We also need the following definitions.

Definition 1.2.2. Assume that $i,j \in \mathbb{Z}$ and $(\underline{C},w)$ is a weight structure.

1. The full subcategory $\underline{Hw}$ of $\underline{C}$ whose objects are $\underline{C}_{w=0} = \underline{C}_{w\ge 0} \cap \underline{C}_{w\le 0}$ is called the *heart* of w.
2. $\underline{C}_{w\ge i}$ (resp. $\underline{C}_{w\le i}$, resp. $\underline{C}_{w=i}$) will denote the class $\underline{C}_{w\ge 0}[i]$ (resp. $\underline{C}_{w\le 0}[i]$, resp. $\underline{C}_{w=0}[i]$).
3. $\underline{C}_{[i,j]}$ denotes $\underline{C}_{w\ge i} \cap \underline{C}_{w\le j}$, so, this class equals $\{0\}$ if $i > j$.
4. We will say that $\underline{C}$ (or $(\underline{C},w)$) is *weight-Karoubian* if $\underline{Hw}$ is Karoubian.
5. Let $\underline{C}'$ be a triangulated category endowed with a weight structure w'; let $F : \underline{C} \to \underline{C}'$ be an exact functor.
   Then F is said to be *weight-exact* with respect to w,w' if it maps $\underline{C}_{w\le 0}$ into $\underline{C}'_{w'\le 0}$ and sends $\underline{C}_{w\ge 0}$ into $\underline{C}'_{w'\ge 0}$.
6. Let $\underline{D}$ be a full triangulated subcategory of $\underline{C}$.
   We will say that *w restricts* to $\underline{D}$ whenever the couple $(\underline{C}_{w\le 0} \cap \operatorname{Obj}\underline{D},\ \underline{C}_{w\ge 0} \cap \operatorname{Obj}\underline{D})$ is a weight structure on $\underline{D}$.
7. We say that M is left (resp., right) *w-degenerate* (or *weight-degenerate* if the choice of w is clear) if M belongs to $\cap_{i\in\mathbb{Z}}\underline{C}_{w\ge i}$ (resp. to $\cap_{i\in\mathbb{Z}}\underline{C}_{w\le i}$).
8. We call $\cup_{i\in\mathbb{Z}}\underline{C}_{w\ge i}$ (resp. $\cup_{i\in\mathbb{Z}}\underline{C}_{w\le i}$) the class of *$w$-bounded below* (resp., *w-bounded above*) objects of $\underline{C}$.
9. We will say that w is *bounded* if every object of $\underline{C}$ is w-bounded both above and below.

Remark 1.2.3. 1. For an arbitrary additive $\underline{B}$ one can take $\underline{C} = K(\underline{B})$ and set $K(\underline{B})_{w^{st}\le 0}$ (resp. $K(\underline{B})_{w^{st}\ge 0}$) to be the class of complexes $\underline{C}$-isomorphic to complexes concentrated in degrees ≥ 0 (resp. ≤ 0); see Remark 1.2.3(1) of [9] for more detail. We will use this notation below.

The heart of this weight structure w^{st} is the retraction-closure of $\underline{B}$ in $K(\underline{B})$; hence it is equivalent to $\operatorname{Kar}(\underline{B})$.

2. The distinguished triangle in axiom (iv) is not determined by M.

Still for every $m \in \mathbb{Z}$ this axiom yields a distinguished triangle

$$w_{\le m}M \to M \to w_{\ge m+1}M \to (w_{\le m}M)[1] \qquad (1.2.1)$$

with some $w_{\le m}M \in \underline{C}_{w\le m}$ and $w_{\ge m+1}M \in \underline{C}_{w\ge m+1}$. We notate this triangle with $WD_m(M)$ and call $WD_m(M)$ an *m-weight decomposition* of M.

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We will often use this notation below even though $w_{\geq m+1}M$ and $w_{\leq m}M$ are not canonically determined by M. We call any possible choice either of $w_{\geq m+1}M$ or of $w_{\leq m}M$ a *weight truncation* of M. Moreover, when we write arrows of the type $w_{\leq m}M \to M$ or $M \to w_{\geq m+1}M$ we always assume that they come from some m-weight decomposition of M.

3. In the current paper (along with several previous ones) we use the 'homological convention' for weight structures, whereas in [4] the cohomological convention was used. In the latter convention, the roles of $\underline{C}_{w\leq 0}$ and $\underline{C}_{w\geq 0}$ are interchanged, that is, one takes $\underline{C}^{w\leq 0} = \underline{C}_{w\geq 0}$ and $\underline{C}^{w\geq 0} = \underline{C}_{w\leq 0}$.

We also recall that weight structures were independently introduced in [24]; D. Pauksztello has called them co-t-structures.

Proposition 1.2.4. *Let $m \leq n \in \mathbb{Z}$, $M, M' \in \operatorname{Obj}\underline{C}$.*

1. *The axiomatics of weight structures is self-dual, that is, on $\underline{C}' = \underline{C}^{op}$ (so $\operatorname{Obj}\underline{C}' = \operatorname{Obj}\underline{C}$) there exists the opposite weight structure w^{op} for which $\underline{C}'_{w^{op}\leq 0} = \underline{C}_{w\geq 0}$ and $\underline{C}'_{w^{op}\geq 0} = \underline{C}_{w\leq 0}$.*
2. *$\underline{C}_{w\geq 0} = (\underline{C}_{w\leq -1})^{\perp}$ and $\underline{C}_{w\leq 0} = {}^{\perp}\underline{C}_{w\geq 1}$.*
3. *$\underline{C}_{w\leq 0}$ is closed with respect to all coproducts that exist in $\underline{C}$.*
4. *$\underline{C}_{w\leq 0}$, $\underline{C}_{w\geq 0}$, and $\underline{C}_{w=0}$ are additive.*
5. *If $M \in \underline{C}_{w\geq m}$, then $w_{\leq n}M \in \underline{C}_{[m,n]}$ for every n-weight decomposition of M. Dually, if $M \in \underline{C}_{w\leq n}$, then $w_{\geq m}M \in \underline{C}_{[m,n]}$.*
6. *Assume $g \in \underline{C}(M, M')$. If $M' \in \underline{C}_{w\geq m}$, then g factors through $w_{\geq m}M$ for any choice of the latter object.*

   Dually, if $M \in \underline{C}_{w\leq m}$, then g factors through $w_{\leq m}M'$.
7. *If $\underline{C}$ is Karoubian, then it is also weight-Karoubian.*

Proof. Assertions 1–5 were proved in [4] (cf. Remark 1.2.3(4) of [9] and pay attention to Remark 1.2.3(3) above!), whereas the easy assertion 6 is given by Proposition 1.2.4(8) of [7].

Assertion 7 follows from axiom (i) in Definition 1.2.1. □

Now we will study certain morphisms between weight decompositions.

Definition 1.2.5. Adopt the notation and assumptions of Remark 1.2.3(2); let $m' \in \mathbb{Z}$. Then a morphism of triangles

$$\begin{array}{ccccc} w_{\leq m}M & \xrightarrow{c} & M & \longrightarrow & w_{\geq m+1}M \\ \downarrow h & & \downarrow g & & \downarrow j \\ w_{\leq m'}M' & \longrightarrow & M' & \longrightarrow & w_{\geq m'+1}M' \end{array} \tag{1.2.2}$$

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will be said to be a morphism $WD_m(M) \to WD_{m'}(M')$ that *extends* $g: M \to M'$. We will say that the morphisms h and j are *w-truncations* of g.

Remark 1.2.6. Clearly, one can compose morphisms of weight decompositions. That is, for a morphism $WD_m(M) \to WD_{m'}(M')$ as above, $m'' \in \mathbb{Z}$, and any morphism $WD_{m'}(M') \to WD_{m''}(M'')$ that extends any $g' \in \underline{C}(M', M'')$ one can compose the corresponding arrows (provided that the corresponding choices of $WD_{m'}(M')$ coincide) to obtain a morphism $WD_m(M) \to WD_{m''}(M'')$ that extends $g' \circ g$.

Lemma 1.2.7. *Adopt the notation of Definition 1.2.5, and assume $m' \geq m$.*

1. *Then an extension of g to a morphism $WD_m(M) \to WD_{m'}(M')$ exists. Moreover, this extension is unique if $m' > m$; consequently, in this case the morphism h (resp. j) is the only one that makes the left- (resp. right-) hand square in diagram 1.2.2 commutative.*
2. *Assume that the rows of diagram 1.2.2 are equal, $g = \mathrm{id}_M$, $m = m'$, $h^2 = h$, and $(\underline{C}, w)$ is weight-Karoubian. Then there exists a decomposition $w_{\leq m} M \cong M_1 \bigoplus M_0$ such that h corresponds to $\mathrm{id}_{M_1} \bigoplus 0_{M_0}$, and the rows of (1.2.2) can be presented as the direct sum of the corresponding two arrows in some m-weight decomposition $M_1 \to M \to M_2$ with $(M_0 \to 0 \to M_0[1])$.*

Proof. 1. This is Lemma 1.5.1(1,2) of [4].

2. Take a triangulated category $\underline{C}'$ that is equivalent to $\mathrm{Kar}(\underline{C})$ and contains $\underline{C}$ as a (full) strict subcategory. Consider the decomposition $w_{\leq m} M \cong M_1 \bigoplus M_0$ corresponding to h in $\underline{C}'$. Since the diagram 1.2.2 is commutative, $c = c \circ h$; thus, c factors through M_1. Hence, the rows of diagram 1.2.2 can be decomposed into the direct sum of the $\underline{C}'$-distinguished triangle $M_0 \to 0 \to M_0[1]$ with a distinguished triangle $M_1 \to M \to M_2$. Thus, M_0 is a retract of $w_{\geq m+1} M[-1]$ as well. Hence, the morphism id_{M_0} factors through some morphism $a: w_{\leq m} M \to w_{\geq m+1} M[-1]$. Next, Proposition 1.2.4(6) implies that a factors through $N = w_{\geq m}(w_{\leq m} M)$; thus, M_0 is a retract of N. Since N belongs to $\underline{C}_{w=m}$ by Proposition 1.2.4(5) and $\underline{Hw}$ is Karoubian, M_0 belongs to $\underline{C}_{w=m} \subset \mathrm{Obj}\,\underline{C}$. It follows that M_1 and M_2 are objects of $\underline{C}$ as well. Applying axiom (i) of Definition 1.2.1 we obtain $M_1 \in \underline{C}_{w \leq m}$ and $M_2 \in \underline{C}_{w \geq m+1}$; hence, $M_1 \to M \to M_2 \to M_1[1]$ is an m-weight decomposition of M indeed. □

1.3. On weight complexes and weak homotopy equivalences

To define the weight complex functor, we need the following definition. Recall here that $\underline{B}$ is an additive category and $C(\underline{B})$ is the category of $\underline{B}$-complexes.

Definition 1.3.1. Let M and N be objects of $K(\underline{B})$ and $m_1, m_2 \in C(\underline{B})(M, N)$.

1. We write $m_1 \backsim m_2$ if $m_1 - m_2 = d_N x + y d_M$ for some collections $x^*, y^* \in \underline{B}(M^*, N^{*-1})$, where d_M and d_N are the corresponding differentials. We call this relation the *weak homotopy equivalence* relation.
2. Assume $k \leq l \in (\{-\infty\} \cup \mathbb{Z} \cup \{+\infty\})$; also, $k \in \mathbb{Z}$ if $k = l$.

Then we write $m_1 \backsim_{[k,l]} m_2$ if $m_1 - m_2$ is weakly homotopic to $m_0 \in C(M, N)$ such that $m_0^i = 0$ for $k \leq i \leq l$ (and $i \in \mathbb{Z}$).

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We need the following properties of these equivalence relations.

Lemma 1.3.2. *Adopt the notation of Definition 1.3.1.*

1. *Factoring morphisms in $K(\underline{B})$ by the weak homotopy relation yields an additive category $K_{\mathfrak{w}}(\underline{B})$. Moreover, the corresponding full functor $p_w : K(\underline{B}) \to K_{\mathfrak{w}}(\underline{B})$ is additive and conservative.*
2. *Let $\mathcal{A} : \underline{B} \to \underline{A}$ be an additive functor, where $\underline{A}$ is any abelian category, and m_1 is weakly homotopic to m_2. Then m_1 and m_2 induce equal morphisms of the homology $H_*(\mathcal{A}(M^i)) \to H_*(\mathcal{A}(N^i))$.*

   Hence, the correspondence $N \mapsto H_0(\mathcal{A}(N^i))$ gives a well-defined functor $K_{\mathfrak{w}}(\underline{B}) \to \underline{A}$.
3. *Applying an additive functor $F : \underline{B} \to \underline{B}'$ to complexes termwisely, one obtains an additive functor $K_{\mathfrak{w}}(F) : K_{\mathfrak{w}}(\underline{B}) \to K_{\mathfrak{w}}(\underline{B}')$.*
4. *$m_1 \backsim_{[k,l]} m_2$ if and only if $m_1 \backsim_{[i,i]} m_2$ for any $i \in \mathbb{Z}$ such that $k \le i \le l$.*
5. *If $k \in \mathbb{Z}$, then $m_1 \backsim_{[k,k]} 0$ if and only if there exists $m_0 \in C(\underline{B})(M,N)$ such that $m_1 = m_0$ in $K(\underline{B})(M,N)$ and $m_0^k = 0$.*
6. *M belongs to $K(\underline{B})_{w^{st} \ge 0}$ if and only if $\mathrm{id}_M \backsim_{[1,+\infty]} 0_M$, and $M \in K(\underline{B})_{w^{st} \le 0}$ if and only if $\mathrm{id}_M \backsim_{[-\infty,-1]} 0_M$.*

Proof. Assertion 3 is obvious, and the remaining ones are contained in Proposition B.2 of [7]. □

Let us describe the approach to weight complexes that we use below.

Remark 1.3.3.

1. In the current paper, we consider an additive *weight complex* functor $t : \underline{C} \to K_{\mathfrak{w}}(\underline{Hw})$ for any triangulated category $\underline{C}$ endowed with a weight structure w. Still to define a canonical functor of this sort one has to replace $\underline{C}$ by a certain equivalent category $\underline{C}_w$; see §1.3 of [7] (and cf. Remark A.2.1(3) of ibid. where the inaccuracies made in [4, §3] are discussed). Thus, to define t one should compose the additive 'canonical weight complex functor' $t_{can} : \underline{C}_w \to K_{\mathfrak{w}}(\underline{Hw})$ with a splitting s of the canonical equivalence $\underline{C}_w \to \underline{C}$ (see Proposition 1.3.4(3,6) of [7]). Clearly, any two splittings s of this sort are isomorphic; thus, we can assume that s is chosen, and so $t = t_{can} \circ s$ is fixed.
2. Moreover, we have no need to describe weight complexes of all morphisms in $\underline{C}$ explicitly. We prefer to list a collection of properties of t instead. So, we only sketch the description of $t(M)$ for $M \in \mathrm{Obj}\,\underline{C}$. The details can be found in loc. cit.

   We choose arbitrary weight truncations $w_{\le n}M$ of M for all $n \in \mathbb{Z}$ and take $g_n : w_{\le n}M \to w_{\le n+1}M$ to be the corresponding w-truncations of id_M in the sense of Definition 1.2.5. Denote $\mathrm{Cone}(g_n)$ by $M^{-n-1}[n+1]$. It is easily seen that $M^i \in \underline{C}_{w=0}$ for all $i \in \mathbb{Z}$ and that the distinguished triangles coming from g_n connect these objects to form a complex (M^i).

Clearly, this complex depends on the choice of the objects $w_{\le n}M$. However, for any choice of this form, we have $(M^i) \cong t(M)$ in $K(\underline{Hw})$. This isomorphism becomes canonical in $K_{\mathfrak{w}}(\underline{Hw})$.

3. t can 'usually' be enhanced to an exact 'strong' weight complex functor $t^{st} : \underline{C} \to K(\underline{Hw})$. This is currently known to be the case if $\underline{C}$ possesses an ∞-enhancement and w is either *bounded* (see Definition 1.2.2(9) and Corollary 3.5 of [26]) or is *purely compactly generated* as in Theorem 4.1.2 below (see Remark 3.6 of ibid.).[3]

   Consequently, the reader will not lose much if she assumes that t^{st} exists throughout the paper.

4. On the other hand, if $K_{\mathfrak{w}}(\underline{B})$ differs from $K(\underline{B})$ (cf. Proposition 4.1.7(1) below) then $K_{\mathfrak{w}}(\underline{B})$ cannot be endowed with a triangulated category structure compatible with that for $K(\underline{B})$, that is, the functor $p_w : K(\underline{B}) \to K_{\mathfrak{w}}(\underline{B})$ cannot be exact.

   Indeed, Lemma 1.3.2(1) allows to reduce this claim to the following statement: If $p : \underline{C} \to \underline{C}'$ is a full conservative exact functor, then p is also faithful. Now, assume that $p(m) = 0$ for some $\underline{C}$-morphism $m : X \to Y$ and take the corresponding distinguished triangle

$$X \xrightarrow{m} Y \xrightarrow{n} Z \to X[1].$$

   Then $p(n)$ is split injective. Since p is full and conservative, the easy Lemma 1.5.2(1) of [7] says that n is split injective as well. Hence, $m = 0$, and we obtain faithfulness.

Now we list the main properties of our functor $t : \underline{C} \to K_{\mathfrak{w}}(\underline{Hw})$.

Proposition 1.3.4. *Assume that $(\underline{C}, w)$ is a weight structure, and $M \xrightarrow{g} M' \xrightarrow{h} \operatorname{Cone}(g)$ are two sides of a $\underline{C}$-distinguished triangle.*

1. *Then $t \circ [n]_{\underline{C}} \cong [n]_{K_{\mathfrak{w}}(\underline{Hw})} \circ t$, where $[n]_{K_{\mathfrak{w}}(\underline{Hw})}$ is the obvious shift by $[n]$ endofunctor of the category $K_{\mathfrak{w}}(\underline{Hw})$.*
2. *There exists a lift of the $K_{\mathfrak{w}}(\underline{Hw})$-morphism chain*

$$t(M) \xrightarrow{t(g)} t(M') \xrightarrow{t(h)} t(\operatorname{Cone}(g))$$

   to two sides of a distinguished triangle in $K(\underline{Hw})$.

3. *If $M \in \underline{C}_{w \le n}$ (resp. $M \in \underline{C}_{w \ge n}$), then $t(M)$ belongs to $K(\underline{Hw})_{w^{st} \le n}$ (resp. to $K(\underline{Hw})_{w^{st} \ge n}$).*

   Moreover, if M is left or right w-degenerate (see Definition 1.2.2(7)), then $t(M) = 0$.

[3] Actually, as was originally noted by O. Schnürer, one has to change the signs of differentials in complexes to make the weight complex functor of ibid. compatible with our 'weak' version; see Remark 1.3.5(3) of [7] or §2 and Definition 5.7 of [25]. However, this subtlety does not appear to affect any of the applications of weight complexes known to the author; hence, the reader may probably ignore it.

Moreover, t^{st} also exists whenever w is bounded and $\underline{C}$ is the underlying category of a stable derivator; combine Theorem 7.1 of ibid. with Theorem 7 of [22].

4. *Let $\underline{C}'$ be a triangulated category endowed with a weight structure w'; let $F : \underline{C} \to \underline{C}'$ be a weight-exact functor. Then the composition $t' \circ F$ is isomorphic to $K_{\mathfrak{w}}(\underline{HF}) \circ t$, where t' is a weight complex functor corresponding to w', and the functor $K_{\mathfrak{w}}(\underline{HF}) : K_{\mathfrak{w}}(\underline{Hw}) \to K_{\mathfrak{w}}(\underline{Hw'})$ is defined as in Lemma 1.3.2(3).*
5. *For any morphism of triangles*

$$\begin{array}{ccccc} w_{\le n-1}M & \xrightarrow{a} & w_{\le n}M & \longrightarrow & \mathrm{Cone}(a) \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle h} \\ w_{\le n-1}M' & \xrightarrow{b} & w_{\le n}M' & \longrightarrow & \mathrm{Cone}(b) \end{array} \tag{1.3.1}$$

where a, b, c and d are the corresponding w-truncations of id_M, $\mathrm{id}_{M'}$ and g (see Definition 1.2.5), respectively, we have $\mathrm{Cone}(a), \mathrm{Cone}(b) \in \underline{C}_{w=n}$. Moreover, $t(g)$ is isomorphic as a $K_{\mathfrak{w}}(\underline{Hw})$-arrow to a morphism x whose $-n$th component $x^{-n} \in \mathrm{Mor}(\underline{Hw})$ equals $h[-n]$.

Furthermore, if $t(g)$ is isomorphic to a $K_{\mathfrak{w}}(\underline{Hw})$-morphism y such that $y^{-n} = 0$, then any choice of the rows in diagram 1.3.1 can be completed to the whole diagram with $h = 0$ in it.

Proof. Assertions 1–4 are given by Proposition 1.3.4(7,9,10,12) of [7].

Lastly, the first two parts of assertion 5 follow from the definition of t in ibid. and its 'furthermore' part easily follows from Proposition 1.3.4(13) of ibid. along with Lemma 1.3.2(5) above. □

2. On morphisms killing weights

In this section, we introduce and study the main new notions of this paper.

In §2.1, we define morphisms killing weights $m,\dots,n$ and objects without these weights. We give several equivalent definitions of these notions.

In §2.2, we establish several interesting properties of our notions. In particular, we prove that any object without weights $m,\dots,n$ admits a weight *decomposition avoiding these weights* (in the sense defined by J. Wildeshaus) if $\underline{C}$ is weight-Karoubian.

In §2.3, we prove that M is without weights $m,\dots,n$ if and only if $t(M)$ possesses this property.

In §2.4, we relate killing a weight m and objects without weights in a range to *pure* functors; those were introduced in §2.1 of [7].

2.1. Killing weights: equivalent definitions

Proposition 2.1.1. *Let $M,N \in \mathrm{Obj}\,\underline{C}$, $g \in \underline{C}(M,N)$ and $m \le n \in \mathbb{Z}$. Then the following conditions are equivalent.*

1. *There exists a choice of $w_{\le n}M$ and $w_{\ge m}N$ such that the composed morphism $w_{\le n}M \xrightarrow{x} M \xrightarrow{g} N \xrightarrow{y} w_{\ge m}N$ is zero (here x and y come from the corresponding weight decompositions; see Remark 1.2.3(2)).*

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2. *There exists a choice of* $w_{\leq n}M$ *and* $w_{\leq m-1}N$ *and of a morphism* h *making the following square commutative:*

$$\begin{array}{ccc} w_{\leq n}M & \xrightarrow{x} & M \\ \downarrow h & & \downarrow g \\ w_{\leq m-1}N & \longrightarrow & N. \end{array} \tag{2.1.1}$$

3. *There exists a choice of* $w_{\geq n+1}M$ *and* $w_{\geq m}N$ *and of a morphism* j *making commutative the square*

$$\begin{array}{ccc} M & \longrightarrow & w_{\geq n+1}M \\ \downarrow g & & \downarrow j \\ N & \xrightarrow{y} & w_{\geq m}N. \end{array} \tag{2.1.2}$$

4. *Any choices of* $WD_n(M)$ *and* $WD_{m-1}(N)$ *can be joined by a morphism that extends* g *in the sense of Definition 1.2.5.*

5. *For any choice of* $WD_n(M)$ *and* $WD_{m-1}(N)$ *and for the corresponding truncations* $a: w_{\leq m-1}M \to w_{\leq n}M$ *and* $b: w_{\leq m-1}N \to w_{\leq n}N$ *of* id_M *and* id_N *(see Lemma 1.2.7(1)), there exists a commutative diagram*

$$\begin{array}{ccccc} w_{\leq m-1}M & \xrightarrow{a} & w_{\leq n}M & \longrightarrow & M \\ \downarrow c & & \downarrow d & & \downarrow g \\ w_{\leq m-1}N & \xrightarrow{b} & w_{\leq n}N & \longrightarrow & N \end{array} \tag{2.1.3}$$

along with $h \in \underline{C}(w_{\leq n}M, w_{\leq m-1}N)$ *that turns the corresponding 'halves' of the left-hand square of diagram 2.1.3 into commutative triangles.*

6. *For any choice of the diagram 2.1.3 as above, its left-hand commutative square can be completed to a morphism of triangles as follows:*

$$\begin{array}{ccccc} w_{\leq m-1}M & \xrightarrow{a} & w_{\leq n}M & \longrightarrow & \mathrm{Cone}(a) \\ \downarrow c & & \downarrow d & & \downarrow 0 \\ w_{\leq m-1}N & \xrightarrow{b} & w_{\leq n}N & \longrightarrow & \mathrm{Cone}(b). \end{array} \tag{2.1.4}$$

7. *There exists a choice of diagram 2.1.3 whose left-hand square can be completed to a morphism of triangles as in diagram 2.1.4.*

Proof. Conditions 1, 2 and 3 are equivalent by Proposition 1.1.9 of [2].

Loc. cit. also implies that any of these conditions implies the existence of a morphism $WD_g: WD'_n(M) \to WD'_{m-1}(N)$ that extends g, where $WD'_n(M)$ (resp. $WD'_{m-1}(N)$) is the corresponding n-weight decomposition of M (resp. $m-1$-weight decomposition of N).

Next, Lemma 1.2.7(1) gives the existence of some choices of 'modification of weight decomposition' morphisms $WD_n(M) \to WD'_n(M)$ and $WD'_{m-1}(N) \to WD_{m-1}(N)$ that

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extend id_M and id_N, respectively. Composing WD_g with these morphisms (see Remark 1.2.6) we obtain condition 4. Conversely, this condition obviously implies conditions 1, 2 and 3.

Next, condition 5 clearly implies condition 2. Conversely, to obtain the commutative diagrams in condition 5 we apply Lemma 1.2.7(1) and take a and b to be the canonical truncations $w_{\le m-1}M \to w_{\le n}M$ and $w_{\le m-1}N \to w_{\le n}N$ of id_M and id_N, respectively. We also take $c = h \circ a$, and $d = b \circ h$.

Next, condition 6 clearly yields condition 7. Now, consider the exact sequence $\underline{C}(w_{\le n}M, w_{\le m-1}N) \to \underline{C}(w_{\le n}M, w_{\le n}N) \to \underline{C}(w_{\le n}M, \mathrm{Cone}(b))$ (for an arbitrary choice of diagram 2.1.3). If condition 7 is fulfilled, the composed morphism $w_{\le n}M \stackrel{d}{\to} w_{\le n}N \to \mathrm{Cone}(b)$ is zero; hence, there exists a morphism $h \in \underline{C}(w_{\le n}M, w_{\le m-1}N)$ making the corresponding triangle commutative. Combining this with the commutativity of the right-hand square in diagram 2.1.3 we obtain condition 2.

It remains to verify that condition 5 implies condition 6. The aforementioned long exact sequence gives the vanishing of the corresponding composed morphism $w_{\le n}M \to \mathrm{Cone}(b)$, whereas the exact sequence

$$\begin{aligned}\underline{C}(w_{\le n}M, w_{\le m-1}N) \to \underline{C}(w_{\le m-1}M, w_{\le m-1}N)\\ \to \underline{C}(\mathrm{Cone}(a)[-1], w_{\le m-1}N) \to \dots\end{aligned}$$

yields the vanishing of the composed morphism $\mathrm{Cone}(a)[-1] \to w_{\le m-1}N$. We obtain that diagram 2.1.4 is a morphism of triangles indeed. □

Now, we give the main original definitions of this paper.

Definition 2.1.2. Let $m \le n \in \mathbb{Z}$, $g \in \underline{C}(M,N)$.

1. We will say that g *kills weights* $m,\dots,n$ (and also that g *kills weight* m if $n = m$) if it satisfies the equivalent conditions of Proposition 2.1.1; denote the class of $\underline{C}$-morphisms killing weights $m,\dots,n$ by $\mathrm{Mor}_{\cancel{[m,n]}}\underline{C}$.

2. We say that M is *without weights* $m,\dots,n$ if id_M kills weights $m,\dots,n$; the class of $\underline{C}$-objects without weights $m,\dots,n$ will be denoted by $\underline{C}_{w\notin[m,n]}$.

Remark 2.1.3. Proposition 1.2.4(1) easily implies that these definitions are self-dual in the following natural sense: $g \in \mathrm{Mor}_{\cancel{[m,n]}}\underline{C}$ (resp. $M \in \underline{C}_{w\notin[m,n]}$) if and only if g kills w^{op}-weights $-n,\dots,-m$ (resp. M is without w^{op}-weights $-n,\dots,-m$) in $\underline{C}' = \underline{C}^{op}$.

2.2. Basic properties of our notions

Theorem 2.2.1. *Let $M,N,O \in \mathrm{Obj}\,\underline{C}$, $h \in \underline{C}(N,O)$, and assume that a morphism $g \in \underline{C}(M,N)$ kills weights $m,\dots,n$ for some $m \le n \in \mathbb{Z}$. Then the following statements are valid.*

1. *Assume $m \le m' \le n' \le n$. Then g also kills weights $m',\dots,n'$.*
2. *$\mathrm{Mor}_{\cancel{[m,n]}}\underline{C}$ is closed with respect to direct sums and retracts (i.e., $\bigoplus g_i$ kills weights $m,\dots,n$ if and only if all g_i do that).*
3. *$\mathrm{Mor}_{\cancel{[m,n]}}\underline{C}$ is a two-sided ideal of morphisms, that is, for any $h' \in \underline{C}(O,M)$ both $h \circ g$ and $g \circ h'$ kill weights $m,\dots,n$.*

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4. *Assume that* h *kills weights* $m',\ldots,m-1$ *for some* $m' < m$. *Then* $h \circ g$ *kills weights* $m',\ldots,n$.
5. *Let* $F : \underline{C} \to \underline{D}$ *be a weight-exact functor (with respect to a certain weight structure on* $\underline{D}$*) and assume that* h *kills weights* $m,\ldots,n$. *Then* $F(h)$ *kills these weights as well.*
6. *For* F *and* h *as in the previous assertion, assume that* F *is a full embedding and* $F(h) \in \mathrm{Mor}_{\not[m,n]}\underline{D}$. *Then* $h \in \mathrm{Mor}_{\not[m,n]}\underline{C}$.
7. *Assume that* O *is without weights* $m,\ldots,n$ *and also is without weights* $n+1,\ldots,n'$ *for some* $n' > n$. *Then* $O \in \underline{C}_{w \notin [m,n']}$.
8. O *is without weights* $m,\ldots,n$ *if and only if* O *is without weight* i *whenever* $m \le i \le n$.
9. *Assume that there exists a distinguished triangle*

$$X \to O \to Y \to X[1] \tag{2.2.1}$$

with $X \in \underline{C}_{w \le m-1}$, $Y \in \underline{C}_{w \ge n+1}$. *We call it a decomposition avoiding weights* $m,\ldots,n$ *for* M.

Then triangle 2.2.1 gives an l*-weight decomposition of* O *for every* $l \in \mathbb{Z}$, $m-1 \le l \le n$. *Moreover,* O *is without weights* $m,\ldots,n$, *and this triangle is unique up to a canonical isomorphism.*
10. *Assume that* $\underline{C}$ *is weight-Karoubian. Then the converse to the previous assertion is also valid, that is, every* O *without weights* $m,\ldots,n$ *admits a decomposition avoiding weights* $m,\ldots,n$.

Proof.

1. This is easy from Lemma 1.2.7(1); see Remark 1.2.6 and condition 2 of Proposition 2.1.1.
2. Proposition 1.2.4(4) implies that direct sums of l-weight decompositions are l-weight decompositions. Here, we take $l = n, m-1$. This implies the assertion easily; see condition 1 in Proposition 2.1.1.

Assertions 3 and 4 easily follow from Lemma 1.2.7(1) as well; see Remark 1.2.6 and condition 2 in Proposition 2.1.1.

5. If we vanish in condition 1 of Proposition 2.1.1, then we obtain this condition for $F(h)$.

6. For any choice of $w_{\le n}M$ and $w_{\ge m}N$, the composed morphism

$$F(w_{\le n}M) \to F(M) \stackrel{F(h)}{\to} F(N) \to F(w_{\ge m}N)$$

is zero (see condition 2 of Proposition 2.1.1); hence, $h \in \mathrm{Mor}_{\not[m,n]}\underline{C}$.

7. Since $\mathrm{id}_O \circ \mathrm{id}_O = \mathrm{id}_O$, the statement follows from assertion 4.

8. If O is without weight i whenever $m \le i \le n$, then, iterating the previous assertion, we obtain that O is without weights $m,\ldots,n$. Conversely, if $O \in \underline{C}_{w \notin [m,n]}$ and $m \le i \le n$, then id_O kills weight i (i.e., O is without weight i) according to assertion 1.

9. Each statement in this assertion easily follows from previous ones.

Indeed, triangle 2.2.1 gives the corresponding l-weight decompositions of O just by definition. We obtain that O is without weights $m,\dots,n$ immediately. Here, we can use either condition 2 or 3 of Proposition 2.1.1. This triangle (2.2.1) is canonical by Lemma 1.2.7(1); we take $M = M' = O$, $g = \mathrm{id}_O$, $m = n-1$, and $m' = n$ in it.

10. The idea is to ‘modify’ any fixed n-decomposition of O using Lemma 1.2.7(2).

We also fix an m-weight decomposition of O. According to condition 2 in Proposition 2.1.1, there exists a commutative square

$$\begin{array}{ccc} w_{\le n}O & \longrightarrow & O \\ \downarrow{\scriptstyle z} & & \downarrow{\scriptstyle \mathrm{id}_O} \\ w_{\le m-1}O & \longrightarrow & O. \end{array}$$

Next, Lemma 1.2.7(1) gives the existence and uniqueness of the square

$$\begin{array}{ccc} w_{\le m-1}O & \longrightarrow & O \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \mathrm{id}_O} \\ w_{\le n}O & \longrightarrow & O. \end{array}$$

Now, if we ‘compose’ these squares in the sense of Remark 1.2.6, then the aforementioned uniqueness statement implies $t = t \circ z \circ t$. Thus, the endomorphism $u = t \circ z$ is idempotent, and the square

$$\begin{array}{ccc} w_{\le n}O & \longrightarrow & O \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \mathrm{id}_O} \\ w_{\le n}O & \longrightarrow & O \end{array}$$

is commutative. Now, we apply Lemma 1.2.7(2); for X being the image of u, we obtain an n-weight decomposition $X \to O \to Y$. It remains to note that $X \in \underline{C}_{w\le m-1}$ since u factors through $w_{\le m-1}O$. □

Remark 2.2.2.

1. In the original Definition 1.10 of [28] O was said to be without weights $m,\dots,n$ if there exists a decomposition (2.2.1) for it. Thus, our definition of this notion is equivalent to loc. cit. if $\underline{C}$ is weight-Karoubian. Recall here that this is automatically the case if $\underline{C}$ is Karoubian according to the easy Proposition 1.2.4(7). Yet in §3.3 below, we demonstrate that this equivalence fails in general; see also §3.3.2 of [6] for a w-bounded (cf. Definition 1.2.2(9)) example of this sort.

   Hence, the uniqueness statement in Theorem 2.2.1(9) coincides with Corollary 1.9 of [28]. Moreover, Lemma 1.2.7 and Theorem 2.2.1(9) imply Proposition 1.7 of ibid. that is essentially as follows: If $X_i \to O_i \to Y_i$ are decompositions avoiding weights $m,\dots,n$, then any $g : O_1 \to O_2$ uniquely extends to a morphism of these triangles.
2. Combining parts 2 and 3 of our theorem, one immediately obtains that the sum of any two parallel morphisms killing weights $m,\dots,n$ kills these weights as well.

Moreover, part 2 implies that $\underline{C}_{w\notin[m,n]}$ is additive and retraction-closed in $\underline{C}$, whereas part 3 yields that every $\underline{C}$-morphism from M kills weights $m,\dots,n$ if (and only if) M is without these weights.

2.3. Relation to the weight complex functor

Theorem 2.3.1. *Let $g \in \underline{C}(M,N)$ (for some $M,N \in \operatorname{Obj}\underline{C}$); $m \le n \in \mathbb{Z}$.*

1. *Then g kills weight m if and only if $t(g) \backsim_{[-m,-m]} 0$; see Definition 1.3.1 for this notation.*
2. *If $\{f_i\}$ for $n \ge i \ge m$ form a chain of composable $\underline{C}$-morphism such that $t(f_i) \backsim_{[-i,-i]} 0$ for all i in this range, then $f_m \circ \cdots \circ f_{n-1} \circ f_n$ kills weights $m,\dots,n$.*
3. *M is without weights $m,\dots,n$ if and only if $t(\mathrm{id}_M) \backsim_{[-n,-m]} 0$.*
4. *Assume in addition that $\underline{C}$ is weight-Karoubian. Then M is without weights $m,\dots,n$ if and only if $t(M)$ is homotopy equivalent to a complex $C = (C^i)$ with $C^i = 0$ for $-n \le i \le -m$.*

Proof.

1. Immediate from Proposition 1.3.4(5); see condition 7 in Proposition 2.1.1.
2. Straightforward from assertion 1 combined with Theorem 2.2.1(4).
3. If $M \in \underline{C}_{w\notin[m,n]}$, then combining assertion 1 with Lemma 1.3.2(4) we obtain $t(\mathrm{id}_M) \backsim_{[-n,-m]} 0$. Conversely, if $t(\mathrm{id}_M) \backsim_{[-n,-m]} 0$, then $t(\mathrm{id}_M) \backsim_{[i,i]} 0$ for all i between $-n$ and $-m$; thus, applying the previous assertion to the composition $\mathrm{id}_M^{\circ n-m+1}$ we obtain that M is without weights $m,\dots,n$.
4. The 'if' implication follows from the previous assertion immediately; cf. Definition 1.3.1(2).

   Conversely, assume that M is without weights $m,\dots,n$. By Theorem 2.2.1(10), M possesses a decomposition avoiding weights $m,\dots,n$. Then for the corresponding objects X and Y (see diagram 2.2.1) Proposition 1.3.4(3) says that $t(X) \in K(\underline{Hw})_{w^{st} \le m-1}$ and $t(Y) \in K(\underline{Hw})_{w^{st} \ge n+1}$. Recalling the definition of w^{st} in Remark 1.2.3(1) and applying Proposition 1.3.4(2), we obtain a $K(\underline{Hw})$-distinguished triangle

$$T_X \to t(M) \to T_Y \to T_X[1],$$

   where T_X and T_Y have zero terms in degrees at most $-m$ and at least $-n$, respectively. Thus, we obtain the 'only if' implication. □

Remark 2.3.2. 1. Let us demonstrate that Theorem 2.3.1(2) generalizes Theorem 3.3.1(II) of [4]. The latter says that for $f = f_m \circ \cdots \circ f_{n-1} \circ f_n$ we have $f = 0$ whenever f_i are $\underline{C}_{[m,n]}$-morphisms (see Definition 1.2.2(3)) such that $t(f_i) = 0$.

Now, if this is the case, then $t(f_i) \backsim_{[-i,-i]} 0$ for $m \le i \le n$; hence, f kills weights $m,\dots,n$ by Theorem 2.3.1(2). Next, if $f \in \underline{C}(M,N)$ for $M,N \in \underline{C}_{[m,n]}$, then we can take $w_{\le n}M = M$ and $w_{\ge m}N = N$. Thus, $f = 0$ indeed; see condition 4 in Proposition 2.1.1.

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Thus, we obtain an alternative proof of loc. cit. that does not depend either on it or on Proposition 3.2.4 of ibid. (cf. Remark A.2.1(3) of [7]).

2. More criteria for killing weights for morphisms and for the absence of weights in objects are given by Theorems 2.1.2 and 2.2.2 of [12]. These matters are also studied in Theorem 3.1.1 of ibid., where the additional assumption of the existence of a t-structure adjacent to w (see Definition 1.2.2(3) of ibid.) was imposed. Note here that this assumption is fulfilled whenever w is purely compactly generated (see Theorem 4.1.2 below). In particular, it is valid if $w = w_{\Lambda}^{G}$ (see Proposition 4.2.1(2) and Remark 4.2.3(6) below).

Let us now improve the conservativity property of weight complexes given by Theorem 3.3.1(V) of [4].

Definition 2.3.3. 1. We say that w is left (resp., right) *nondegenerate* if all left (resp. right) w-degenerate objects (see Definition 1.2.2(7)) are zero.

We say that w is *nondegenerate* if it is both left and right nondegenerate.

2. We will say that $M \in \operatorname{Obj}\underline{C}$ is *w-degenerate* or *weight-degenerate* if $t(M)$ is zero in $K_{\mathfrak{w}}(\underline{Hw})$; hence, it is zero in $K(\underline{Hw})$ as well.

Theorem 2.3.4. *Let $g: M \to M'$ be a $\underline{C}$-morphism, $m \in \mathbb{Z}$.*

I.1. *Then $t(g)$ is an isomorphism if and only if* $\operatorname{Cone}(g)$ *is w-degenerate.*

2. *Any extension of a left w-degenerate object Y of $\underline{C}$ by a right w-degenerate object X is w-degenerate.*

3. *If M is an extension of a left w-degenerate object Y by $X_m \in \underline{C}_{w\le m}$ (resp. an extension of $Y_m \in \underline{C}_{w\ge m}$ by a right w-degenerate object X), then $t(M) \in K(\underline{Hw})_{w^{st}\le m}$ (resp. $t(M) \in K(\underline{Hw})_{w^{st}\ge m}$; see Remark 1.2.3(1)).*

II. *Assume that $\underline{C}$ is weight-Karoubian.*

1. *Then M is w-degenerate if and only if M is an extension of a left w-degenerate Y by a right w-degenerate X; cf. assertion I.2.*

2. *$t(M) \in K(\underline{Hw})_{w^{st}\le m}$ (resp. $t(M) \in K(\underline{Hw})_{w^{st}\ge m}$) if and only if M is an extension of a left w-degenerate Y by $X_m \in \underline{C}_{w\le m}$ (resp. an extension of $Y_m \in \underline{C}_{w\ge m}$ by a right w-degenerate X; cf. assertion I.3).*

Proof. I.1. Immediate from Proposition 1.3.4(2) combined with the conservativity of the projection functor $p_w : K(\underline{B}) \to K_{\mathfrak{w}}(\underline{B})$ given by Lemma 1.3.2(1).

2. If N is left or right w-degenerate, then $t(N) = 0$ according to Proposition 1.3.4(3). Hence, the assertion follows from the previous one.

3. Similarly to the previous assertion, it suffices to combine Proposition 1.3.4(3) with assertion I.1.

II. If $t(M) \in K(\underline{Hw})_{w^{st}\le m}$, then for every $n > m$ we have $\operatorname{id}_{t(M)} \backsim_{[-n,-m-1]} 0$; see Lemma 1.3.2(6).

Since $\underline{C}$ is weight-Karoubian, for every $n > 0$ Theorem 2.2.1(10) gives a distinguished triangle

$$X_m \to M \to Y_{n+1} \to X_m[1] \tag{2.3.1}$$

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with $X_m \in \underline{C}_{w \le m}$ and $Y_{n+1} \in \underline{C}_{w \ge n+1}$. All of these triangles are isomorphic to the one for $n = m+1$ by the uniqueness statement in Theorem 2.2.1(9). Hence, $T = Y_{m+1}$ is left w-degenerate and we obtain the 'only if' implication for this case. Next, the 'if' implication is given by assertion I.3.

The proofs of the two remaining statements are similar and left to the reader. □

Remark 2.3.5. 1. Thus, we get a precise answer to the question when $t(g)$ is an isomorphism in the weight-Karoubian case. In particular, the weight complex functor is conservative if and only if w is nondegenerate.

2. To obtain the latter statement in the general case, one should combine part I.1 of our theorem with Theorem 3.1.3 below. Moreover, that theorem contains several equivalent conditions for $t(M)$ to belong to $K(\underline{Hw})_{w^{st} \le 0}$. However, those formulations require Definition 3.1.1(2); see §3.3 below.

2.4. On the relation to pure functors

Let us prove that the assumption that a $\underline{C}$-morphism g kills a given weight m can be expressed in terms of pure functors.

Definition 2.4.1. Assume that $\underline{C}$ is endowed with a weight structure w.

We will say that a (co)homological functor H from $\underline{C}$ into an abelian category $\underline{A}$ is *w-pure* or just *pure* if H kills both $\underline{C}_{w \ge 1}$ and $\underline{C}_{w \le -1}$.

Theorem 2.4.2. 1. *Let $\mathcal{A} : \underline{Hw} \to \underline{A}$ be an additive functor, where $\underline{A}$ is an abelian category. For $M \in \operatorname{Obj} \underline{C}$ and $t(M) = (M^j)$, we set $H(M) = H^{\mathcal{A}}(M)$ to be the zeroth homology of the complex $(\mathcal{A}(M^j))$. Then $H(-)$ yields a pure homological functor, and the assignment $\mathcal{A} \mapsto H^{\mathcal{A}}$ is natural in $\mathcal{A}$.*

2. *The correspondence $\mathcal{A} \to H^{\mathcal{A}}$ is an equivalence of categories between the following categories of functors:* $\operatorname{AddFun}(\underline{Hw}, \underline{A})$ *and the category of pure homological functors from $\underline{C}$ into $\underline{A}$.*

3. *Dually, the correspondence sending a contravariant functor $\mathcal{A}'$ into the functor $H_{\mathcal{A}'}$ that maps M into the zeroth homology of the complex $(\mathcal{A}'(M^{-j}))$ gives an equivalence of categories between* $\operatorname{AddFun}(\underline{Hw}^{op}, \underline{A})$ *and the category of pure cohomological functors from $\underline{C}$ into $\underline{A}$.*

Proof. Assertions 1 and 2 are contained in Theorem 2.1.2 of [7], and assertion 3 is their dual; cf. Proposition 1.2.4(1) or Remark 2.1.3(1) of ibid. □

We also need the following definitions.

Definition 2.4.3. Assume that $\underline{C}$ is *smashing*, that is, closed with respect to (small) coproducts.

1. We will say that $\underline{C}$ is a *Brown category* if any cohomological functor from $\underline{C}$ into Ab that converts $\underline{C}$-coproducts into products of groups is representable in $\underline{C}$.

2. We will say that a weight structure w on $\underline{C}$ is *smashing* if the class $\underline{C}_{w \ge 0}$ is closed with respect to $\underline{C}$-coproducts (cf. Proposition 1.2.4(3)).

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Proposition 2.4.4. *Assume that $\underline{C}$ is endowed with a weight structure w, $g: M \to N$ is a $\underline{C}$-morphism, and $j \in \mathbb{Z}$.*

I. *Then the following assumptions are equivalent to $g \in \mathrm{Mor}_{\not[j,j]}\underline{C}$.*

1. *$H^j_{\mathcal{A}}(g) = 0$ for every pure cohomological functor $H_{\mathcal{A}}$ as above.*

2. *$H^j_{\mathcal{A}}(g) = 0$ for every pure cohomological functor $H_{\mathcal{A}}$ coming from $\mathcal{A} \in \mathrm{AddFun}(\underline{Hw}^{op}, \mathrm{Ab})$ such that $\mathcal{A}$ converts $\underline{Hw}$-coproducts into products of groups.*

3. *$H_j^{\mathcal{A}}(g) = 0$ for every pure homological functor $H^{\mathcal{A}}$.*

II. *Assume in addition that $\underline{C}$ is a Brown category and w is smashing. Then the functors $H_{\mathcal{A}}$ as in condition I.3 are the pure representable ones.*

Proof. I. Proposition 1.3.4(1) enables us to assume $j = 0$.

Next, if $g \in \mathrm{Mor}_{\not[0,0]}\underline{C}$, then condition 1 is fulfilled by Proposition 1.3.4(2) combined with Theorem 2.3.1(1). Moreover, condition 1 clearly implies condition 2.

Furthermore, if $g \notin \mathrm{Mor}_{\not[0,0]}\underline{C}$, then $t(g) \not\backsim_{[0,0]} 0$ by Theorem 2.3.1(1); here, we use the notation of Definition 1.3.1(2). Thus, Proposition B.2(8) of [7] gives the existence of a functor $\mathcal{A}$ as in assertion 2 such that $H_{\mathcal{A}}(g) \neq 0$. Hence, condition 2 implies $g \in \mathrm{Mor}_{\not[0,0]}\underline{C}$.

Lastly, $g \in \mathrm{Mor}_{\not[j,j]}\underline{C}$ if and only if condition I.3 is fulfilled since I.3 is the categorical dual of condition I.1; see also Remark 2.1.3.

II. This is Proposition 2.3.2(8) of ibid. □

3. Non-Karoubian generalizations and a conservativity application

In §3.1, we extend Theorem 2.3.4(II) to the case where $\underline{C}$ is not necessarily weight-Karoubian.

In §3.2, we apply our results to prove that certain weight-exact functors are 'conservative up to weight-degenerate objects'; we also discuss the relation of this proposition to the corresponding results of [29] and [7].

In §3.3, we consider an example; it demonstrates that the modifications made in §3.1 to generalize Theorem 2.3.4(II) cannot be avoided.

3.1. Main statements in non-weight-Karoubian categories

Definition 3.1.1. 1. We call a triangulated category $\underline{C}' \subset \mathrm{Kar}(\underline{C})$ a *weight-Karoubian extension* of $\underline{C}$ if $\underline{C}'$ contains $\underline{C}$, the retraction-closures of $\underline{C}_{w\le 0}$ and $\underline{C}_{w\ge 0}$ in $\underline{C}'$ give a weight structure w' on it, and $\underline{Hw}'$ is Karoubian.

2. We say that an object M of $\underline{C}$ is *essentially w-positive* (resp. *essentially w-negative*) if M is a retract of some $\tilde{M} \in \mathrm{Obj}\,\underline{C}$ that is an extension of an element of $\underline{C}_{w\ge 0}$ by a right w-degenerate object of $\underline{C}$ (resp. of a left w-degenerate object by an element of $\underline{C}_{w\le 0}$; see Definition 1.2.2(7)).

Proposition 3.1.2. 1. *Let $\underline{C}'$ be a weight-Karoubian extension of $\underline{C}$. Then the embedding $\underline{C} \to \underline{C}'$ is weight-exact with respect to (w, w'), and $\underline{C}'_{w'=0}$ equals the retraction-closure of $\underline{C}_{w=0}$ in $\underline{C}'$.*

2. *Any $(\underline{C}, w)$ possesses a weight-Karoubian extension.*

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Proof. Assertion 1 immediately follows from Theorem 2.2.2(I.1) of [9], and invoking part III.1 of loc. cit., one also obtains assertion 2. □

Now, we generalize Theorem 2.3.4(II) to arbitrary weighted categories; see Definition 2.3.3, Remark 1.3.3, and Theorem 2.4.2 for the definitions.

Theorem 3.1.3. *The following assumptions on $M \in \operatorname{Obj} \underline{C}$ are equivalent.*

1. *M is w-degenerate (resp. $t(M) \in K(\underline{Hw})_{w^{st} \leq 0}$).*
2. *M can be presented as an extension of a left w'-degenerate object Y' of $\underline{C}$ by a right w'-degenerate object X' (resp. by $X'_0 \in \underline{C}'_{w' \leq 0}$; cf. diagram 2.3.1) in a weight-Karoubian extension $(\underline{C}', w')$ of $(\underline{C}, w)$.*
3. *In any weight-Karoubian extension of $\underline{C}$, there exists a presentation of M as in condition 2.*
4. *M is a $\underline{C}$-retract of an extension $\tilde{M}$ of a left w-degenerate object Y of $\underline{C}$ by a right w-degenerate X (resp. M is essentially w-negative in the sense of Definition 3.1.1(2)).*
5. *The object $M \bigoplus M[-1]$ is an extension of a left w-degenerate $Y \in \operatorname{Obj} \underline{C}$ by a right w-degenerate X (resp. by $X_0 \in \underline{C}_{w \leq 0}$).*
6. *M is without weight i for all $i \in \mathbb{Z}$ (resp. for all $i > 0$).*
7. *$H_i(M) = 0$ for all $i \in \mathbb{Z}$ (resp. $i > 0$) and every w-pure homological functor H from $\underline{C}$.*
8. *$H^i_{\mathcal{A}}(M) = \{0\}$ for all $i \in \mathbb{Z}$ (resp. for $i > 0$) and every additive functor $\mathcal{A} : \underline{Hw}^{op} \to \mathrm{Ab}$ that respects products; here we use the notation of Theorem 2.4.2(3).*

Proof. Let us study the conditions in brackets (that correspond to the essential w-negativity of M).

Clearly, the corresponding version of condition 5 implies condition 4. 2 follows from 3 since a weight-Karoubian extension $(\underline{C}', w')$ of $\underline{C}$ exists; see Proposition 3.1.2(2).

Condition 1 is easily seen to be equivalent to condition 6 according to Theorem 2.3.1(3,1) combined with Lemma 1.3.2(4,6). Moreover, condition 1 is equivalent to $t(M) \backsim_{[i,i]} 0$ for all $i < 0$ by the latter lemma; thus, applying Proposition 2.4.4(I), we obtain that this condition is also equivalent to conditions 7 and 8.

Next, for any weight-Karoubian extension $\underline{C}'$ of $\underline{C}$ the complex $t_w(M)$ belongs $K(\underline{Hw})_{w^{st} \leq 0}$ if and only if it belongs to $K(\underline{Hw}')_{w^{st} \leq 0}$; see Proposition 1.2.5(1) of [10] that easily follows from Proposition 1.2.4(2) above. Applying Proposition 1.3.4(4), we obtain that condition 4 implies 1.

Now, we fix some $(\underline{C}', w')$ and apply Theorem 2.3.4(II.2) to $\underline{C}'$. We obtain that condition 1 implies condition 3.

It remains to deduce condition 5 from condition 2. Any $N' \in \operatorname{Obj} \underline{C}'$ is the image of an idempotent $p \in \underline{C}(N, N)$ for some $N \in \operatorname{Obj} \underline{C}$ (see §1.1), and $\operatorname{Cone}(p) \cong N' \bigoplus N'[1] \in \operatorname{Obj} \underline{C}$; cf. Lemma 2.2 of [27]. Hence, the direct sum of the $\underline{C}'$-'decomposition' of M given by condition 2 with its shift by $[-1]$ yields condition 5.

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The equivalence of the conditions corresponding to $t(M)=0$ is similar; note that the functors $K(\underline{Hw}) \to K(\underline{Hw'}) \to K_{\mathfrak{w}}(\underline{Hw'})$ are conservative. □

Corollary 3.1.4. *If w is a weight structure on $\underline{C}$ and M is its object, then the following conditions are equivalent.*

1. $t(M) \in K(\underline{Hw})_{w^{st}\geq 0}$.
2. *M is without weight i for all $i<0$.*
3. *M is an extension of some $Y_0' \in \underline{C}'_{w'\geq 0}$ by a right w'-degenerate object X' in some weight-Karoubian extension $(\underline{C}', w')$ of $\underline{C}$.*
4. *In any weight-Karoubian extension of $\underline{C}$ there exists a presentation of M as in condition 3.*
5. *M is essentially w-positive in the sense of Definition 3.1.1(2).*
6. *The object $M \bigoplus M[1]$ is an extension of $Y_0 \in \underline{C}_{w\geq 0}$ by a right w-degenerate X.*
7. *$H^i_{\mathcal{A}}(M) = \{0\}$ for all $i<0$ and all additive functors $\mathcal{A}: \underline{Hw}^{op} \to \mathrm{Ab}$ that respect products.*

Proof. Conditions 1–6 are the categorical duals of the corresponding conditions in Theorem 3.1.3, whereas conditions 2 and 7 are equivalent according to Proposition 2.4.4(I). □

Remark 3.1.5. 1. Clearly, the formulation of conditions 8 in Theorem 3.1.3 and condition 7 of Corollary 3.1.4 can be combined with Proposition 2.4.4(II) to obtain the following statement: If $\underline{C}$ is a Brown category and w is smashing, then an object M is w-degenerate (resp. essentially w-negative, resp. essentially w-positive) if and only if $H^i(M) = \{0\}$ for any pure representable H and any $i \in \mathbb{Z}$ (resp. any $i>0$, resp. any $i<0$).

2. [9] contains a lot of information on weight-Karoubian extensions and related matters. In particular, the smallest strict triangulated subcategory $\underline{C}'$ of $\mathrm{Kar}(\underline{C})$ that contains both $\underline{C}$ and $\mathrm{Kar}(\underline{Hw})$ is essentially the minimal weight-Karoubian extension of $\underline{C}$; thus, it makes sense to apply conditions 3 of Theorem 3.1.3 and 4 of Corollary 3.1.4 for this choice of $\underline{C}'$.

3. Moreover, in §3.1 of ibid. it was demonstrated that there does not have to exist a weight structure on $\mathrm{Kar}(\underline{C})$ whose restriction to $\underline{C}$ equals w; thus, weight-Karoubian extensions are necessary for our arguments.

Now, let us prove a few results closely related to our theorem. The reader may consult Definition 1.2.2(8, 6) for some notions mentioned below.

Proposition 3.1.6. *Let w be a weight structure on $\underline{C}$.*

1. *Assume that w is left (resp. right) nondegenerate.*

Then every weight-degenerate object of $\underline{C}$ is right (resp. left) w-degenerate, and every essentially w-negative (resp. essentially w-positive) object belongs to $\underline{C}_{w\leq 0}$ (resp. to $\underline{C}_{w\geq 0}$).

2. *Assume that w is left nondegenerate, an object M of $\underline{C}$ is weight-degenerate and either M is w-bounded below or w is also right nondegenerate. Then M is zero.*

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3. *For an object M of $\underline{C}$, assume that $\underline{Hw}$-complexes $t_j = (M_j^i)$ for $j = 1,2$ are $K(\underline{Hw})$-isomorphic to $t(M)$, and $F : \underline{C} \to (\underline{C}', w')$ is a weight-exact functor that annihilates the groups $\underline{C}(M_1^i, M_2^i)$ for all $i \in \mathbb{Z}$.*

Then $F(M)$ is w'-degenerate.

4. *Assume that $\underline{C}_1$, $\underline{C}_2$, and $\underline{C}_3$ are full triangulated subcategories of $\underline{C}$ such that w restricts to them; suppose that the Verdier localization functor $F : \underline{C} \to \underline{C}' = \underline{C}/\underline{C}_3$ exists (i.e., all morphism classes in this localization are sets) and all $\underline{Hw}$-morphisms between elements of the corresponding classes $\underline{C}_{1,w_1=0}$ and $\underline{C}_{2,w_2=0}$ are killed by F.*

Then there is a unique weight structure w' on $\underline{C}'$ such that F is weight-exact, and for any $M \in \operatorname{Obj}\underline{C}_1 \cap \operatorname{Obj}\underline{C}_2$ the object $F(M)$ is w'-degenerate. Moreover, if w' is left nondegenerate, then $F(M)$ is w'-right degenerate, and M belongs to $\operatorname{Kar}_{\underline{C}}(\operatorname{Obj}\underline{C}_3)$ whenever M is w-bounded below.

Proof. 1. By axiom 1.2.1(i), retracts of left (resp. right) w-degenerate objects are left (resp. right) w-degenerate. The assertion follows easily.

2. According to the previous assertion, M is right weight-degenerate, that is, it belongs to $\cap_{i\in\mathbb{Z}}\underline{C}_{w\le i}$. On the other hand, M belongs to $\underline{C}_{w\ge i+1}$ for some $i \in \mathbb{Z}$ (and it actually belongs to all of these classes in the second case by the previous assertion). Since $\underline{C}_{w\le i} \perp \underline{C}_{w\ge i+1}$, $M \perp M$; hence, $M = 0$.

3. Let m be a $K_{\mathfrak{w}}(\underline{Hw})$-isomorphism $t_1 \to t_2$. Then, for the functor $K_{\mathfrak{w}}(\underline{HF}) : K_{\mathfrak{w}}(\underline{Hw}) \to K_{\mathfrak{w}}(\underline{Hw}')$ given by Lemma 1.3.2(3), we clearly have $K_{\mathfrak{w}}(\underline{HF})(m) = 0$. Here, $\underline{HF}$ is the restriction of F to hearts. Since $K_{\mathfrak{w}}(\underline{HF})(m)$ is also an isomorphism, we obtain $K_{\mathfrak{w}}(\underline{HF})(t_1) = 0$. On the other hand, by Proposition 1.3.4(4) we have $t_{w'}(F(M)) \cong K_{\mathfrak{w}}(\underline{HF})(t(M)) \cong K_{\mathfrak{w}}(\underline{HF})(t_1)$; hence, $t_{w'}(F(M)) = 0$, as desired.

4. w' exists according to Proposition 8.1.1(1) of [4], and w' is uniquely determined by w according to Proposition 3.1.1(1) of [10].

Next, Proposition 1.3.4(4) gives the existence of $\underline{Hw}$-complexes t_1 and t_2 such that $t(M) \cong t_1 \cong t_2$ (both in $K_{\mathfrak{w}}(\underline{Hw})$ and in $K(\underline{Hw})$; see Lemma 1.3.2(1)). The terms M_1^i belong to $\underline{C}_{1,w_1=0}$, and the terms of t_2 belong to $\underline{C}_{2,w_2=0}$. Thus, $F(M)$ is w'-degenerate by assertion 3.

By assertion 1, it follows that $F(M)$ is right w'-degenerate whenever w' is left nondegenerate. Lastly, if M is weight-bounded below, then $F(M)$ also is. Hence, our assumptions imply that $F(M) = 0$ according to assertion 2. Thus, M belongs to $\operatorname{Kar}_{\underline{C}}(\operatorname{Obj}\underline{C}_3)$. Here, we apply the well-known Lemma 2.1.33 of [23]. □

We will describe some consequences of Proposition 3.1.6(4) in Corollary 4.1.4 and Remark 4.1.5 below.

3.2. A conservativity application

Our results imply that certain weight-exact functors are 'almost conservative'.

Proposition 3.2.1. *Let $F : (\underline{C}, w) \to (\underline{C}', w')$ be a weight-exact functor.*

Assume that the induced functor $\underline{HF} : \underline{Hw} \to \underline{Hw}'$ is full. Every $\underline{Hw}$-endomorphism killed by $\underline{HF}$ is nilpotent, and for some $M \in \operatorname{Obj}\underline{C}$ the object $F(M)$ belongs to $\underline{C}'_{w'\notin[m,n]}$ for some $m \le n \in \mathbb{Z}$ (resp. $F(M)$ is w'-degenerate, resp. essentially w'-positive, resp. essentially w'-negative).

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Then M is without weights $m,\dots,n$ (resp. M is weight-degenerate, resp. essentially w-positive, resp. essentially w-negative).

Consequently, $M=0$ if $F(M)$ is w'-degenerate and w is nondegenerate.

Proof. Let us prove the first statement in our proposition for $m=n$; thus, assume that $F(M)$ is without weight m. We should prove that this assumption implies $M\in \underline{C}_{w\notin[m,m]}$; we call this implication Claim (*).

According to Theorem 2.3.1(3), this claim is equivalent to the following one: $t(\mathrm{id}_M)\backsim_{[-m,-m]}0$ whenever $t_{w'}(\mathrm{id}_{F(M)})\backsim_{[-m,-m]}0$.

Now, we can assume that $t_{w'}(F(M))$ is obtained from the weight complex $t(M)=(M^i,d^i)$; see Proposition 1.3.4(4). Thus, there exist morphisms $h'\in \underline{Hw}'(F(M^{-m}),F(M^{-m-1}))$ and $j'\in \underline{Hw}'(F(M^{1-m}),F(M^{-m}))$ such that $\mathrm{id}_{F(M^{-m})}=j'\circ F(d^{-m})+F(d^{-m-1})\circ h'$. Since the restriction of F to $\underline{Hw}$ is full and conservative, we can lift h' and j' to some $\underline{Hw}$-morphisms h and j, and for lifts of this sort the endomorphism

$$\varepsilon=\mathrm{id}_{M^{-m}}-j\circ d^{-m}-d^{-m-1}\circ h:M^{-m}\to M^{-m}$$

is nilpotent. Hence, there exists $n>0$ such that

$$\mathrm{id}_{M^{-m}}=(j\circ d^{-m}+d^{-m-1}\circ h)(\mathrm{id}_{M^{-m}}+\varepsilon+\varepsilon^2+\cdots+\varepsilon^{n-1}).$$

Therefore, $\mathrm{id}_{M^{-m}}$ can be presented in the form $a\circ d^{-m}+d^{-m-1}\circ b$ for some $a\in \underline{Hw}(M^{-m},M^{-m-1})$ and $b\in \underline{Hw}(M^{1-m},M^{-m})$; one can just write down explicit formulas for a and b in this setting. Thus, $t(\mathrm{id}_M)\backsim_{[-m,-m]}0$.

Next, the general case of the 'without weights $m,\dots,n$ part' follows from Claim (*) immediately according to Theorem 2.2.1(8).

Lastly, our remaining statements follow from Claim (*) as well; see condition 6 in Theorem 3.1.3, condition 2 in Corollary 3.1.4 and Proposition 3.1.6(2). □

Remark 3.2.2.

1. Our proposition essentially says that F is 'conservative (and detects weights; cf. Remark 1.5.3(1) of [7]) up to weight-degenerate objects'. The latter feature of the result is unavoidable. Indeed, arguing similarly to Proposition 4.2.1(1) of [10] one can easily prove that for any set W of weight-degenerate objects of $\underline{C}$ the localization of $\underline{C}$ by the triangulated subcategory generated by W gives a weight-exact functor that restricts to a full embedding $\underline{Hw}\to\underline{C}'$.
2. Let us relate our proposition to earlier results.

   In Theorem 1.5.1(1,2) of [7], only the case where M is bounded either above or below was considered. On the other hand, $\underline{HF}$ was just assumed to be full and conservative. Thus, neither our proposition implies loc. cit. nor the converse is valid. Note also that, in the case where the endomorphisms killed by $\underline{HF}$ are nilpotent, all the conclusions of loc. cit. can be easily deduced from our proposition.

   Next, we recall that in Theorem 2.8 of [29] (as well as in the weaker Theorem 2.5 of ibid.) it was assumed that F is weight-exact, $\underline{HF}$ is full and conservative, $\underline{Hw}$ is Karoubian and *semiprimary* and w is bounded (see Definition 1.2.2(9)). Now, these assumptions imply that endomorphisms killed by $\underline{HF}$ are nilpotent. Indeed, the conservativity of $\underline{HF}$ means that these endomorphisms belong to the *radical* of $\underline{Hw}$, and semiprimality means that all elements of this radical are nilpotent. Thus, our proposition implies Theorem 2.8 of ibid.

3. One can obtain plenty of examples to our proposition by taking $F = K(G) : K(\underline{B}) \to K(\underline{B}')$; here, $G : \underline{B} \to \underline{B}'$ is a full additive functor such that every $\underline{B}$-endomorphism killed by G is nilpotent, and one takes w and w' to be the corresponding stupid weight structures. Note that in this case we have $\underline{HF} \cong \mathrm{Kar}(G)$ (see Remark 1.2.3(1)); thus, $\underline{HF}$ fulfills our assumptions as well.

   Since w is nondegenerate, our proposition implies that F is conservative. The author wonders whether a proof of this statement 'without killing weights' exists. This may allow to modify the assumptions on $\underline{HF}$ in Proposition 3.2.1.

   Other examples to our proposition are given by Corollary 4.1.4(2).

3.3. An 'indecomposable' weight-degenerate object

Let $K^b(L-\mathrm{vect})$ be the category of bounded complexes of vectors spaces over a field L. We take $\underline{C}' = (K^b(L-\mathrm{vect}))^3$ and $\underline{C}$ to be the subcategory of $\underline{C}'$ consisting of objects whose 'total Euler characteristic' is even in the following sense: The sum of dimensions of all homology of all the three components of $M = (M_1, M_2, M_3)$ should be even. We define $\underline{C}'_{w' \le 0}$ as the class of all $M = (M_1, M_2, M_3)$ such that $M_1 = 0$ and M_2 is acyclic in negative degrees; $M \in \underline{C}'_{w' \ge 0}$ if $M_3 = 0$ and M_2 is acyclic in positive degrees. We set $\underline{C}_{w \le 0} = \underline{C}'_{w \le 0} \cap \mathrm{Obj}\,\underline{C}$ and $\underline{C}_{w \ge 0} = \underline{C}'_{w \ge 0} \cap \mathrm{Obj}\,\underline{C}$. Obviously, w' is a weight structure on $\underline{C}'$. w is a weight structure on $\underline{C}$ since for $(M_1, M_2, M_3) \in \mathrm{Obj}\,\underline{C}$ any triangle of the form

$$(0, M', M_3) \to (M_1, M_2, M_3) \to (M_1, M'', 0),$$

where $M' \to M_2 \to M''$ is a w^{st}-decomposition of M_2 with the corresponding parities of the Euler characteristics, gives a w-decomposition. Clearly, $(\underline{C}', w')$ is a weight-Karoubian extension of $\underline{C}$.

Take $M = (L, 0, L)$ and a distinguished triangle $X' \to M \to Y'$ with $X' = (0, 0, L)$ and $Y' = (L, 0, 0)$. Then X' (resp. Y') is right (resp. left) w'-degenerate. Thus, M is weight-degenerate and without weight 0 both in $\underline{C}'$ and in $\underline{C}$; see conditions 1 and 2 in Theorem 3.1.3 and Theorem 2.2.1(9,6).

Next, for a distinguished triangle $X \to M \to Y$ with $X \in \underline{C}'_{w' \le -1}$, $Y \in \underline{C}'_{w' \ge 1}$, we have $X \cong X'$ and $Y \cong Y'$ by Theorem 2.2.1(9). Since X' (as well as Y') is not an object of $\underline{C}$, X is not an object of $\underline{C}$ either. Thus, neither of the parts of Theorem 2.3.4(II) nor Theorem 2.2.1(10) extends to $\underline{C}$.

Consequently, the notions of essential w-positivity and w-negativity are necessary for Theorem 3.1.3 and Corollary 3.1.4.

Note also that $t_w(M) = 0$; hence, Theorem 2.3.1(4) does not extend to $\underline{C}$ as well.

4. On 'topological' examples and converse Hurewicz theorems

In this section, we discuss the applications of our results to equivariant stable homotopy categories as well as to general *purely compactly generated* weight structures. We significantly extend the main results of [7, §4].

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In §4.1, we recall some properties of weight structures generated by sets of compact objects in their hearts and apply the main results of the previous sections to this setting. As an application, we establish important prerequisites for [5].

In §4.2, we apply our results to prove some new properties of the spherical weight structure w^G on the equivariant stable homotopy category $SH(G)$ of G-spectra (w^G was introduced in [7]; actually, we work in a somewhat more general context). We obtain a certain converse Hurewicz theorem in this setting. Moreover, the theory of objects without weights gives canonical 'decompositions' of spectra whose singular homology vanishes in two subsequent degrees; see Theorem 4.2.4(5) and Remark 4.2.3(2) below.

4.1. On purely compactly generated weight structures

In this section, we always assume that $\underline{C}$ is a smashing triangulated category; see Definition 2.4.3.

Definition 4.1.1. Let $\mathcal{P}$ be a full subcategory of $\underline{C}$.

1. We will say that $\mathcal{P}$ is *connective* in $\underline{C}$ if $\mathcal{P} \perp (\cup_{i>0}\mathcal{P}[i])$.[4]
2. Let $\mathcal{P}'$ be the category of 'formal coproducts' of objects of $\mathcal{P}$, that is, the objects of $\mathcal{P}'$ are of the form $\coprod_i P_i$ for families of $P_i \in \operatorname{Obj}\mathcal{P}$, and
$$\mathcal{P}'(\coprod M_i, \coprod N_j) = \prod_i (\bigoplus_j \mathcal{P}(M_i, N_j)). \tag{4.1.1}$$
Then we will call the idempotent completion of $\mathcal{P}'$ the *smashing idempotent completion* of $\mathcal{P}$.
3. We will say that a full triangulated subcategory $\underline{D} \subset \underline{C}$ is *localizing* whenever $\underline{D}$ is closed with respect to $\underline{C}$-coproducts. Respectively, we will call the smallest localizing subcategory of $\underline{C}$ that contains a given subcategory $\mathcal{P} \subset \underline{C}$ the *localizing subcategory of $\underline{C}$ generated by $\mathcal{P}$*.
4. An object M of $\underline{C}$ is said to be *compact* if the functor $H^M = \underline{C}(M,-) : \underline{C} \to \mathrm{Ab}$ respects coproducts.
5. We will say that $\underline{C}$ is *compactly generated* by its subcategory $\mathcal{P}$ if $\mathcal{P}$ is small, generates $\underline{C}$ as its own localizing subcategory and objects of $\mathcal{P}$ are compact.

Now, let us recall the main properties of *purely compactly generated* weight structures. These are the ones provided by the following theorem.

Theorem 4.1.2. *Let $\mathcal{P}$ be a connective subcategory of $\underline{C}$ that compactly generates it. Then the following statements are valid; cf. Definition 2.4.3.*

1. *$\underline{C}$ is a Karoubian Brown category.*
2. *There exists a unique smashing weight structure w on $\underline{C}$ such that $\mathcal{P} \subset \underline{Hw}$; w is left nondegenerate.*

[4]In earlier texts of the author, connective subcategories were called *negative* ones; another related notion is *silting*.

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3. *The corresponding* $\underline{C}_{w\le 0}$ *(resp.* $\underline{C}_{w\ge 0}$*) is the smallest subclass of* $\operatorname{Obj}\underline{C}$ *that is closed with respect to coproducts and extensions and contains* $\operatorname{Obj}\mathcal{P}[i]$ *for* $i\le 0$ *(resp. for* $i\ge 0$*), and* $\underline{Hw}$ *is equivalent to the smashing idempotent completion of* $\mathcal{P}$*.*
   Moreover, $\underline{C}_{w\ge 0} = (\cup_{i<0}\mathcal{P}[i])^{\perp}$*.*
4. *Let* H *be a cohomological functor from* $\underline{C}$ *into an abelian category* $\underline{A}$ *that converts all small coproducts into products. Then it is pure if and only if it kills* $\cup_{i\ne 0}\mathcal{P}[i]$*.*
5. *Let* $F:\underline{C}\to\underline{D}$ *be an exact functor that respects coproducts, where* $\underline{D}$ *is a triangulated category endowed with a smashing weight structure* v*. Then* F *is weight-exact if and only if it sends* $\mathcal{P}$ *into* $\underline{Hv}$*.*
6. *The category* $\underline{Ht}\subset\underline{C}$ *of* w*-pure representable functors from* $\underline{C}$ *is equivalent to the category* $\underline{A}_{\mathcal{P}}$ *of additive contravariant functors from* $\mathcal{P}$ *into* Ab *(i.e., we take those functors that respect the addition of morphisms).*[5] *Moreover,* $\underline{A}_{\mathcal{P}}$ *(and thus also* $\underline{Ht}$*) is Grothendieck abelian and has enough projectives. Consequently,* $\underline{A}_{\mathcal{P}}$ *is cogenerated by an object* I *that is injective in it; we fix the choice of* I*.*
   Furthermore, restricting functors representable by objects of $\mathcal{P}$ *to* $\underline{Hw}$ *one obtains a fully faithful functor* $\mathcal{A}_{\mathcal{P}}:\underline{Hw}\to\underline{A}_{\mathcal{P}}$ *whose essential image is the subcategory of projective objects of* $\underline{A}_{\mathcal{P}}$*.*
7. *The following assumptions on an object* M *of* $\underline{C}$ *are equivalent.*
   (i). $t(M)\in K(\underline{Hw})_{w^{st}\ge 0}$.
   (ii). $H_j^{\mathcal{A}_{\mathcal{P}}}(M)=0$ *for* $j<0$*; see Theorem 2.4.2(3) for this notation.*
   (iii). $M\perp(\cup_{j<0}\{I[j]\})$.
8. *Assume that there exists an integer* $j>0$ *such that* $\mathcal{P}\perp\cup_{i\ge j}\mathcal{P}[-i]$*. Then* w *is nondegenerate.*

Proof. Assertions 1–7 were mostly established in [23] and [10]; see §3.2 of [7] for the detail.

Next, if $\mathcal{P}\perp(\cup_{i\ge j}\mathcal{P}[-i])$, then the compactness of elements of $\mathcal{P}$ along with the description of $\underline{C}_{w\le 0}$ in assertion 3 imply that $\cup_{i\ge j}\mathcal{P}[i]\perp\underline{C}_{w\le 0}$. Thus, every right weight-degenerate element of $\underline{C}$ belongs to $(\cup_{i\in\mathbb{Z}}\mathcal{P}[i])^{\perp}$. Now, the well-known Proposition 8.4.1 of [23] says that the latter class is zero since $\underline{C}$ is compactly generated by $\mathcal{P}$; hence, w is right nondegenerate. Since w is also left nondegenerate by assertion 2, we obtain assertion 8. □

Remark 4.1.3. We will now discuss certain examples to our theorem. Note, however, that the spherical weight structure on SH is purely compactly generated and still right weight-degenerate; see Remark 4.2.3(4) below.

Corollary 4.1.4. *Let* $n>0$*.*

1. *The assumptions of Theorem 4.1.2 are fulfilled if* $\underline{C}$ *is one of the following categories:*

(i) *the derived category of a small differential graded category* $\underline{B}$ *(see §3.2 [16]) such that the complex* $\underline{B}^*(M,N)$ *is acyclic in positive degrees whenever* $M,N\in\underline{B}$*. Here, we take* $\mathcal{P}$ *to be the subcategory of* $\underline{C}$ *corresponding to* $\underline{B}$*;*

[5] According to Proposition 4.3.3 of [7], the category $\underline{Ht}$ is actually the *heart of a t-structure* on $\underline{C}$ (cf. Remark 4.2.3(6) below), whence the notation.

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(ii) *the derived category of E-modules, where E is an S-algebra and $\pi_i(E)=\{0\}$ if $i<0$; see Example 1.2.3(f) of [15] for the detail. In this case, we take $\mathcal{P}=\{E\}$.*

Moreover, the additional assumption of Theorem 4.1.2(8) is fulfilled in these cases as well whenever all the complexes $\underline{B}^*(M,N)$ are acyclic in degrees less than $-n$ or if $\pi_i(E)=\{0\}$ for $i>n$, respectively.

2. *Let A and B be differential graded algebras whose cohomology $H^*(A)$ and $H^*(B)$ is concentrated in negative degrees, and assume that B is a right A-module.*

Take $\underline{C}=D(A)$, w_A to be the weight structure corresponding to $\mathcal{P}=\{A\}$ according to assertion 1 and Theorem 4.1.2(2), $\underline{C}'=D(B)$, w_B to be the weight structure corresponding to $\mathcal{P}'=\{B\}$. Assume that n-th power of the kernel of the ring homomorphism $\underline{C}(A,A)\to\underline{C}'(B,B)$ induced by F is zero.

Then for the exact functor $F:\underline{C}=D(A)\to\underline{C}'=D(B)$ described in §3.8 of [16], the assumptions of Proposition 3.2.1 are fulfilled. Here, we take $X=B$ and consider X as an $A-B$-bimodule when we apply loc. cit.

3. *Assume that $\underline{C}$ is compactly purely generated by its subcategory $\mathcal{P}$, and that for $i=1,2,3$ we have full subcategories $\mathcal{P}_i$ of $\mathcal{P}$ such that all morphisms between $\mathcal{P}_1$ and $\mathcal{P}_2$ factor through $\mathcal{P}_3$. Denote by $\underline{C}_i$ the localizing triangulated subcategory of $\underline{C}$ generated by $\mathcal{P}_i$ $(i=1,2,3)$.*

Then the localization functor $F:\underline{C}\to\underline{C}'=\underline{C}/\underline{C}_3$ exists (cf. Proposition 3.1.6(4)), there is a unique weight structure w' on $\underline{C}'$ such that F is weight-exact and for any $M\in\operatorname{Obj}\underline{C}_1\cap\operatorname{Obj}\underline{C}_2$, the object $F(M)$ is right w'-degenerate. Moreover, M is an object of $\underline{C}_3$ whenever $M\in\operatorname{Obj}\underline{C}_1\cap\operatorname{Obj}\underline{C}_2$ and M is w-bounded below.

Proof. 1. In case (i), the objects of $\underline{C}$ coming from $\underline{B}$ are compact by Corollary 3.7 of [16]. Moreover, in the discussion preceding loc. cit. the equality $(\cup_{i\in\mathbb{Z}}\mathcal{P}[i])^{\perp}=\{0\}$ is mentioned. Hence, $\mathcal{P}$ compactly generates $\underline{C}$ by Proposition 8.4.1 of [23]. Next, formula (3) in [16, §3.2] says that for every $j\in\mathbb{Z}$ and $M,N\in\underline{B}$ the jth cohomology of the complex $(\underline{B}^i(M,N))$ is isomorphic to $\underline{C}(M,N[j])$. This easily implies all the remaining statements for this case.

The statement that $\{E\}$ compactly generates $\underline{C}$ in case (ii) can be found in Example 1.2.3(f) of [15]. Furthermore, loc. cit. states that $\underline{C}(E,X)$ is the zeroth homotopy group of the underlying spectrum X for any object X of $\underline{C}$. This yields all the remaining statements for this case.

2. The description of F given in §3.8 of [16] immediately implies that F is exact, respects coproducts and $F(A)\cong B$. Applying Theorem 4.1.2(5), we obtain that F is weight-exact.

Next, the hearts of w_A and w_B are equivalent to the categories of projective modules over $\underline{C}(A,A)$ and $\underline{C}'(B,B)$, respectively, according to Theorem 4.1.2(3). Hence, the nilpotence condition in Proposition 3.2.1 follows from our nilpotence of the kernel assumption.

3. It clearly suffices to prove that $\underline{C}_3$ is retraction-closed in $\underline{C}$, $(\underline{C},w,\underline{C}_i)$ satisfy the assumptions of Proposition 3.1.6(4), and the corresponding w' is left nondegenerate.

$\underline{C}_3$ is retraction-closed in $\underline{C}$ since $\underline{C}_3$ is Karoubian (see Theorem 4.1.2(1)). Next, w restricts to $\underline{C}_1$ and $\underline{C}_2$ by Theorem 4.1.2(5).

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The localization functor F is well-known to exist and (also) to respect coproducts and the compactness of objects; see Proposition 9.1.19, Theorem 8.3.3, Corollary 3.2.11 and Theorem 4.4.9 of [23]. Thus, Theorem 4.1.2(3) implies that all $\underline{Hw}$-morphisms between elements of the corresponding classes $\underline{C}_{1,w_1=0}$ and $\underline{C}_{2,w_2=0}$ are killed by F; see equality 4.1.1.

Lastly, by Proposition 3.1.6(4) there exists a unique weight structure w' such that F is weight-exact. Thus, $F(\mathcal{P}) \subset \underline{C}'_{w'=0}$; hence, $F(\mathcal{P})$ is a connective subcategory of $\underline{C}'$. Since $\mathcal{P}$ compactly generates $\underline{C}$, $F(\mathcal{P})$ compactly generates $\underline{C}'$ and applying Theorem 4.1.2(5), we obtain that w' is purely compactly generated by $F(\mathcal{P})$. Hence, w' is left nondegenerate by Theorem 4.1.2(2). □

Remark 4.1.5. Corollary 4.1.4(3) slightly generalizes Proposition 1.9 of [5] which seriously depended on an earlier version of the current paper. Thus, we obtain an unconditional proof of loc. cit. This is crucial for ibid.

Now, let us relate purely compactly generated weight structures to the main definitions of the current paper.

Corollary 4.1.6. *Adopt the notation and the assumptions of Theorem 4.1.2. Let $m \le n \in \mathbb{Z}$.*

1. *The class of essentially w-positive objects coincides with ${}^{\perp}\cup_{j<0}\{I[j]\}$, where I is an injective cogenerator of $\underline{A}_{\mathcal{P}}$. This class is also characterized by the vanishing of $H_j^{\mathcal{A}_{\mathcal{P}}}(-)$ for $j<0$.*
2. *The class of w-degenerate objects coincides with ${}^{\perp}\cup_{j\in\mathbb{Z}}\{I[j]\}$ and also with ${}^{\perp}\cup_{j\in\mathbb{Z}}\{\underline{Ht}[j]\}$. Moreover, this class is characterized by the vanishing of $H_j^{\mathcal{A}_{\mathcal{P}}}(-)$ for all $j \in \mathbb{Z}$.*
3. *A $\underline{C}$-morphism g kills weight m if and only if $H^m(g)=0$ for every pure representable functor H.*
4. *An object M of $\underline{C}$ is without weights $m,\dots,n$ if and only if $H^j(M)=0$ whenever H is pure and representable and $m \le j \le n$.*
5. *$\underline{C}_{w\le 0} = {}^{\perp}(\cup_{j>0}\underline{Ht}[j])$. Moreover, this class is also annihilated by H_i for all $i>0$ and for every w-pure homological functor H from $\underline{C}$.*

Proof. 1. According to Corollary 3.1.4, an object M of $\underline{C}$ is essentially w-positive if and only if $t(M) \in K(\underline{Hw})_{w^{st}\ge 0}$. Combining this fact with Theorem 4.1.2(7), we obtain our assertion.

2. w is left nondegenerate by Theorem 4.1.2(2); hence, Proposition 3.1.6(1) implies that M is essentially w-positive if and only if it belongs to $\underline{C}_{w\ge 0}$. On the other hand, M is weight-degenerate if and only if it is right weight-degenerate. Thus, M is weight-degenerate if and only if $M[j]$ is essentially w-positive for all $j \in \mathbb{Z}$. Hence, our assertion follows from the previous one.

3. Since w is smashing and $\underline{C}$ is a Brown category, the assertion follows from Proposition 2.4.4 immediately.

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4. This is a straightforward consequence of the previous assertion combined with Theorem 2.3.1(1,3) along with Lemma 1.3.2(4).

5. Since w is left nondegenerate, $\underline{C}_{w\le 0}$ coincides with the class of all essentially w-negative objects according to Proposition 3.1.6(1). Thus, Theorem 3.1.3 gives the result in question; cf. Remark 3.1.5(1). □

Now, let us demonstrate that one can say more on this setting if an additional assumption is imposed.

Proposition 4.1.7. *Adopt the notation and assumptions of Theorem 4.1.2, and suppose in addition that the category $\underline{A}_{\mathcal{P}}$ is of projective dimension at most 1, that is, any its object has a projective resolution of length 1. Let $m \le n \in \mathbb{Z}$ and $g : E \to E'$ be a $\underline{C}$-morphism.*

1. *Then the category $K_{\mathfrak{w}}(\underline{Hw})$ equals $K(\underline{Hw})$, and the natural functor $K(\underline{Hw}) \to D(\underline{A}_{\mathcal{P}})$ is an equivalence.*
2. *For any objects C and C' in $D(\underline{A}_{\mathcal{P}})$, we have natural isomorphisms $C \cong \coprod H_j(C)[j]$ and*
$$D(\underline{A}_{\mathcal{P}})(C,C') \cong \prod_{j\in\mathbb{Z}} \underline{A}_{\mathcal{P}}(H_j(C),H_j(C')) \bigoplus (\prod_{j\in\mathbb{Z}} \operatorname{Ext}^1_{\underline{A}_{\mathcal{P}}}(H_j(C),H_{j+1}(C'))).$$
3. *g kills weight m if and only if $H_m^{\mathcal{A}_{\mathcal{P}}}(g) = 0$, the class of $t(g)$ in the group $\operatorname{Ext}^1_{\underline{A}_{\mathcal{P}}}(H_{m-1}(t(E)),H_m(t(E')))$ (here we use the identification provided by the previous two assertions) vanishes, and the morphism $H_{m-1}^{\mathcal{A}_{\mathcal{P}}}(g)$ factors through a projective object of $\underline{A}_{\mathcal{P}}$.*
4. *E is without weights $m,\dots,n$ (resp. $E \in \underline{C}_{w\le m-1}$) if and only if $H_j^{\mathcal{A}_{\mathcal{P}}}(E) = 0$ for $m \le j \le n$ (resp. for $j \ge m$) and $H_{m-1}^{\mathcal{A}_{\mathcal{P}}}(E)$ is a projective object of $\underline{A}_{\mathcal{P}}$.*

Proof. 1. These statements easily follow from Theorem 4.1.2(6) according to Remark 3.3.4 of [4].

2. The first splitting statement is well-known, and the second statement easily follows from the vanishing of higher extension groups in $\underline{A}_{\mathcal{P}}$.

3. Once again, we should check whether $t(g) \backsim_{[-m,-m]} 0$; see Theorem 2.3.1(1). Now, assertion 2 implies that
$$t(E) \cong_{K(\underline{Hw})} \coprod_{i\in\mathbb{Z}} C_i, \tag{4.1.2}$$
where $C_i = \operatorname{Cone}(A_i \xrightarrow{f_i} B_i)[i]$, and f_i are $\underline{Hw}$-monomorphisms; thus, f_i are also $\underline{A}_{\mathcal{P}}$-monomorphisms between projective objects. Similarly, we present $t(E')$ as $\coprod C'_j$, where $C'_j = \operatorname{Cone}(A'_j \xrightarrow{f'_j} B'_j)[j]$. Clearly, it suffices to treat all our assumptions on $t(g)$ separately for the morphisms $g_{ij} : C_i \to C'_j$ induced by $t(g)$, for $\{i,j\} \subset \{m,m-1\}$.

Next, $C_m \perp C'_{m-1}$ since f'_{m-1} is monomorphic. It remains to verify that the three remaining cases of (i,j) give our three conditions on g.

Assertion 2 implies that $K(\underline{Hw})(C_{m-1},C'_m) \cong \operatorname{Ext}^1_{\underline{A}_{\mathcal{P}}}(H_{m-1}^{\mathcal{A}_{\mathcal{P}}}(E),H_m^{\mathcal{A}_{\mathcal{P}}}(E'))$. Moreover, $g_{m-1,m} \backsim_{[l,l]} 0$ for all $l \ne -m$; thus, $g_{m-1,m} = 0$ if and only if $g_{m-1,m}$ kills weight m.

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Hence, $g_{m,m-1}$ kills weight m if and only if the class of $t(g)$ in $\mathrm{Ext}^1_{\underline{\mathrm{A}}_{\mathcal{P}}}(H^{\mathcal{A}_{\mathcal{P}}}_{m-1}(E),H^{\mathcal{A}_{\mathcal{P}}}_m(E'))$ vanishes.

Next, if g kills weight m, then $H^{\mathcal{A}_{\mathcal{P}}}_m(g)=0$ by Proposition 2.4.4(I). Conversely, $K(\underline{Hw})(C_m,C'_m)\cong \underline{A}_{\mathcal{P}}(H^{\mathcal{A}_{\mathcal{P}}}_m(E),H^{\mathcal{A}_{\mathcal{P}}}_m(E'))$; thus, if $H^{\mathcal{A}_{\mathcal{P}}}_m(g)=0$, then $g_{mm}=0$.

Lastly, $g_{m-1,m-1}\backsim_{[-m,-m]}0$ if and only if $g_{m-1,m-1}$ factors through the obvious morphism $B'_{m-1}[m-1]\to C'_{m-1}$. The latter condition clearly implies that $H^{\mathcal{A}_{\mathcal{P}}}_{m-1}(g)$ factors through B'_{m-1}. Conversely, if $H^{\mathcal{A}_{\mathcal{P}}}_{m-1}(g)$ factors through a projective object of $\underline{A}_{\mathcal{P}}$, then $g_{m-1,m-1}$ factors through $\underline{C}_{w=m-1}$. Note here that $\underline{A}_{\mathcal{P}}$ embeds into $K(\underline{Hw})_{w^{st}\geq m-1}$ via projective resolutions. Thus, $g_{m-1,m-1}$ factors through $B'_{m-1}[m-1]$. Here, we apply Proposition 1.2.4(6) to the category $\underline{C}=K(\underline{Hw})$ endowed with the stupid weight structure.

Assertion 4 follows from the previous one more or less easily. Combining Corollary 3.1.4 (see condition 2 in it) with Proposition 3.1.6(1) we obtain that E belongs to $\underline{C}_{w\leq m-1}$ if and only if id_E kills weight i for all $i\geq m$. Recall also that E is without weights $m,\ldots,n$ if and only if id_E kills weight i for $m\leq i\leq n$; see Theorem 2.2.1(8). Hence, it remains to note that for every $i\in\mathbb{Z}$ the morphism $H^{\mathcal{A}_{\mathcal{P}}}_i(\mathrm{id}_E)$ clearly equals the corresponding identity, whereas the classes of $t(\mathrm{id}_E)$ in all the groups $\mathrm{Ext}^1_{\underline{\mathrm{A}}_{\mathcal{P}}}(H_i(E),H_{i+1}(E))$ (see assertion 2) are zero. □

Remark 4.1.8. 1. Let us describe a motivic example to our proposition.

One can take $\underline{C}$ to be the localizing subcategory DTM generated by the Tate motives $\mathbb{Z}(i)$, $i\in\mathbb{Z}$, in the category DM of motives with integral coefficients over any perfect field k; see §4.2 of [13]. Then the category $\mathcal{P}=\{\mathbb{Z}(i)[2i],\ i\in\mathbb{Z}\}$ is connective in DTM since $\mathbb{Z}(i)[2i]\perp\mathbb{Z}(j)[2j]$ for every $i\neq j\in\mathbb{Z}$, whereas $DM(\mathbb{Z}(i)[2i],\mathbb{Z}(i)[2i])\cong\mathbb{Z}$ for all i; see Corollary 6.7.3 of [3]. Hence, the corresponding category $\underline{A}_{\mathcal{P}}$ is equivalent to $\mathrm{Ab}^{\mathbb{Z}}$; thus, $\underline{A}_{\mathcal{P}}$ is of projective dimension 1.

Moreover, $\underline{A}_{\mathcal{P}}$ is of projective dimension 1 for the R-linear motivic category DTM_R whenever R is a Dedekind domain or a field. Note also that for any R the corresponding functor $H^{\mathcal{A}_{\mathcal{P}}}$ essentially computes the so-called Chow-weight homology of Tate motives (inside $DM_R\supset DM_R^{eff}$); see [8] and [11].

2. The case $E\in\underline{C}_{w\leq m-1}$ in part 4 of our proposition is an abstract generalization of [20, Proposition 6.16] (where $\underline{C}=SH$ was considered; see Theorem 4.2.4 below). Respectively, Proposition 6.17 of loc. cit. is the corresponding case of the orthogonality axiom for the weight structure w^{sph}. Note, however, that the methods of loc. cit. do not seem to extend to the case of a general G as well as to the setting of Theorem 4.1.2.

4.2. Equivariant spectra and converse Hurewicz theorems

Let us generalize Theorem 4.1.1 of [7] to the setting of S-local equivariant stable homotopy categories and extend it. We need some notation.

- Choose a set of prime numbers $S\subset\mathbb{Z}$, and denote $\mathbb{Z}[S^{-1}]$ by Λ. S may be empty. Then $\Lambda=\mathbb{Z}$, and this case is quite important.
- G is a fixed compact Lie group. We will write $SH(G)$ for the stable homotopy category of G-spectra indexed by a complete G-universe; see [19] and [21] for lots of detail on these categories.

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- We write $SH_\Lambda(G)$ for the full subcategory of S-local objects of $SH(G)$, that is, $M \in \operatorname{Obj} SH_\Lambda(G)$ whenever the morphism $p\,\mathrm{id}_M$ is invertible for every $p \in S$. According to Proposition A.2.8 of [17], $SH_\Lambda(G)$ is a triangulated subcategory of $SH(G)$ and there exists an exact left adjoint $l_S = (-)[S^{-1}]$ to the embedding $SH_\Lambda(G) \to SH(G)$.
- We take $\mathcal{P}$ to be the set of spectra of the form $l_S(S^0_B)$, where B is a closed subgroup of G (cf. Definition I.4.3 of [19]. Recall that $S^0_B = \sum^\infty G/B^+$ is constructed starting from the G-space G/B). We will write $\mathcal{P}$ for the corresponding preadditive subcategory of $SH_\Lambda(G)$. If $\Lambda = \mathbb{Z}$, then $\mathcal{P}$ is the (stable) *orbit category* of ibid.

  Recall also that $S^0_B[n] \in \operatorname{Obj} SH(G)$ is the corresponding sphere spectrum S^n_B essentially by definition; see loc. cit.
- The equivariant homotopy groups of an object E of $SH(G) \supset SH_\Lambda(G)$ are defined as $\pi^B_n(E) = SH(G)(S^n_B, E)$ (for all $n \in \mathbb{Z}$; see §I.6 and Definition I.4.4(i) of ibid.).
- We will write EM_G for the full subcategory of $SH_\Lambda(G)$ whose object class is $(\cup_{i \in \mathbb{Z} \setminus \{0\}} \mathcal{P}[i])^\perp$. In the case $\Lambda = \mathbb{Z}$, the objects of EM_G are the *Eilenberg–MacLane* G-spectra; see §XIII.4 of [21].
- We write $\mathcal{M}_G$ for the category of additive contravariant functors from $\mathcal{P}$ into Ab (cf. Theorem 4.1.2(6)). We call its objects *Mackey functors*. Respectively, $\mathcal{A}_\mathcal{P} : \mathcal{P} \to \mathcal{M}_G$ is the Yoneda embedding.
- We call ${}^\perp(\cup_{i \in \mathbb{Z}} \mathrm{EM}_G[i]) \subset \operatorname{Obj} SH_\Lambda(G)$ the class of *acyclic* spectra; that is, a spectrum is acyclic if it is annihilated by H^i for all H represented by S-local Eilenberg–MacLane spectra and $i \in \mathbb{Z}$.

Now, let us describe a weight structure on $SH_\Lambda(G)$.

Proposition 4.2.1. *Let $n \in \mathbb{Z}$. Then the following statements are valid.*

1. *The restriction of l_S to the category $SH_\Lambda(G) \subset SH(G)$ is fully faithful. Moreover, $SH_\Lambda(G)(l_S(S^n_B), l_S(C)) \cong SH(G)(S^n_B, l_S(C)) \cong \pi^B_n(C) \otimes_\mathbb{Z} \Lambda$ for any closed subgroup B of G and an object C of $SH(G)$.*
2. *The category $\underline{C} = SH_\Lambda(G)$ and the class $\mathcal{P}$ specified above satisfy the assumptions of Theorem 4.1.2. The heart $\underline{Hw}^G_\Lambda$ of the corresponding weight structure w^G_Λ on $SH_\Lambda(G)$ consists of retracts of coproducts of elements of $\mathcal{P}$, and $\underline{Hw}^G_\Lambda$ is equivalent to the smashing idempotent completion of $\mathcal{P}$; see Definition 4.1.1(2).*
3. *The class of $n-1$-connected S-local spectra (see Definition I.4.4(iii) of [19]; that is, this is the class $(\cup_{i<n} \mathcal{P}[i])^\perp$) coincides with $SH_\Lambda(G)_{w^G_\Lambda \geq n}$. In particular, $SH_\Lambda(G)_{w^G_\Lambda \geq 0}$ is the class of S-local connective spectra. It is the smallest class of objects of $SH_\Lambda(G)$ that contains $\cup_{i \geq 0} \mathcal{P}[i]$ and is closed with respect to coproducts and extensions.*
4. *$SH_\Lambda(G)_{w^G_\Lambda \leq 0}$ is the smallest subclass of $\operatorname{Obj} SH_\Lambda(G)$ that is closed with respect to coproducts and extensions and contains $\mathcal{P}[i]$ for $i \leq 0$. This class also equals ${}^\perp(\cup_{i \geq 0} \mathrm{EM}_G[i]) = {}^\perp SH_\Lambda(G)_{w^G_\Lambda \geq 1}$. Moreover, it is annihilated by H_i for all $i > 0$ and for every pure homological functor H from $SH_\Lambda(G)$.*

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5. *A (co)homological functor from $SH_\Lambda(G)$ into an abelian category $\underline{A}$ that respects coproducts (resp. converts them into products) is w_Λ^G-pure if and only if it kills $\cup_{i\neq 0}\mathcal{P}[i]$.*
   In particular, all objects of EM_G *represent w_Λ^G-pure functors.*
6. *The category* EM_G *is naturally equivalent to $\mathcal{M}_G$ via Yoneda; thus,* EM_G *is Grothendieck abelian and has an injective cogenerator I.*

Proof. 1. The restriction of l_S to $SH_\Lambda(G)$ is fully faithful since l_S is left adjoint to the embedding $SH_\Lambda(G) \to SH(G)$. Next, the only nontrivial isomorphism for morphism groups in the assertion is given by Corollary A.2.13 of [17].

2. The fact that objects of the form S_B^0 form a connective subcategory of $SH(G)$ that compactly generates this category can be deduced from the results of [19]; see [7, Theorem 4.1.1] for more detail. Applying the previous assertion, we obtain that $\mathcal{P}$ is connective in $SH_\Lambda(G)$. Next, the functor l_S sends any family of compact generators of $SH(G)$ into a one for $SH_\Lambda(G)$ according to Proposition A.2.8 of [17]. Lastly, we apply Theorem 4.1.2(3) to compute the heart of w_Λ^G.

3. By definition, a G-spectrum N is $n-1$-connected whenever $\pi_i^B(N) \cong SH(G)(S_B^i, N) = \{0\}$ for all $i < n$ and every closed subgroup B of G. Since $SH(G)(S_B^n, N) \cong SH_\Lambda(G)(l_S(S_B^n), l_S(N))$ (see assertion 1), it remains to apply Theorem 4.1.2(3) to obtain all the statements in question.

4. The first of these descriptions of $SH_\Lambda(G)_{w_\Lambda^G \le 0}$ is given by Theorem 4.1.2(1). Next, $SH_\Lambda(G)_{w_\Lambda^G \le 0} = {}^\perp SH_\Lambda(G)_{w_\Lambda^G \ge 1}$ according to Proposition 1.2.4(2). It remains to apply Theorem 4.1.2(5) to conclude the proof.

Assertion 5 follows from Theorem 4.1.2(4) immediately. Lastly, assertion 6 is just the corresponding case of Theorem 4.1.2(6). □

Now, we apply the results of this paper to w_Λ^G; consult Definitions 2.3.3, 2.1.2, 3.1.1(2) and 2.4.1 for the notions mentioned in it.

Theorem 4.2.2. *Let $E \in \operatorname{Obj} SH(G)$; I is an injective cogenerator of* EM_G.

1. *The following conditions are equivalent.*
   (i). *E is acyclic.*
   (ii). *E is w_Λ^G-degenerate.*
   (iii). *E is right w_Λ^G-degenerate.*
   (iv). *$E \perp SH_\Lambda(G)_{w_\Lambda^G \ge i}$ for every $i \in \mathbb{Z}$.*
   (v). *$H_i^{\mathcal{A}_\mathcal{P}}(E) = 0$ for every $i \in \mathbb{Z}$.*
   (vi). *$E \perp (\cup_{i\in\mathbb{Z}} I[i])$.*
2. *The following assumptions on a $SH_\Lambda(G)$-morphism h are equivalent.*
   (i). *h kills weight* 0.
   (ii). *$H(h) = 0$ for every w_Λ^G-pure functor H from $SH_\Lambda(G)$.*
   (iii). *For every $J \in \mathrm{EM}_G$ and $P = SH_\Lambda(G)(-, J)$, we have $P(h) = 0$.*
3. *The following conditions are equivalent as well.*
   (i). *E is essentially w_Λ^G-positive.*
   (ii). *E is an extension of a connective spectrum by an acyclic one.*

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(iii). *$H(E[j])=0$ for every w_Λ^G-pure functor H from $SH_\Lambda(G)$ and $j>0$.*

(iv). *$E\perp I[j]$ for all $j<0$.*

(v). *$H_j^{\mathcal{A}_{\mathcal{P}}}(E)=0$ for all $j<0$.*

4. *Let $m\le n\in\mathbb{Z}$. Then the following conditions are equivalent.*

(i). *E is without weights $m\dots n$.*

(ii). *There exists a distinguished triangle $E_1\to E\to E_2\to E_1[1]$ such that $E_1\in SH_\Lambda(G)_{w_\Lambda^G\le m-1}$ and $E_2\in SH_\Lambda(G)_{w_\Lambda^G\ge n+1}$. Moreover, if this is the case, then this triangle is canonically determined by E.*

(iii). *$E\perp(\cup_{m\le i\le n}\mathrm{EM}_G[-i])$.*

(iv). *E is annihilated by H_i whenever $m\le i\le n$ and H is a pure homological functor from $SH_\Lambda(G)$.*

Proof. 1. According to Proposition 4.2.1(2), we can apply Corollary 4.1.6(2) to obtain that conditions (i), (ii), (v) and (vi) are equivalent. Next, conditions (ii) and (iii) are equivalent since w_Λ^G is left nondegenerate (see Theorem 4.1.2(2) and Proposition 3.1.6(1)), and applying Proposition 1.2.4(2) we obtain the equivalence of (iii) and (iv).

Assertion 2 immediately follows from Proposition 2.4.4.

3. Since $SH_\Lambda(G)$ is Karoubian, conditions (i) and (ii) are equivalent according to Corollary 3.1.4(II) combined with Proposition 4.2.1(3) and assertion 1. Moreover, (i) is equivalent to (iv) and (v) by Theorem 4.1.2(1), and condition (iii) clearly implies condition (iv).

4. By definition, E is without weights $m\dots n$ if and only if id_E kills these weights. Hence, applying assertion 2 we obtain the equivalence of conditions (i), (iii) and (iv). Lastly, conditions (i) and (ii) are equivalent according to Theorem 2.2.1(9,10). □

Remark 4.2.3.

1. Let us explain that part 3 of our theorem is a certain converse to the natural Hurewicz theorem for this context.

   We recall that the ‘usual’ stable Hurewicz theorem essentially says that in the case $G=\{e\}$ (and $\Lambda=\mathbb{Z}$, so $SH(G)=SH$) a $w_\Lambda^G=w^{sph}$-bounded below spectrum E is connective if and only if its singular homology is concentrated in nonnegative degrees. An equivariant version of this statement is given by Theorem 2.1(i) of [18] (cf. also Theorem 1.11 of ibid. and Proposition 7.1.2(f) of [15]). Note that one replaces singular homology by $H^{\mathcal{A}_{\mathcal{P}}}$ in it (cf. part 4 of this remark).

   Now, it is easily seen that those essentially w_Λ^G-positive objects that are w_Λ^G-bounded below are connective. Hence, part 3 of our theorem naturally generalizes the aforementioned converse equivariant Hurewicz theorem to arbitrary objects of $SH_\Lambda(G)$. Our generalization depends on the notion of acyclic spectra, and the corresponding part 1 of our theorem also appears to be quite new; cf. also part 4 of this remark.

2. The notions of killing weights and being without weights $m,\dots,n$ (along with parts 2–4 of our theorem see also Proposition 4.2.1(4)) appear to be new in this context even when restricted to the case $\underline{C}=SH$. Note that one can obtain canonical

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'decompositions' of some spectra (see condition 4. (ii) in our theorem) by looking at their cohomology with coefficients in Eilenberg–Maclane spectra.

3. Recall that if $\Lambda = \mathbb{Z}$, then the pure homological functor $H^{\mathcal{A}_{\mathcal{P}}}$ is the equivariant ordinary homology with Burnside ring coefficients functor H_0^G considered in [18] (cf. also Definition X.4.1 of [21]), and for every Mackey functor M the corresponding pure functor H_M coincides with $H_G^0(-,M)$ in Definitions X.4.2 and §XIII.4 of ibid. Clearly, it follows that the functors $H^{\mathcal{A}_{\mathcal{P}}}$ and H_M are closely related to these notions as well.
4. None of the descriptions of acyclic spectra in part 1 of our theorem characterizes them 'explicitly'. In the case $G = \{e\}$, our definition of this notion coincides with the definition in [20]; see Theorem 4.2.4(2) below. Thus, Theorem 16.17 of ibid. gives an explicit example of a nonzero acyclic spectrum.
5. Proposition 4.1.7 certainly gives some more information on ('weights of') objects of $SH_\Lambda(G)$ whenever the corresponding category $\mathcal{M}_G$ is of projective dimension at most 1. So we note that this assumption is fulfilled whenever G is a finite group of order n and $1/n \in \Lambda$. One should join Theorem 2.1 of [14] with the finite projective dimension statement established in §6 of ibid. to obtain this fact.
6. Let us mention some nice properties of w^G.

   Firstly, it restricts to the subcategory of compact objects of $SH_\Lambda(G)$; cf. Theorem 4.1.1(2) of [7]. Next, the class $SH_\Lambda(G)_{w^G\geq 0}$ can also be described as $SH_\Lambda(G)_{t\geq 0}$ for a certain *Postnikov t-structure* t (see Definition 4.3.1(I) and Proposition 4.3.3 of ibid.), yet this t-structure does not restrict to compact objects of $SH_\Lambda(G)$.

   Note also that our theory gives a certain converse Hurewicz-type theorem (and also other 'decompositions' as well as several new definitions; see part 2 of this remark) for the so-called connective stable homotopy theory as discussed in §7 of [15]; cf. version (ii) of Corollary 4.1.4(1) and see Remark 4.3.4(2) of [7] for more detail.

Now, we apply our results to the stable homotopy category SH. Some of these statements were already stated in Theorem 4.2.1 of [7]. This corresponds to the case of trivial G and S in Proposition 4.2.1, so we will write EM for EM_G and w^{sph} for w_Λ^G in this case and use the remaining notation from the proposition.

Theorem 4.2.4. *Set $\mathcal{P} = \{S^0\}$, and assume $m \leq n \in \mathbb{Z}$ and $g: E \to E'$ is an SH-morphism.*

1. *Then the functor $SH(S^0,-)$ gives equivalences $\underline{Hw}^{sph} \to \mathrm{FreeAb}$ (free abelian groups) and $\mathrm{EM} \to \mathrm{Ab}$; thus, $\underline{A}_{\mathcal{P}}$ is equivalent to* Ab *as well.*
2. *The last of the aforementioned equivalences makes the functor $\mathcal{A}_{\mathcal{P}}$ isomorphic to the singular homology functor $H^{sing} = H^{sing}(-,\mathbb{Z})$. Consequently, acyclic spectra in SH are characterized by the vanishing of their singular homology (cf. §6.2 of [20]).*
3. *For every abelian group Γ and the corresponding spectrum $\mathrm{EM}^\Gamma \in \mathrm{Obj}\,\mathrm{EM}$, the functor $SH(-,\mathrm{EM}^\Gamma)$ is isomorphic to the singular cohomology with coefficients in Γ.*

4. *g kills weight m if and only if $H^{sing}_m(g) = 0$, the class of $t(g)$ in the group $\mathrm{Ext}^1_{\mathrm{Ab}}(H^{sing}_{m-1}(E), H^{sing}_m(E'))$ (see Proposition 4.1.7(3)) vanishes and the morphism $H^{sing}_{m-1}(g)$ factors through a free abelian group.*
   This condition is also equivalent to the vanishing of $H^m_{sing}(-,\Gamma)(g)$ for every abelian group Γ.
5. *E is an extension of an n-connected spectrum Y by $X_{m-1} \in SH_{w^{sph} \le m-1}$ [6] if and only if $H^{sing}_j(E) = \{0\}$ for $m \le j \le n$, and $H^{sing}_{m-1}(E)$ is a free abelian group.*
6. *E is an extension of a connective spectrum by an acyclic one if and only if $H^{sing}_j(E) = \{0\}$ for all $j < 0$. This is also equivalent to the vanishing of $H^j_{sing}(E, \mathbb{Q}/\mathbb{Z})$ for all $j < 0$.*

Proof. Assertions 1–3 easily follow from Theorem 4.1.2. They are also contained in Theorem 4.2.1 of [7]. These facts also yield that Theorem 4.1.2(3) implies our assertion 6.

Next, the category $\mathrm{Ab} \cong \underline{A}_{\mathcal{P}}$ is of cohomological dimension 1; hence, we can combine Proposition 4.1.7(3,4) with the preceding assertions along with Theorem 4.1.2(2,4) to obtain assertions 4 and 5. □

Now, we can relate w^{sph} to central notions of [20].

Theorem 4.2.5. 1. *The following assumptions on a spectrum E (that is, an object of SH) are equivalent.*

(i) *$E \in SH_{w^{sph} \le n}$;*

(ii) *$H^{sing}_i(E) = \{0\}$ for $i > n$ and $H^{sing}_n(E)$ is a free abelian group;*

(iii) *$H^i_{sing}(E, \Gamma) = \{0\}$ for every $i > n$ and every abelian group Γ;*

(iv) *E is an n-skeleton (of some spectrum) in the sense of [20, §6.3].*

2. *A w^{sph}-Postnikov tower (see Definition 1.3.3 of [7]) for E is the same thing as a cellular tower for E in the sense of [20, §6.3].*

Proof. Applying Theorem 4.2.4(1–3) we obtain that Proposition 4.2.1(4) gives the equivalence of conditions 1(i) and 1(iii), and also that Proposition 4.1.7(4) implies the equivalence of conditions 1(i) and 1(ii). Moreover, the latter equivalence statement implies that these conditions are fulfilled if and only if E is an n-skeleton, and also that assertion 2 is valid according to Theorem 4.2.1(4,5) of [7]. □

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[6] $SH_{w^{sph} \le m-1}$ consists precisely of $m-1$-*skeleta*; see Theorem 4.2.5(1).

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