ISSN 1066-369X, Russian Mathematics, 2024, Vol. 68, No. 6, pp. 58–67. © Allerton Press, Inc., 2024.
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Conditions for Ultimate Boundedness of Solutions and Permanence for a Hybrid Lotka–Volterra System

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Received May 10, 2023; revised May 29, 2023; accepted September 26, 2023

Abstract—In the paper, a generalized Lotka–Volterra-type system with switching is considered. The conditions for the ultimate boundedness of solutions and the permanence of the system are studied. With the aid of the direct Lyapunov method, the requirements for the switching law are established to guarantee the necessary dynamics of the system. An attractive compact invariant set is constructed in the phase space of the system, and a given region of attraction for this set is provided. A distinctive feature of the work is the use of a combination of two different Lyapunov functions, each of which plays its own special role in solving the problem.

Keywords: generalized Lotka–Volterra system, switching, ultimate boundedness of solutions, permanence

DOI: 10.3103/S1066369X24700440

INTRODUCTION

Lotka–Volterra type systems are an important class of dynamical systems widely used to model various interactions between entities in a biological, chemical, or economic environment. Such systems have been of great interest for many years (see, e.g., [1–11]). The possible dynamics of solutions to such systems are analyzed; in particular, issues of stability, dissipativity, convergence, etc., were examined. Both continuous systems and their difference and differential-difference analogues were considered. Both stationary and nonstationary systems, as well as systems with random coefficients, were studied. Functions of interspecific connections in Lotka–Volterra systems can be linear (classical systems) and nonlinear (generalized systems), additive and multiplicative. Systems can describe both concentrated and distributed interactions caused by migration or diffusion effects.

In this paper, we consider a generalized Lotka–Volterra system with switching. Switching systems are hybrid systems consisting of a family of subsystems, each of which specifies the functioning of the system in a certain mode [12]. The switching law is a certain function that determines at each moment which of the subsystems is active. Many works are devoted to the study of Lotka–Volterra systems with switching (see, e.g., [13–16] and the references cited there).

One of the important problems associated with the theory of dynamical systems is the problem of the ultimate boundedness of solutions. Within the framework of this problem, it is important to determine conditions that guarantee that the solutions of the system do not go to infinity, but eventually fall into some bounded invariant region. Note that in real practical problems it is enough to consider only solutions starting in some finite neighborhood of the origin. This allows significantly weakening the required conditions for the ultimate boundedness of solutions. In addition to that, in biological, chemical, and economic processes it is often important to ensure the constant conservation of interacting entities. Mathematically, this means that any solution to the system must be component-wise separated from zero. When this condition is met, the system under study is called persistent. Dynamic systems with ultimately bounded solutions and the property of persistence are usually called permanent [4].

To solve the problems posed, the work uses the method of Lyapunov functions in combination with the theory of differential inequalities. Because the problems under consideration belong to the field of global analysis, choosing one Lyapunov function that satisfies the necessary conditions in the entire space is problematic. Therefore, the work involves the idea of splitting the phase space into two parts and using different Lyapunov functions in these parts. This idea has already proven its effectiveness in a number of works, see [16, 17].

1. PROBLEM FORMULATION

Let us consider a generalized Lotka–Volterra type system

$$\dot{x}_i = g_i(x_i)\left(a_i^{(\sigma)} + \sum_{j=1}^{n} b_{ij}^{(\sigma)} f_j(x_j)\right), \quad i = 1,\ldots,n. \tag{1}$$

Systems of this type are most often used to model the interaction of several populations in a biological community [1–3]. Here, the variables $x_1(t),\ldots,x_n(t)$ describe the size of populations at a time instance $t \geq 0$. A piecewise-constant function $\sigma = \sigma(t)\colon [0,+\infty) \to S = \{1,\ldots,N\}$ sets the law of switching between various possible modes of system operation. Switches can be caused both by changes in the environmental conditions in which populations live and be a result of targeted human influence on a given biological community. Constant coefficients $a_i^{(s)}$ and $b_{ij}^{(s)}$ characterize the rate of natural growth or decline in populations, as well as intraspecific and interspecific interactions; $i, j = 1,\ldots,n$; $s = 1,\ldots,N$. Functions $g_i(x_i)$ and $f_i(x_i)$ are usually selected from some given class of functions based on available observations; $i = 1,\ldots,n$.

Note that systems of the form (1) are also widely used to describe some chemical and economic processes [1].

According to the physical meaning of the variables, we investigate system (1) in a nonnegative orthant $K^+ = \{\mathbf{x} = (x_1,\ldots,x_n)^{\mathrm{T}} : x_i \geq 0,\ i = 1,\ldots,n\}$. By $K_0^+ = \{\mathbf{x} = (x_1,\ldots,x_n)^{\mathrm{T}} : x_i > 0,\ i = 1,\ldots,n\}$ we denote the interior of the orthant K^+.

We assume that the functions $g_i(x_i)$ and $f_i(x_i)$, $i = 1,\ldots,n$, satisfy the following standard conditions (see [1–3]): $g_i(x_i)$ and $f_i(x_i)$ are continuous for $x_i \geq 0$, and $f_i(x_i)$ is continuously differentiable for $x_i > 0$; system (1) has the property of uniqueness of solutions to the Cauchy problem in the orthant K^+; $f_i(0) = g_i(0) = 0$, and $f_i(x_i) > 0$ and $g_i(x_i) > 0$ for $x_i > 0$; $f_i'(x_i) > 0$ for $x_i > 0$, and $f_i(x_i) \to +\infty$ as $x_i \to +\infty$.

Note that K^+ and K_0^+ are invariant sets of system (1).

Definition. A compact set G_1 is called *attracting* for system (1) with the attraction region G_2, $G_1 \subset G_2 \subset K^+$, if for any $t_0 \geq 0$ and $\mathbf{x}_0 \in G_2$ there exists a $\rho \geq 0$ such that $\mathbf{x}(t) \in G_1$ for all $t \geq t_0 + \rho$. Here, $\mathbf{x}(t)$ is the solution to system (1) emerging from the point $\mathbf{x}_0$ at a time instance t_0. If the value ρ is independent of choice of t_0, then the set G_1 is referred to as *uniformly attractive*. If the domain of attraction is the entire nonnegative orthant ($G_2 = K^+$), then it is said that system (1) is *dissipative* in K^+. Finally, if $G_1 \subset K_0^+$, then system (1) is called *permanent*.

Carrying out the definition for some compact set G_1 means that solutions starting in the area G_2, are ultimately bounded (population numbers do not exceed certain finite values). Permanence means that the specified solutions are, in addition, separated from the orthant boundary K^+ (populations do not die out).

The goal of this work is to find conditions that guarantee the ultimate boundedness of the solutions to system (1) and its permanence, as well as an estimate of the size of the corresponding regions G_1 and G_2.

2. CONSTRUCTION OF LYAPUNOV FUNCTIONS

Let us consider a family of subsystems that form a hybrid system (1):

$$\dot{x}_i = g_i(x_i)\left(a_i^{(s)} + \sum_{j=1}^{n} b_{ij}^{(s)} f_j(x_j)\right), \quad i = 1,\ldots,n, \quad s = 1,\ldots,N. \tag{2}$$

In [16] we assumed that subsystems (2) have a general equilibrium position $\overline{\mathbf{x}} = (\overline{x}_1,\ldots,\overline{x}_n)^{\mathrm{T}} \in K_0^+$ and proposed to analyze the behavior of solutions to system (1) using Lyapunov functions given by

$$V_1(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \int_{\overline{x}_i}^{x_i} \frac{f_i^{\omega}(\tau)}{g_i(\tau)} d\tau, \qquad V_2(\mathbf{x}) = \sum_{i=1}^{n} \lambda_i \int_{\overline{x}_i}^{x_i} \frac{\left(f_i(\tau) - f_i(\overline{x}_i)\right)^{\omega}}{g_i(\tau)} d\tau,$$

where $\lambda_1,\dots,\lambda_n$ are positive constants, ω is a positive rational number with odd numerator and denominator. In this work, we use Lyapunov functions of a different type, which allow weakening some of the assumptions made in [16] and obtaining simpler sufficient conditions that provide the required dynamics of solutions to the system under study.

Assumption 1. *Suppose that for some nonempty subset $\hat{S}^- \subset S$ the system of inequalities*

$$\mathbf{B}_s\hat{\theta} < \mathbf{0}, \quad s \in \hat{S}^-, \tag{3}$$

has a positive solution for $\hat{\theta}$. Here, $\mathbf{B}_s = \left\{\hat{b}_{ij}^{(s)}\right\}_{i,j=1}^n$, $\hat{b}_{ii}^{(s)} = b_{ii}^{(s)}$, and $\hat{b}_{ij}^{(s)} = \max\left\{b_{ij}^{(s)};0\right\}$ at $i \neq j$. Inequalities (3) are understood componentwise.

To analyze the ultimate boundedness of solutions to system (1), we construct a Lyapunov function given by [14]

$$\hat{V}(\mathbf{x}) = \max_{i=1,\dots,n}\frac{f_i(x_i)}{\hat{\theta}_i},$$

where for $\hat{\theta} = (\hat{\theta}_1,\dots,\hat{\theta}_n)^{\mathrm{T}}$ we take a positive solution to system (3).

Let us choose an arbitrary point $\hat{\mathbf{x}} = (\hat{x}_1,\dots,\hat{x}_n)^{\mathrm{T}} \in K_0^+$. Consider a solution $\mathbf{x}(t) = (x_1(t),\dots,x_n(t))^{\mathrm{T}}$ to a subsystem with number s from family (2) emerging from the point $\hat{\mathbf{x}}$ at a time instance $t = 0$. Denote

$$M = \max_{i=1,\dots,n}\frac{f_i(\hat{x}_i)}{\hat{\theta}_i}.$$

Let us define a subset of indices $I \subset \{1,\dots,n\}$ such that $f_i(\hat{x}_i)/\hat{\theta}_i = M$ at $i \in I$ and $f_i(\hat{x}_i)/\hat{\theta}_i < M$ at $i \notin I$. For each $i \in I$ we have

$$\frac{d}{dt}\left(\frac{f_i(x_i(t))}{\hat{\theta}_i}\right)\bigg|_{t=0} = \frac{f_i'(\hat{x}_i)}{\hat{\theta}_i}g_i(\hat{x}_i)\left(a_i^{(s)} + \sum_{j=1}^n b_{ij}^{(s)}f_j(\hat{x}_j)\right) \le \frac{f_i'(\hat{x}_i)}{\hat{\theta}_i}g_i(\hat{x}_i)\left(a_i^{(s)} + \frac{f_i(\hat{x}_i)}{\hat{\theta}_i}\sum_{j=1}^n \hat{b}_{ij}^{(s)}\hat{\theta}_j\right).$$

Taking into account Assumption 1, there exist constants $\delta_1 > 0$ and $\delta_2 \ge 0$ such that

$$D^+\hat{V}(\hat{\mathbf{x}})\Big|_{(s)} \le \max_{i\in I}\left\{\frac{f_i'(\hat{x}_i)}{\hat{\theta}_i}g_i(\hat{x}_i)\left(\hat{a}_1 - \delta_1\hat{V}(\hat{\mathbf{x}})\right)\right\}, \quad s \in \hat{S}^-,$$

$$D^+\hat{V}(\mathbf{x})\Big|_{(s)} \le \max_{i\in I}\left\{\frac{f_i'(\hat{x}_i)}{\hat{\theta}_i}g_i(\hat{x}_i)\left(\hat{a}_2 + \delta_2\hat{V}(\hat{\mathbf{x}})\right)\right\}, \quad s \in \hat{S}^+.$$

Here, $D^+\hat{V}(\mathbf{x})\Big|_{(s)}$ is the upper right Dini derivative [18] of the function $\hat{V}(\mathbf{x})$ by virtue of the sth subsystem from family (2), $\hat{S}^+ = S\backslash\hat{S}^-$, $\hat{a}_1 = \max\limits_{s\in\hat{S}^-}\max\limits_{i=1,\dots,n} a_i^{(s)}$, and $\hat{a}_2 = \max\limits_{s\in\hat{S}^+}\max\limits_{i=1,\dots,n} a_i^{(s)}$.

Let us choose an $\hat{H} > 0$ such that $\hat{V}(\hat{\mathbf{x}}) > \hat{a}_1/\delta_1$ for all $\hat{\mathbf{x}}$ satisfying the condition $\|\hat{\mathbf{x}}\| \ge \hat{H}$. From now on, unless otherwise stated, the norm of a vector is understood to be the Euclidean norm.

Given the properties of functions $f_i(x_i)$ and $g_i(x_i)$, $i = 1,\dots,n$, for any $H_1 \ge \hat{H}$ and for any number γ we can choose positive coefficients $c_1(H_1,\gamma)$ and $c_2(H_1,\gamma)$ in such a way that the inequalities

$$c_1(H_1,\gamma)\hat{V}^{\gamma}(\hat{\mathbf{x}}) \le \frac{f_i'(\hat{x}_i)}{\hat{\theta}_i}g_i(\hat{x}_i) \le c_2(H_1,\gamma)\hat{V}^{\gamma}(\hat{\mathbf{x}}) \tag{4}$$

hold at $\hat{H} \le \|\hat{\mathbf{x}}\| \le H_1$, $i \in I$.

Thus, in the region

$$K_1(H_1) = \left\{\mathbf{x} \in K_0^+ : \hat{H} \le \|\mathbf{x}\| \le H_1\right\} \tag{5}$$

we arrive at the estimates

$$D^{+}\hat{V}(\mathbf{x})\Big|_{(s)} \le -\hat{\alpha}(H_1,\gamma)\hat{V}^{1+\gamma}(\mathbf{x}), \quad s \in \hat{S}^{-}, \tag{6}$$

$$D^{+}\hat{V}(\hat{\mathbf{x}})\Big|_{(s)} \le \hat{\beta}(H_1,\gamma)\hat{V}^{1+\gamma}(\mathbf{x}), \quad s \in \hat{S}^{+}, \tag{7}$$

where $\hat{\alpha}(H_1,\gamma)$ and $\hat{\beta}(H_1,\gamma)$ are constant coefficients, depending, generally speaking, on the choice of values H_1 and γ; $\hat{\alpha}(H_1,\gamma) > 0$ and $\hat{\beta}(H_1,\gamma) \ge 0$.

Remark 1. Below, estimates (6) and (7) are used to establish restrictions on the switching law in system (1) guaranteeing the ultimate boundedness of solutions and ensuring the required sizes of regions G_1 and G_2 (see definition). Obviously, these results depend on the choice of values H_1 and γ. This allows posing the problem of optimizing the choice of these values to obtain the best results in some sense. This problem can be solved through numerical analysis. Moreover, in the left- and right-hand sides of (4) we can use different parameter values γ; then, the degrees of the function $\hat{V}(\mathbf{x})$ in differential inequalities (6) and (7) also are different. Using the approaches described in [16], we can also successfully study this more general case. However, because the problem of optimizing the selection of γ is not solved in this article, in what follows, we limit ourselves to considering a simpler situation when the parameter γ is chosen uniformly in inequalities (4) and, correspondingly, in (6), (7).

To find the conditions for the permanence of system (1), we introduce additional assumptions.

Assumption 2. *Let* $\int_0^1 \frac{d\tau}{g_i(\tau)} = +\infty$, $i = 1,\dots,n$.

Assumption 3. *Suppose that for some nonempty subset* $\tilde{S}^{-} \subset S$ *the system of inequalities*

$$\tilde{\mathbf{B}}_s^{\mathrm{T}}\tilde{\theta} < \mathbf{0}, \quad s \in \tilde{S}^{-}, \tag{8}$$

has a positive solution for $\tilde{\theta}$. *Here*, $\tilde{\mathbf{B}}_s = \left\{\tilde{b}_{ij}^{(s)}\right\}_{i,j=1}^{n}$, $\tilde{b}_{ii}^{(s)} = b_{ii}^{(s)}$, *and* $\tilde{b}_{ij}^{(s)} = \left|b_{ij}^{(s)}\right|$ *at* $i \ne j$. *Inequalities* (8) *are understood componentwise*.

Remark 2. Necessary and sufficient conditions for the solvability of linear inequalities of form (3) and (8) with Metzler matrices $\hat{\mathbf{B}}_s$ and $\tilde{\mathbf{B}}_s^{T}$ were obtained in [19]. Note that the existence of a positive solution to the system $\hat{\mathbf{B}}_s\hat{\theta} < \mathbf{0}$ for some set of indices s, generally speaking, does not imply that there exists a positive solution to the system $\tilde{\mathbf{B}}_s^{\mathrm{T}}\tilde{\theta} < \mathbf{0}$ for the same index values s, and vice versa. Therefore, subsets $\hat{S}^{-}$ and $\tilde{S}^{-}$ in Assumptions 1 and 3 may be different.

Assumption 4. *Let a point* $\tilde{\mathbf{x}} = (\tilde{x}_1,\dots,\tilde{x}_n)^{\mathrm{T}} \in K_0^{+}$ *exist such that the inequalities are satisfied*:

$$\sum_{i=1}^{n}\tilde{\theta}_i\left|a_i^{(s)} + \sum_{j=1}^{n} b_{ij}^{(s)} f_j(\tilde{x}_j)\right| < \min_{j=1,\dots,n}\delta_j^{(s)} f_j(\tilde{x}_j), \quad s \in \tilde{S}^{-}, \tag{9}$$

where $\tilde{\theta} = (\tilde{\theta}_1,\dots,\tilde{\theta}_n)^{\mathrm{T}}$ *is the positive solution to system* (8) *and* $\delta_j^{(s)} = -\sum_{i=1}^{n}\tilde{b}_{ij}^{(s)}\tilde{\theta}_i$.

Remark 3. According to inequalities (8) we have $\delta_j^{(s)} > 0$, $j = 1,\dots,n$, $s \in \tilde{S}^{-}$. Note also that using the change of variables $y_j = f_j(\tilde{x}_j)$, $j = 1,\dots,n$, inequalities (9) can be rewritten in a linear form. Thus, checking the solvability of system (9) is not difficult to perform using both analytical and numerical approaches. For example, Assumption 4 is satisfied if subsystems (2) with numbers from the subset $\tilde{S}^{-}$ have equilibrium positions located in the positive orthant sufficiently close to each other.

Let us now construct a Lyapunov function of the form

$$\tilde{V}(\mathbf{x}) = \sum_{i=1}^{n}\tilde{\theta}_i\left|\int_{\tilde{x}_i}^{x_i}\frac{d\tau}{g_i(\tau)}\right|,$$

selecting the parameters $\tilde{\theta}_i$ and $\tilde{x}_i$, $i = 1,\dots,n$, according to Assumptions 3 and 4.

We get

$$D^+\widetilde{V}(\mathbf{x})\Big|_{(s)} \le \sum_{i=1}^n \tilde{\theta}_i \operatorname{sgn}(x_i - \tilde{x}_i)\left(a_i^{(s)} + \sum_{j=1}^n b_{ij}^{(s)} f_j(x_j)\right)$$

$$\le \sum_{i=1}^n \tilde{\theta}_i \left|a_i^{(s)} + \sum_{j=1}^n b_{ij}^{(s)} f_j(\tilde{x}_j)\right| + \sum_{j=1}^n \left(\sum_{i=1}^n \tilde{b}_{ij}^{(s)} \tilde{\theta}_i\right) \left|f_j(x_j) - f_j(\tilde{x}_j)\right|.$$

Denote

$$P_s = \left\{\mathbf{x} = (x_1, \ldots, x_n)^T : \sum_{j=1}^n \delta_j^{(s)} \left|f_j(x_j) - f_j(\tilde{x}_j)\right| \le M_s\right\}, \quad s \in \widetilde{S}^-, \tag{10}$$

where $M_s = \sum_{i=1}^n \tilde{\theta}_i \left|a_i^{(s)} + \sum_{j=1}^n b_{ij}^{(s)} f_j(\tilde{x}_j)\right|$. Sets (10) are compact, and, when inequalities (9) hold, we have $P_s \subset K_0^+$ and $s \in \widetilde{S}^-$. This means that we can construct a compact set P so that

$$\bigcup_{s \in \widetilde{S}^-} P_s \subset P \subset K_0^+, \tag{11}$$

and for any $H_2 > 0$ in the region $K_2(H_2)\backslash P$ the estimate holds:

$$D^+\widetilde{V}(\mathbf{x})\Big|_{(s)} \le -\tilde{\alpha}(H_2), \quad s \in \widetilde{S}^-. \tag{12}$$

Here, $\tilde{\alpha}(H_2)$ is some positive coefficient and

$$K_2(H_2) = \left\{x \in K_0^+ : \|x\| \le H_2\right\}. \tag{13}$$

Similarly, we can choose a nonnegative coefficient $\tilde{\beta}(H_2)$ so that in the region $K_2(H_2)\backslash P$ the inequalities hold:

$$D^+\widetilde{V}(\mathbf{x})\Big|_{(s)} \le \tilde{\beta}(H_2), \quad s \in \widetilde{S}^+, \tag{14}$$

where $\widetilde{S}^+ = S\backslash\widetilde{S}^-$.

3. ANALYSIS OF DYNAMICS OF SOLUTIONS TO HYBRID SYSTEM

Let us first establish sufficient conditions for the ultimate boundedness of solutions of system (1) starting in some arbitrarily chosen neighborhood of the origin.

Denote

$$A(u) = \max_{\mathbf{x} \in K_0^+ : \|\mathbf{x}\| = u} \hat{V}(\mathbf{x}), \quad B(v) = \max_{\mathbf{x} \in K_0^+ : \hat{V}(\mathbf{x}) = v} \|\mathbf{x}\|.$$

The functions $A(u)$ and $B(v)$ are defined and strictly increasing on the interval $[0, +\infty)$, and $A(u) \to +\infty$ and $B(v) \to +\infty$ as $u, v \to +\infty$.

Assumption 5. Let the duration of the time intervals during which switching in system (1) occurs only between subsystems from a set $\hat{S}^-$ be bounded from below by a positive constant $\hat{L}_1$, and suppose that the duration of time intervals during which switching occurs only between subsystems from a set $\hat{S}^+$ is bounded from above by a positive constant $\hat{L}_2$.

Theorem 1. *Let Assumptions* 1 *and* 5 *hold, and suppose that for some* $H_1 \ge \hat{H}$ *in the region* $K_1(H_1)$ *estimates* (6) *and* (7) *are constructed and the following conditions are true*:

(1) *There exist numbers* Δ_1 *and* Δ_2 *such that*

$$\hat{H} < B(A(\hat{H})) < \Delta_1 < \Delta_2 < B(A(\Delta_2)) < H_1. \tag{15}$$

(2) *Constants* $\hat{L}_1$ *and* $\hat{L}_2$ *satisfy the inequalities*

$$-\hat{\alpha}(H_1,\gamma)\hat{L}_1 + \hat{\beta}(H_1,\gamma)\hat{L}_2 < 0, \tag{16}$$

$$\hat{L}_2 \le \min\{\hat{L}_{21}; \hat{L}_{22}\}, \tag{17}$$

where

$$\hat{L}_{21} = \begin{cases} \gamma^{-1}\hat{\beta}^{-1}(H_1,\gamma)\Big(\big(A(\Delta_2)\big)^{-\gamma} - \Big(B^{(-1)}(H_1)\Big)^{-\gamma}\Big), & \textit{if}\ \ \gamma \ne 0 \\ \hat{\beta}^{-1}(H_1,0)\ln\Big(B^{(-1)}(H_1)/A(\Delta_2)\Big), & \textit{if}\ \ \gamma = 0, \end{cases}$$

$$\hat{L}_{22} = \begin{cases} \gamma^{-1}\hat{\beta}^{-1}(H_1,\gamma)\Big(\big(A(\hat{H})\big)^{-\gamma} - \Big(B^{(-1)}(\Delta_1)\Big)^{-\gamma}\Big), & \textit{if}\ \ \gamma \ne 0 \\ \hat{\beta}^{-1}(H_1,0)\ln\Big(B^{(-1)}(\Delta_1)/A(\hat{H})\Big), & \textit{if}\ \ \gamma = 0, \end{cases}$$

and $B^{(-1)}(\cdot)$ *is the function inverse to the function* $B(\cdot)$.

Then the set $K_2(\Delta_1)$ *is attractive for system* (1) *with attraction region* $K_2(\Delta_2)$. *Here*, *the regions* $K_1(\cdot)$ *and* $K_2(\cdot)$ *are constructed according to formulas* (5) *and* (13).

Proof. Let us build on the interval $[0,+\infty)$ a piecewise-constant function $\hat{\eta}(t)$ such that $\hat{\eta}(t) = -\hat{\alpha}(H_1,\gamma)$ if $\sigma(t) \in \hat{S}^-$ and $\hat{\eta}(t) = \hat{\beta}(H_1,\gamma)$ if $\sigma(t) \in \hat{S}^+$. Let us set $t_0 \ge 0$ and $\mathbf{x}_0 \in K_2(\Delta_2)$. Let us consider the solution $\mathbf{x}(t)$ to system (1) emerging from the point $\mathbf{x}_0$ at a time instance t_0. Obviously, it is enough to examine the situation when $\|\mathbf{x}_0\| > \Delta_1$.

Taking into account differential inequalities (6) and (7), we obtain (see [18]) the following: so far the solution $\mathbf{x}(t)$ remains in the region $K_1(H_1)$, the estimates take place:

$$\hat{V}^{-\gamma}(\mathbf{x}(t)) \ge \hat{V}^{-\gamma}(\mathbf{x}_0) - \gamma\int_{t_0}^{t}\hat{\eta}(t)dt, \quad \text{if} \quad \gamma > 0,$$

$$\hat{V}^{-\gamma}(\mathbf{x}(t)) \le \hat{V}^{-\gamma}(\mathbf{x}_0) - \gamma\int_{t_0}^{t}\hat{\eta}(t)dt, \quad \text{if} \quad \gamma < 0,$$

$$\hat{V}(\mathbf{x}(t)) \le \hat{V}(\mathbf{x}_0)e^{\int_{t_0}^{t}\hat{\eta}(t)dt}, \quad \text{if} \quad \gamma = 0.$$

Then from conditions (15)–(17) we get the desired result. In fact, inequality (16) provides the tendency $\int_{t_0}^{t}\hat{\eta}(t)dt \to -\infty$ as $t \to +\infty$. This means that at some time instance the solution $\mathbf{x}(t)$ is in the region $K_2(\hat{H})$. The condition $\hat{L}_2 \le \hat{L}_{21}$ guarantees that this solution at any time remains in the region $K_2(H_1)$. Finally, the requirement $\hat{L}_2 \le \hat{L}_{22}$ entails the fact that all solutions to system (1) starting in the region $K_2(\hat{H})$ do not leave the region $K_2(\Delta_1)$. Inequalities (15) imply the correctness of estimate (17).

□

Remark 4. It is easy to see that for the existence of constants Δ_1 and Δ_2 satisfying inequalities (15), it is necessary and sufficient that the condition $B(A(B(A(\hat{H})))) < H_1$ holds. This condition is satisfied if the value H_1 is set large enough. Estimates (16) and (17) represent restrictions on the switching law in a hybrid system (1), providing the required dynamics of its solutions. These restrictions are determined by the size of the region of initial values of solutions ($K_2(\Delta_2)$) and the region of limiting values of solutions ($K_2(\Delta_1)$) for chosen values of Δ_1 and Δ_2.

Remark 5. Note that the accuracy of the resulting numerical estimates can be increased if, when analyzing the ultimate boundedness of solutions for a specific system (1), as a vector norm, we use not the Euclidean norm, but put $\|\mathbf{x}\| = \hat{V}(\mathbf{x})$. This eliminates unnecessary roughness in the estimates. In this case we have $A(u) = u$ and $B(v) = v$, then it suffices to choose the constants Δ_1 and Δ_2 in Theorem 1 on the basis of the conditions $\hat{H} < \Delta_1 < \Delta_2 < H_1$.

Remark 6. If $\hat{S}^- = S$ under Assumption 1, then system (1) is uniformly dissipative in K^+ under any switching law. In this case the function $\hat{V}(\mathbf{x})$ is the general Lyapunov function for subsystems (2) satisfying Yoshizawa's theorem on uniform dissipativity.

Remark 7. If a $\gamma \le 0$ exists such that the coefficients $\hat{\alpha}$ and $\hat{\beta}$ in differential inequalities (6) and (7) can be chosen independent of H_1, then, if condition (16) is met, system (1) is uniformly dissipative in K^+. Indeed, in this case we can put $H_1 = \Delta_2 = +\infty$. For example, if the functions $f_i'(x_i)g_i(x_i)$, $i = 1,\dots,n$, are bounded from below and above by positive constants on an infinite interval $[\hat{H},+\infty)$, then, assuming $\gamma = 0$, we arrive at the indicated conditions of uniform dissipativity. This situation is typical, in particular, for saturation systems [20].

Let us now additionally involve the Lyapunov function $\tilde{V}(\mathbf{x})$ in our analysis to establish sufficient conditions for the permanence of system (1).

Assumption 6. Let the duration of the time intervals during which switching in system (1) occurs only between subsystems from a set $\tilde{S}^-$ be bounded from below by a positive constant $\tilde{L}_1$, and suppose that the duration of time intervals during which switching occurs only between subsystems from a set $\tilde{S}^+$ is bounded from above by a positive constant $\tilde{L}_2$.

Theorem 2. *Let Assumptions* 1–6 *be fulfilled and suppose that the following conditions hold*:

(1) *For some* $H_1 \ge \hat{H}$ *in the region* $K_1(H_1)$ *estimates* (6) *and* (7) *are constructed*, *there exist numbers* Δ_1 *and* Δ_2 *satisfying inequalities* (15), *and for constants* $\hat{L}_1$ *and* $\hat{L}_2$ *relations* (16) *and* (17) *are satisfied*.

(2) *For* $H_2 = \Delta_1$ *in the region* $K_2(H_2)\backslash P$ *estimates* (12) *and* (14) *are constructed and constants* $\tilde{L}_1$ *and* $\tilde{L}_2$ *satisfy the relation*

$$-\tilde{\alpha}(\Delta_1)\tilde{L}_1 + \tilde{\beta}(\Delta_1)\tilde{L}_2 < 0. \tag{18}$$

Then the set

$$G = \left\{\mathbf{x} \in K_0^+ : \tilde{V}(\mathbf{x}) \le \max_{\mathbf{x}\in P\cap K_2(\Delta_1)} \tilde{V}(\mathbf{x}) + \tilde{\beta}(\Delta_1)\tilde{L}_2\right\} \cap K_2(\Delta_1) \tag{19}$$

is attractive for system (1) *with the attraction region* $K_2(\Delta_2)$. *Here*, *the regions* $K_1(\cdot)$ *and* $K_2(\cdot), P$ *are constructed according to formulas* (5), (13), *and* (11).

Proof. According to Theorem 1, any solution to system (1) starting in the area $K_2(\Delta_2)$ at some time instance occurs in the region $K_2(\Delta_1)$ and will never leave it.

Let us build on the interval $[0,+\infty)$ a piecewise-constant function $\tilde{\eta}(t)$ such that $\tilde{\eta}(t) = -\tilde{\alpha}(\Delta_1)$ if $\sigma(t) \in \tilde{S}^-$ and $\tilde{\eta}(t) = \tilde{\beta}(\Delta_1)$ if $\sigma(t) \in \tilde{S}^+$. We prescribe some $t_0 \ge 0$ and $\mathbf{x}_0 \in K_2(\Delta_1)$. Consider the solution $\mathbf{x}(t)$ to system (1) emerging from the point $\mathbf{x}_0$ at a time instance t_0. Let us explore the situation when $\mathbf{x}_0 \notin P$.

In accordance with differential inequalities (12) and (14), so far the solution $\mathbf{x}(t)$ remains in the region $K_2(\Delta_1)\backslash P$, the estimate holds:

$$\tilde{V}(\mathbf{x}(t)) \le \tilde{V}(\mathbf{x}_0) + \int_{t_0}^{t} \tilde{\eta}(t)dt. \tag{20}$$

Due to inequality (18) we get $\int_{t_0}^{t} \tilde{\eta}(t)dt \to -\infty$ as $t \to +\infty$. This means that at some time instance the solution $\mathbf{x}(t)$ appears in the region P. At the same time, taking into account estimate (20), any solution to the system starting in the region P will not leave the region G.

□

Remark 8. Note that the conditions of permanence in this work can only be obtained by jointly using two Lyapunov functions $\hat{V}(\mathbf{x})$ and $\tilde{V}(\mathbf{x})$. Here, we use gluing of these functions. The function $\hat{V}(\mathbf{x})$ is suitable only for analyzing the ultimate boundedness of solutions to system (1). And with just one function $\tilde{V}(\mathbf{x})$, without some additional more burdensome assumptions, it is not be possible to guarantee either the permanence or even simply the utmost boundedness of solutions. This function is only suitable for very local analysis of the system. In particular, this article does not even assume the fulfillment of the conditions $\int_{\tilde{x}_i}^{x_i} \frac{d\tau}{g_i(\tau)} \to +\infty$ as $x_i \to +\infty$, $i = 1,\ldots,n$, that is, the function $\widetilde{V}(\mathbf{x})$ may not be infinitely large as $\|\mathbf{x}\| \to \infty$.

Remark 9. The conditions for the ultimate boundedness of solutions and the permanence of the system obtained in the article depend on the choice of parameters H_1, γ, Δ_1, and Δ_2. As already noted in Remark 1, to find the best conditions, the choice of these parameters can be optimized, for example, using grid methods.

Example. Consider a family (2) consisting of three subsystems:

$$\begin{aligned} \dot{x}_1 &= x_1\left(2 - 3\ln(1 + x_1^4) + \ln(1 + x_2^4)\right), \\ \dot{x}_2 &= x_2\left(1 + \ln(1 + x_1^4) - 2\ln(1 + x_2^4)\right); \end{aligned} \tag{21}$$

$$\begin{aligned} \dot{x}_1 &= x_1\left(-1 - 2\ln(1 + x_1^4) + \ln(1 + x_2^4)\right), \\ \dot{x}_2 &= x_2\left(-1 - 3\ln(1 + x_1^4) - \ln(1 + x_2^4)\right); \end{aligned} \tag{22}$$

$$\begin{aligned} \dot{x}_1 &= x_1\left(-1 - \ln(1 + x_1^4) + 2\ln(1 + x_2^4)\right), \\ \dot{x}_2 &= x_2\left(1 + \ln(1 + x_1^4) - \ln(1 + x_2^4)\right). \end{aligned} \tag{23}$$

Thus, we have here $n = 2$, $N = 3$, $g_i(x_i) = x_i$, $f_i(x_i) = \ln(1 + x_i^4)$, and $i = 1,2$.

We assume that subsystem (21) corresponds to the first operating mode of the hybrid system, subsystem (22) corresponds to the second mode, and subsystem (23) corresponds to the third one. It is easy to see that Assumption 1 holds for $\hat{S}^- = \{1,2\}$ (we can put $\hat{\theta} = (1,1)^{\mathrm{T}}$). Assumption 3 is satisfied for $\widetilde{S}^- = \{1\}$ (we can put $\tilde{\theta} = (1,1)^{\mathrm{T}}$). Assumptions 2 and 4 are also fulfilled. In fact, subsystem (21) has an equilibrium position $\left(\sqrt[4]{e-1}, \sqrt[4]{e-1}\right)^{\mathrm{T}}$ in K_0^+ and, accordingly, it can be taken as a point $\tilde{x}$ from Assumption 4 (see Remark 3).

According to the notation introduced earlier, we can put $\hat{H} = 3.9$. For any $H_1 \geq \hat{H}$ in the region $K_1(H_1)$ the differential inequalities hold:

$$D^+\hat{V}(\mathbf{x})\Big|_{(s)} \leq -1.9\hat{V}(\mathbf{x}), \quad s \in \hat{S}^-,$$

$$D^+\hat{V}(\mathbf{x})\Big|_{(s)} \leq 5\hat{V}(\mathbf{x}), \quad s \in \hat{S}^+,$$

that is, here, $\gamma = 0$, and the coefficients $\hat{\alpha}(H_1,0) = 1.9$ and $\hat{\beta}(H_1,0) = 5$ are independent of choice of H_1. This means (see Remark 7) that we can take $H_1 = \Delta_2 = +\infty$ and establish sufficient conditions for uniform dissipativity of the hybrid system under study.

We have $A(\hat{H}) = 5.4$, $B(A(\hat{H})) = 5.5$. Let us set, for example, $\Delta_1 = 10$. Then $B^{(-1)}(\Delta_1) = 7.8$. Let Assumption 5 be satisfied. Writing out relations (16) and (17), we find by Theorem 1 that, if $\hat{L}_1 > 2.6\hat{L}_2$ and $\hat{L}_2 \leq 0.07$, then the set $K_2(10)$ is attractive for the hybrid system under consideration, and the attraction region of this set is the entire orthant K_0^+.

Let us now define a compact set

$$P = \left\{\mathbf{x} = (x_1, x_2)^{\mathrm{T}} : 1 \leq x_i \leq 10,\, i = 1,2\right\} \subset K_0^+.$$

In area $K_2(10)\backslash P$ we arrive at differential inequalities

$$D^+\widetilde{V}(\mathbf{x})\Big|_{(s)} \leq -0.9, \quad s \in \widetilde{S}^-,$$

$$D^+\widetilde{V}(\mathbf{x})\Big|_{(s)} \leq 10, \quad s \in \widetilde{S}^+.$$

Let Assumption 6 be satisfied. Then if, in addition to the previously found conditions, the relation $\widetilde{L}_1 > 11.2\widetilde{L}_2$ is also satisfied, then, as follows from Theorem 2, system (1) is permanent. By introducing some restriction on $\widetilde{L}_2$, we can find an estimate of the attracting set using formula (19). For example, assuming $\widetilde{L}_2 \le 0.1$, we get

$$G = \left\{ \mathbf{x} = (x_1, x_2)^{\mathrm{T}} \in K_0^+ : \left| \ln \frac{x_1}{\tilde{x}_1} \right| + \left| \ln \frac{x_2}{\tilde{x}_2} \right| \le 4.7 \right\} \cap K_2(10).$$

Let us assume that subsystems (21)–(23) are activated sequentially one after another. Suppose that $L_1^{(s)}$ and $L_2^{(s)}$, $s = 1, 2, 3$, are lower and upper restrictions on the lengths of subsystem activity intervals (21)–(23), respectively. Then, taking into account the above, we obtain the fact that if the conditions

$$L_1^{(1)} + L_1^{(2)} > 2.6L_2^{(3)}, \quad L_2^{(3)} \le 0.07$$

are met, a given hybrid system is uniformly dissipative in K_0^+ with the attracting set $K_2(10)$. If, in addition, the conditions

$$L_1^{(1)} > 11.2(L_2^{(2)} + L_2^{(3)}), \quad L_2^{(2)} + L_2^{(3)} \le 0.1,$$

hold, then the region serves as a globally attracting set G.

CONCLUSIONS

The sufficient conditions for the ultimate boundedness of solutions and the permanence of the system established in the work were based on the idea of fragmenting the phase space and using its own Lyapunov function in each of the resulting parts. This approach allows more subtly taking into account the nonlinear features of the system during global analysis. In this article, we used two Lyapunov functions of different structures. The task of one of these functions was to drive the solutions of the system into a bounded neighborhood of the origin of coordinates (to ensure that the solutions are ultimately bounded). The task of the other function was to move solutions away from the boundaries of the nonnegative orthant (to ensure the permanence of the system).

FUNDING

This work was supported by ongoing institutional funding. No additional grants to carry out or direct this particular research were obtained.

CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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